Optimal Investment and Risk Control for an Insurer with Stochastic Factor

Lijun Bo* Shihua Wang†

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Abstract

We study an optimal investment and risk control problem for an insurer under stochastic factor. The insurer allocates his wealth across a riskless bond and a risky asset whose drift and volatility depend on a factor process. The risk process is modeled by a jump-diffusion with state-dependent jump measure. By maximizing the expected power utility of the terminal wealth, we characterize the optimal strategy of investment and risk control, analyze classical solutions of HJB PDE and prove the verification theorem.

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1 Introduction

Since the seminal work of Merton [11], portfolio optimization problems have been the subject of considerable investigation. The recent development focuses on the optimal investment problem with stochastic volatility, see e.g. Fouque, et al. [6]. The stochastic volatility model directly relaxes the log-normal assumptions on the price process dynamics which is able to capture empirically observed features of price processes and has been successfully used in several contexts, including stochastic interest rates, see e.g. Brennan and Xia [3], and stochastic volatility, see e.g. Zariphopoulou [20] for a related survey.

The paper considers an optimal investment and risk control problem for an insurer under stochastic factor. The stochastic factor models the evolution of macroeconomic variables such as interest rates, broad share price indices or measures of economic activity or growth. For the case without stochastic factor, Zou and Cadenillas [22] study an optimal investment and risk control for an insurer by selecting the insurance policies. This work is also related to the optimal reinsurance which is raised by the case where the insurer wants to control the reinsurance payout. It has been extensively studied by [9] and [18] in the (jump) diffusion market. Zhuo, et al. [21] take the regime-switching risk into the optimal reinsurance. Peng and Wang [15] consider the optimal strategy of investment and risk control for an insurer who has some inside information on the insurance business. Further, for the general Lévy market model without risk control, Nutz [13] studies power utility maximization for exponential Lévy models with portfolio constraints. For the random utility, Nutz [12] studies utility maximization for power utility random fields with and without intermediate consumption in a general semimartingale model with closed portfolio constraints. For

*Corresponding author. Email: lijunbo@ustc.edu.cn, School of Mathematical Sciences, University of Science and Technology of China, Anhui Province 230026, China, and Wu Wen Tsun Key Laboratory of Mathematics, Chinese Academy of Science, Hefei, Anhui Province 230026, China.

†Email: shwangjy@mail.ustc.edu.cn, School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui Province 230026, China.

the random coefficient driven optimal investment without risk control, Benth, et al. [2] and Delong and Klüppelberg [5] deal with the Merton's case in a Black-Scholes market where the volatility is described as a pure-jump OU process. In our model, an insurer allocates his wealth across a riskless bond and a risky asset where drift and volatility of its price dynamics depend on a diffusion factor. The risk process is described as a general jump-diffusion with state-dependent jump measure. We also allow the correlation among the risky asset price, risk control process and stochastic factor. Differently from the works reviewed above, the appearance of stochastic factor in the model leads that HJB PDE is a fully nonlinear PDE. We then analyze classical solutions to thus equation via a power transformation and then the original HJB PDE can be transformed to a linear one. Since the coefficients of our equation on unbounded domain $\mathbb{R}$ only satisfy local conditions, they are unbounded, have unbounded derivatives and don't satisfy linear growth constraint. Hence standard existence and uniqueness results (see e.g. Chapter 6 of Friedman [7] and Section 2.9 of Krylov [10]) do not apply here. Becherer and Schweizer [1] (see Proposition 2.3) and Health and Schweizer [8] (see Theorem 1 and Lemma 2) provide new sufficient conditions for guaranteeing the existence and uniqueness of global classical solutions of the PDE under some type of local conditions. We then apply the above technique to analyze the global classical solution of the transformed equation.

The rest of the paper is organized as follows. Section 2 formulates the model. Section 3 derives the HJB PDE. Section 4 characterizes the optimal strategies for investment and risk control. Section 5 analyzes the classical solution of HJB PDE and proves the verification theorem. Section 6 presents a numerical analysis.

2 The Model

We fix $T > 0$ to be the finite target horizon and consider a complete filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$. This space also supports a 3-dimensional Brownian motion $(W_t, \hat{W}_t, \bar{W}_t)$ for $t \in [0,T]$ and an independent Poisson random measure $N(du, dt)$ on $\mathcal{U} \times [0,T]$. Here $\mathcal{U}$ is a topological space and the reference filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ is given by the augmented natural filtration generated jointly by Brownian motion and Poisson random measure.

We next describe the market model considered in this paper which consists of four blocks: the stochastic factor model, the riskless-bond, the price processes of risky asset and a risk control model.

Stochastic factor. They are used to model the evolution of macroeconomic variables such as interest rates, price indices, and measures of economic activity or growth. These factors influence both the drift and the volatility of the risky asset price processes. The single stochastic factor process is described as a real-valued diffusion process which satisfies the SDE given by

$$dY_t = b(Y_t)dt + a(Y_t)\left(\rho_1 dW_t + \sqrt{1-\rho_1^2}d\hat{W}_t\right), \quad Y_0 = y \in \mathbb{R}. \tag{1}$$

Here $\rho_1 \in (-1,1)$ is the correlation coefficient. The condition satisfied by coefficients $(a(y), b(y))$ will be imposed in the assumption (**H1**) below so that the SDE (1) admits a unique strong solution.

Riskless bond. Let $r(Y_t)$ be the time-t stochastic interest rate where $r(y)$, $y \in \mathbb{R}$, is a positive and bounded C^1-function. Then the time-t price of the riskless bond is given by $B_t = e^{\int_0^t r(Y_s)ds}$ for $t \in [0,T]$.

Price processes. The price processes of the risky asset (e.g. stock) is denoted by $S = (S_t)_{t\in[0,T]}$ whose dynamics is described as

$$dS_t = S_t\{\mu(Y_t)dt + \sigma(Y_t)dW_t\}, \qquad S_0 > 0, \tag{2}$$

where $\mu(y)$ and $\sigma(y) > 0$ are C^1-functions. The excess-stock-return is given by $\theta(y) := \mu(y) - r(y)$ for $y \in \mathbb{R}$, and hence the market price of risk is $\bar{\theta}(y) := \frac{\theta(y)}{\sigma(y)}$ for $y \in \mathbb{R}$. The additional conditions satisfied by the above coefficients will be given in the assumption (**H2**) below.

Risk control process. The risk model for claims is described as an extensive Cramér-Lundberg model, in which the claim (risk) per policy $C = (C_t)_{t\in[0,T]}$ is given by the following dynamics

$$dC_t = \phi(Y_t)\left(\rho_2 dW_t + \sqrt{1-\rho_2^2}d\bar{W}_t\right) + \int_{\mathcal{U}} g(Y_t,u)N(du,dt), \tag{3}$$

where $\rho_2 \in (-1,1)$ is the correlation coefficient, $\phi(y) > 0$ is a C^1-function, and $g(y,u)$ is a strictly positive C^1-function of jump sizes in $y \in \mathbb{R}$ for all $u \in \mathcal{U}$. Here the compensator of Poisson random measure $N(du,dt)$ is given by $\nu(Y_t,du)dt$ where $\nu(y,du)$ is a sigma-finite Borel measure on $\mathcal{U}$ for each $y \in \mathbb{R}$, which is also C^1 in y. Further it satisfies $\mathbb{E}[\int_0^T g^2(Y_t,u)\nu(Y_t,du)] < +\infty$ so that the above stochastic integral w.r.t. Poisson measure is well-defined (see also Bo, et al. [4]). Let η_t the $\mathbb{G}$-adapted total outstanding number of policies (liabilities) at time t. Thus by extending the model without stochastic factor in Zou and Cadenillas [22], the total risk of the insurer can be describe as

$$dR_t^\eta = \eta_{t-}\left\{c(Y_t)dt + dC_t\right\}. \tag{4}$$

Here $c(y) > 0$ is the premium rate which is also C^1 in $y \in \mathbb{R}$. In addition, we assume that the average premium per liability for the insurer is $p(Y_t)$. Then $p_c(y) := p(y) - c(y)$ is the excess premium per unit of liability for $y \in \mathbb{R}$, and hence the excess premium per unit of liability and per unit of risk is given by $\bar{p}_c(y) := \frac{p_c(y)}{\phi(y)}$. The further conditions satisfied by $\bar{p}_c(y)$ will be given in the assumption (**H2**) below. In terms of the above proposed models, we impose the following assumptions on the coefficients of models:

(**H1**) For the stochastic factor, $b(y)$ is Lipschitz continuous and $a(y) > 0$ is bounded Lipschitz continuous.

(**H2**) For the risky asset and risk control process, $(\bar{\theta}(y), \bar{p}_c(y))$ are C^1 and bounded.

Assumption (**H1**) on $(a(y), b(y))$ imply that SDE (1) admits a unique strong solution using Theorem V.38 in [17]. The condition (**H2**) can guarantee that the coefficient of linear term in Eq. (24) in Section 5 is bounded from above when we analyze the global classical solution of the HJB PDE. We next present an example to illustrate the richness of our model and how it can recover specifications considered in the literature as special cases.

Example 2.1. *The proposed framework is rich enough to incorporate several stochastic factor models considered in the literature. The stochastic factor process is chosen to be of the OU type:*

$$dY_t = (\alpha - \beta Y_t)dt + a\left(\rho_1 dW_t + \sqrt{1-\rho_1^2}d\hat{W}_t\right), \quad Y_0 \in \mathbb{R}.$$

*Here $\alpha \in \mathbb{R}$ and $\beta, a > 0$ are constants. Thus the assumption (**H1**) is satisfied. The price dynamics of the risky asset is given by*

$$dS_t = \mu S_t dt + \sqrt{\vartheta_1(Y_t)}S_t dW_t, \quad S_0 > 0,$$

where the return rate $\mu \in \mathbb{R}$, and the volatility $\vartheta_1(y)$ is positive and C^1. For the risk control model, we assume the premium rate $c > 0$ and the average premium per liability $p > 0$. For the constant interest rate $r > 0$, we have that the excesses of return and average premium per liability are $\theta = \mu - r$ and $p_c = p - c$. We consider a Poisson random measure whose compensator is given by $\nu(y,du) = j(y)\delta_1(du)$ for $u \in \mathcal{U} = \mathbb{R}\setminus\{0\}$ where the amplitude function $j(y)$ is positive and C^1. Then the risk control model is given by

$$dR_t^\eta = \eta_{t-}\left\{cdt + \sqrt{\vartheta_2(y)}\Big(\rho_2 dW_t + \sqrt{1-\rho_2^2}d\bar{W}_t\Big) + \int_{\mathcal{U}} g(Y_t,u)N(du,dt)\right\}.$$

*Here the volatility $\vartheta_2(y)$ is positive and C^1. We next consider two choices for the volatility function ϑ_i, previously considered in the literature (see e.g. Example 5 in [16]). Under both choices, we can show that the coefficients of the model satisfy the above assumption (**H2**):*

(I) *Uniformly elliptic Scott volatility, i.e.* $\vartheta_i(y) = \varepsilon_i + e^{\gamma_i y}$ *for* $\gamma_i, \varepsilon_i > 0$. *Then the market price of risk is* $\bar{\theta}(y) = \frac{\theta}{\sqrt{\varepsilon_1 + e^{\gamma_1 y}}}$ *and hence* $|\bar{\theta}(y)| \leq \frac{|\theta|}{\sqrt{\varepsilon_1}}$ *for all* $y \in \mathbb{R}$. *Similarly we also have* $\bar{p}_c(y) = \frac{p_c}{\sqrt{\varepsilon_2 + e^{\gamma_2 y}}}$ *and* $|\bar{p}_c(y)| \leq \frac{|p_c|}{\sqrt{\varepsilon_2}}$ *for all* $y \in \mathbb{R}$.

(II) *Uniformly elliptic Stein-Stein volatility, i.e.* $\vartheta_i(y) = \varepsilon_i + \gamma_i |y|^2$ *for* $\gamma_i, \varepsilon_i > 0$. *In this case we still have that* $|\bar{\theta}(y)| \leq \frac{|\theta|}{\sqrt{\varepsilon_1}}$ *and* $|\bar{p}_c(y)| \leq \frac{|p_c|}{\sqrt{\varepsilon_2}}$ *for all* $y \in \mathbb{R}$.

3 Dynamic Optimization for an Insurer

This section formulates the optimal portfolio problem of the insurer with risk control and derives the HJB PDE. Recall that the average premium per liability for the insurer is $p(Y_t)$, and η_t the $\mathbb{G}$-adapted total outstanding number of policies (liabilities) at time t introduced in the above section. Then the revenue from selling insurance policies over the time period of $(t, t+dt)$ is given by $p(Y_t)\eta_t dt$. Denote by ϕ_t the time-t amount of the money invested in the risky asset. Then the surplus process of the insurer is given by, for $t \in [0,T]$, $X_t = \phi_t S_t + \big(X_t - \phi_t\big) B_t + \int_0^t p(Y_s)\eta_s ds - R_t^{\eta}$. Let π_t be the fraction of wealth invested in the risky asset at time t, and ℓ_t be the ratio of liabilities over surplus at time t, i.e. $(\phi_t, \eta_t) = X_t^{\pi,\ell}(\pi_t, \ell_t)$. Here $X_t^{\pi,\ell}$ represents the time-t surplus level with strategies $\pi = (\pi_t)_{t\in[0,T]}$ and $\ell = (\ell_t)_{t\in[0,T]}$. Using the self-financing trading strategy, we have from Eq. (2) and Eq. (4) that the dynamics of $X^{\pi,\ell} = (X_t^{\pi,\ell})_{t\in[0,T]}$ is given by

$$\begin{aligned}
\frac{dX_t^{\pi,\ell}}{X_{t-}^{\pi,\ell}} &= \pi_t \frac{dS_t}{S_t} + (1-\pi_t)\frac{dB_t}{B_t} + p(Y_t)\ell_t dt - dR_t^{\ell} \\
&= \Big\{ r(Y_t) + \pi_t \theta(Y_t) + p_c(Y_t)\ell_t \Big\} dt + \Big\{ \pi_t \sigma(Y_t) - \rho_2 \ell_t \phi(Y_t) \Big\} dW_t \\
&\quad - \sqrt{1-\rho_2^2}\,\ell_t \phi(Y_t) d\bar{W}_t - \ell_{t-} \int_{\mathcal{U}} g(Y_t, u) N(du, dt). \qquad (5)
\end{aligned}$$

Here R_t^{ℓ} satisfies that

$$dR_t^{\ell} = \ell_{t-} \left\{ c(Y_t)dt + \phi(Y_t) \left(\rho_2 dW_t + \sqrt{1-\rho_2^2} d\bar{W}_t \right) + \int_{\mathcal{U}} g(Y_t, u) N(du, dt) \right\}. \qquad (6)$$

We next define the admissible control strategies considered in the paper which is given by

Definition 3.1. *Let* $t \in [0,T]$. *The* t*-admissible control set* $\mathcal{A}_t$ *is a class of* $\mathbb{G}$*-adapted strategies* (π_s, ℓ_s) *with* $s \in [t,T]$ *so that SDE* (5) *admits a unique positive solution* $X_s^{\pi,\ell}$ *for* $s \in [t,T]$ *if* $X_t^{\pi,\ell} = x \in \mathbb{R}_+$.

For $(t,x,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$, and the admissible strategy $(\pi,\ell) \in \tilde{\mathcal{A}}_t$, define the objective functional by $J^{\pi,\ell}(t,x,y) := \mathbb{E}[U(X_T^{\pi,\ell}) | X_t^{\pi,\ell} = x, Y_t = y]$. Here the utility function is given by power utility, i.e. $U(x) = \frac{1}{\gamma}x^{\gamma}$, $x > 0$, where $\gamma \in (0,1)$ is the risk-aversion parameter. Thus for $(t,x,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$, the value function is then given by

$$V(t,x,y) := \sup_{(\pi,\ell)\in\mathcal{A}_t} J^{\pi,\ell}(t,x,y). \qquad (7)$$

If $V(t,x,y)$ is $C^{1,2,2}$, using the dynamical programming principle, the HJB PDE is then given by, on $(t,x,y) \in [0,T) \times \mathbb{R}_+ \times \mathbb{R}$,

$$\begin{aligned}
0 &= \frac{\partial V(t,x,y)}{\partial t} + \frac{1}{2}a^2(y)\frac{\partial^2 V(t,x,y)}{\partial y^2} + b(y)\frac{\partial V(t,x,y)}{\partial y} \\
&\quad + \sup_{(\pi,\ell)\in\mathbb{R}^2} \Bigg\{ \frac{\partial V(t,x,y)}{\partial x} x\big[r(y) + \pi\theta(y) + p_c(y)\ell\big] + \rho_1 \frac{\partial^2 V(t,x,y)}{\partial x \partial y} x a(y)\big[\pi\sigma(y) - \rho_2 \ell \phi(y)\big]
\end{aligned}$$

$$+\frac{1}{2}\frac{\partial^2 V(t,x,y)}{\partial x^2}x^2\Big[\big(\pi\sigma(y)-\rho_2\ell\phi(y)\big)^2+(1-\rho_2^2)\ell^2\phi^2(y)\Big] \tag{8}$$
$$+\int_{\mathcal{U}}\Big[V\big(t,x\big(1-\ell g(y,u)\big),y\big)-V(t,x,y)\Big]\nu(y,du)\Big\}$$

with terminal condition $V(T,x,y)=U(x)$ for all $(x,y)\in\mathbb{R}_+\times\mathbb{R}$. Consider the decomposition $V(t,x,y)=U(x)B(t,y)$ where $B(t,y)\in C^{1,2}$ is a positive function which solves the following equation: on $(t,y)\in[0,T)\times\mathbb{R}$,

$$0=\frac{\partial B(t,y)}{\partial t}+\frac{a^2(y)}{2}\frac{\partial^2 B(t,y)}{\partial y^2}+b(y)\frac{\partial B(t,y)}{\partial y}+B(t,y)\sup_{(\pi,\ell)\in\mathbb{R}^2}\mathcal{H}\left(\pi,\ell;y,\frac{\partial B(t,y)}{\partial y}B^{-1}(t,y)\right) \tag{9}$$

with the terminal condition $B(T,y)=1$ for all $y\in\mathbb{R}$. Here for $(\pi,\ell)\in\mathbb{R}^2$ and $(y,\varphi)\in\mathbb{R}\times\mathbb{R}$, the Hamiltonian $\mathcal{H}(\pi,\ell;y,\varphi)$ is

$$\mathcal{H}(\pi,\ell;y,\varphi):=\gamma\big\{r(y)+\pi\theta(y)+p_c(y)\ell\big\}+\rho_1\gamma a(y)\big[\pi\sigma(y)-\rho_2\ell\phi(y)\big]\varphi \tag{10}$$
$$+\frac{1}{2}\gamma(\gamma-1)\Big\{\big(\pi\sigma(y)-\rho_2\ell\phi(y)\big)^2+(1-\rho_2^2)\ell^2\phi^2(y)\Big\}+\int_{\mathcal{U}}\big\{\big(1-\ell g(y,u)\big)^{\gamma}-1\big\}\,\nu(y,du).$$

It can be also verified that if $B(t,y)$ solves Eq. (9), then $V(t,x,y)=U(x)B(t,y)$ is the solution of HJB PDE (8).

4 Optimal Strategies

This section focuses on the characterization of the optimal strategies of investment and risk control of the insurer. By Theorem 11.2.3., pag. 232 in Oksendal [14], it suffices to consider the Markov control in our case. Recall the Hamiltonian given by (10). Then the first-order condition of the Hamiltonian w.r.t. π gives that, for $(\pi,\ell)\in\mathbb{R}^2$ and $(y,\varphi)\in\mathbb{R}\times\mathbb{R}$,

$$\frac{\partial\mathcal{H}(\pi,\ell;y,\varphi)}{\partial\pi}=\gamma\theta(y)+\gamma(\gamma-1)\big(\sigma^2(y)\pi-\rho_2\ell\sigma(y)\phi(y)\big)+\rho_1\gamma\sigma(y)a(y)\varphi=0. \tag{11}$$

The solution of the above first-order condition equation admits, for $(y,\varphi)\in\mathbb{R}\times\mathbb{R}$ and $\ell\in\mathbb{R}$,

$$\pi^*(y,\varphi,\ell)=\frac{\rho_2\phi(y)}{\sigma(y)}\ell-\frac{\rho_1 a(y)}{(\gamma-1)\sigma(y)}\varphi-\frac{\theta(y)}{(\gamma-1)\sigma^2(y)}. \tag{12}$$

Observe that the above π^* also depends on the strategy $\ell\in\mathbb{R}$ which corresponds to the ratio of liabilities over surplus. On the other hand, the first-order condition of the Hamiltonian w.r.t. ℓ gives that, for $(\pi,\ell)\in\mathbb{R}^2$ and $(y,\varphi)\in\mathbb{R}\times\mathbb{R}$,

$$\frac{\partial\mathcal{H}(\pi,\ell;y,\varphi)}{\partial\ell}=\gamma p_c(y)-\gamma(\gamma-1)\rho_2\sigma(y)\phi(y)\pi+\gamma(\gamma-1)\phi^2(y)\ell-\rho_1\rho_2\gamma a(y)\phi(y)\varphi$$
$$-\gamma\int_{\mathcal{U}}\big(1-\ell g(y,u)\big)^{\gamma-1}g(y,u)\nu(y,du)=0. \tag{13}$$

Plugging π^* given by (12) into Eq. (13) yields that

$$p_c(y)+(\gamma-1)\phi^2(y)\ell-\rho_1\rho_2 a(y)\phi(y)\varphi-\int_{\mathcal{U}}\big(1-\ell g(y,u)\big)^{\gamma-1}g(y,u)\nu(y,du)$$
$$-(\gamma-1)\rho_2\sigma(y)\phi(y)\left(\frac{\rho_2\phi(y)}{\sigma(y)}\ell-\frac{\rho_1 a(y)}{(\gamma-1)\sigma(y)}\varphi-\frac{\theta(y)}{(\gamma-1)\sigma^2(y)}\right)=0.$$

Notice that it holds that

$$\rho_2\sigma(y)\phi(y)\frac{\rho_2\phi(y)}{\sigma(y)}\ell=\rho_2^2\phi^2(y)\ell,\quad\text{and}\quad\rho_2\sigma(y)\phi(y)\frac{\rho_1 a(y)}{(\gamma-1)\sigma(y)}\varphi=\rho_1\rho_2\phi(y)a(y)\varphi.$$

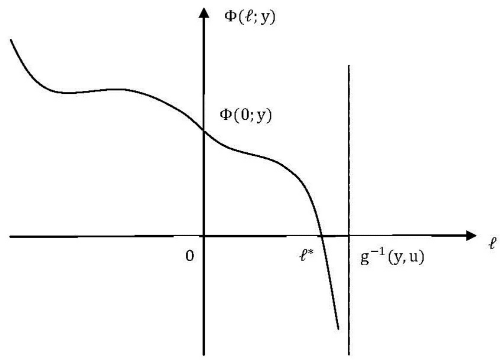

Figure 1: The function $\ell \to \Phi(\ell; y)$ for fixed $y \in \mathbb{R}$.

Then the first-order condition equation (13) continues that

$$\Phi(\ell; y) = 0. \tag{14}$$

Here $\Phi(\ell; y)$ is defined as, for $\ell \in \mathbb{R}$ and $y \in \mathbb{R}$,

$$\Phi(\ell; y) := (\gamma - 1)(1 - \rho_2^2)\phi^2(y)\ell + p_c(y) + \frac{\rho_2\phi(y)\theta(y)}{\sigma(y)} - \int_{\mathcal{U}} \big(1 - \ell g(y,u)\big)^{\gamma-1} g(y,u)\nu(y,du). \tag{15}$$

We next analyze the root of Eq. (14) in the unknown variable ℓ for fixed $y \in \mathbb{R}$. Notice that the root of Eq. (14) corresponds to the optimal ratio of liabilities over surplus. In order to make it admissible, the root denoted by $\ell^* = \ell^*(y)$ is required to satisfy the admissible condition given by

$$1 - \ell^* g(y,u) > 0, \qquad \text{for all } (y,u) \in \mathbb{R} \times \mathcal{U}. \tag{16}$$

In fact, the condition (16) can guarantee that the surplus process $X^{\pi,\ell}$ still keep positive after a (negative) jump raised by the Poisson measure in the risk control process, see also (5). The following lemma presents the result on the root $\ell^* = \ell^*(y)$ for $y \in \mathbb{R}$, which is given by

Lemma 4.1. *Let $y \in \mathbb{R}$ be fixed. There exists a unique root $\ell^* = \ell^*(y)$ satisfying the admissible condition (16) of Eq. (14). Moreover, as a function of $y \in \mathbb{R}$, the root $\ell^* = \ell^*(y)$ is C^1 and for $y \in \mathbb{R}$, it is nonnegative if and only if the following condition holds*

$$p_c(y) + \rho_2\phi(y)\bar{\theta}(y) - \int_{\mathcal{U}} g(y,u)\nu(y,du) \geq 0. \tag{17}$$

Proof. Notice that $\gamma < 1$ and $\rho_2 \in (-1,1)$. Then we have from (15) that for $y \in \mathbb{R}$ fixed, $\ell \longrightarrow \Phi(\ell; y)$ is C^1 and decreasing on $\ell \in \mathbb{R}$. For $(y,u) \in \mathbb{R} \times \mathcal{U}$ fixed, the admissible condition (16) is equivalent to requiring that the root $\ell^* < \frac{1}{g(y,u)}$. Then $\int_{\mathcal{U}} (1 - \ell g(y,u))^{\gamma-1} g(y,u)\nu(y,du) \uparrow +\infty$, and $(1-\rho_2^2)\phi^2(y)\ell \uparrow \frac{(1-\rho_2^2)\phi^2(y)}{g(y,u)}$ as $\ell \uparrow \frac{1}{g(y,u)}$. This yields that $\Phi(\ell; y) \downarrow -\infty$ as $\ell \downarrow \frac{1}{g(y,u)}$. On the other hand, as $\ell \downarrow -\infty$, it holds that $\int_{\mathcal{U}} (1 - \ell g(y,u))^{\gamma-1} g(y,u)\nu(y,du) \downarrow 0$, and $(1-\rho_2^2)\phi^2(y)\ell \downarrow -\infty$. Hence as $\ell \downarrow -\infty$, we obtain $\Phi(\ell; y) \uparrow +\infty$. This concludes that there is a unique root lying in $(-\infty, \frac{1}{g(y,u)})$ of Eq. (14). See Fig. 1 for a further illustration. In terms of the above analysis, for $y \in \mathbb{R}$, the root $\ell^* = \ell^*(y)$ is nonnegative if and only if $\Phi(0; y) \geq 0$, i.e. for $y \in \mathbb{R}$, $\Phi(0; y) = p_c(y) + \frac{\rho_2\phi(y)\theta(y)}{\sigma(y)} - \int_{\mathcal{U}} g(y,u)\nu(y,du) \geq 0$ (see also Fig. 1). The C^1-property of the root $y \longrightarrow \ell^*(y)$ follows from the C^1-property of $\ell \longrightarrow \Phi(\ell; y)$ for $y \in \mathbb{R}$ and the implicit function theorem. Thus we complete the proof of the lemma. □

Now using (12) and Lemma 4.1, we have the following solutions to the first-order condition system of the Hamiltonian w.r.t. (π, ℓ), which is given by, for $(y, \varphi) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{cases} \pi^*(y,\varphi) = \pi^*(y,\varphi,\ell^*(y)) = \frac{\rho_2\phi(y)}{\sigma(y)}\ell^*(y) - \frac{\rho_1 a(y)}{(\gamma-1)\sigma(y)}\varphi - \frac{\theta(y)}{(\gamma-1)\sigma^2(y)}; \\ \ell^* = \ell^*(y) \text{ is the unique root obtained in Lemma 4.1.} \end{cases} \tag{18}$$

We here emphasize that the variable $\varphi \in \mathbb{R}$ corresponds to the proportion of the gradient of the solution and the solution of the HJB PDE. We next prove that $(\pi^*, \ell^*) = (\pi^*(y,\varphi), \ell^*(y))$ for $(y,\varphi) \in \mathbb{R}\times\mathbb{R}$ given by (18) is in fact the optimum. The proof of the following lemma is standard and hence we omit it.

Lemma 4.2. *Let $(y,\varphi) \in \mathbb{R}\times\mathbb{R}$ and the condition (17) hold. For $(\pi,\ell) \in \mathbb{R}\times\mathbb{R}_+$ satisfying the admissible condition (16), then the Hessian matrix of $\mathcal{H}(\pi,\ell;y,\varphi)$ is negative definite.*

5 HJB PDE and Verification Theorem

In this section, we analyze existence and uniqueness of the global classical solution of HJB PDE (9) and then we will prove the corresponding verification theorem.

Recall the optimal strategy (π^*, ℓ^*) given by (18). Plugging it into (9) and we have the following updated HJB PDE given by, on $(t,y) \in [0,T)\times\mathbb{R}$,

$$\begin{aligned} 0 =& \frac{\partial B(t,y)}{\partial t} + \frac{1}{2}a^2(y)\frac{\partial^2 B(t,y)}{\partial y^2} + \left(b(y) - \frac{\rho_1\gamma}{\gamma-1}\frac{a(y)\theta(y)}{\sigma(y)}\right)\frac{\partial B(t,y)}{\partial y} - \frac{\rho_1^2\gamma}{2(\gamma-1)}a^2(y)\frac{\left(\frac{\partial B(t,y)}{\partial y}\right)^2}{B(t,y)} \\ &+ \gamma B(t,y)\left(r(y) + p_c(y)\ell^*(y) + \frac{\rho_2\phi(y)\theta(y)}{\sigma(y)}\ell^*(y) - \frac{\theta^2(y)}{2(\gamma-1)\sigma^2(y)} + \frac{\gamma-1}{2}(1-\rho_2^2)\phi^2(y)|\ell^*(y)|^2\right) \\ &+ B(t,y)\int_{\mathcal{U}}\left\{(1-\ell^*(y)g(y,u))^\gamma - 1\right\}\nu(y,du), \end{aligned} \tag{19}$$

while the terminal condition is given by $B(T,y) = 1$ for all $y \in \mathbb{R}$. We have that

Theorem 5.1. *Under the condition (17), and let assumptions (**H1**) and (**H2**) hold. Then there exists a unique positive and bounded classical solution of Eq. (19).*

Proof. As in Zariphopoulou [19], introduce the transform given by

$$B(t,y) = F^\delta(t,y), \qquad (t,y) \in [0,T]\times\mathbb{R}. \tag{20}$$

Here $\delta \in \mathbb{R}$ is a free parameter which needs to be determined later and Eq. (19) gives that $F(t,y)$ satisfies

$$\begin{aligned} 0 =& \delta F^{\delta-1}(t,y)\frac{\partial F(t,y)}{\partial t} + F^{\delta-2}(t,y)\left(\frac{\partial F(t,y)}{\partial y}\right)^2\left(\frac{\delta(\delta-1)}{2} - \frac{\rho_1^2\gamma\delta^2}{2(\gamma-1)}\right)a^2(y) \\ &+ \frac{\delta}{2}F^{\delta-1}(t,y)a^2(y)\frac{\partial^2 F(t,y)}{\partial y^2} + \delta F^{\delta-1}(t,y)\zeta(y)\frac{\partial F(t,y)}{\partial y} + \psi(y)F^\delta(t,y), \end{aligned} \tag{21}$$

where, for $y \in \mathbb{R}$, the coefficients

$$\begin{aligned} \zeta(y) &:= b(y) - \frac{\rho_1\gamma}{\gamma-1}\frac{a(y)\theta(y)}{\sigma(y)} = b(y) - \frac{\rho_1\gamma}{\gamma-1}a(y)\bar{\theta}(y), \\ \psi(y) &:= \gamma\left(r(y) - \frac{\bar{\theta}^2(y)}{2(\gamma-1)} + \kappa(y)\ell^*(y) + \frac{\gamma-1}{2}(1-\rho_2^2)\phi^2(y)|\ell^*(y)|^2\right) \\ &\qquad + \int_{\mathcal{U}}\left\{(1-\ell^*(y)g(y,u))^\gamma - 1\right\}\nu(y,du). \end{aligned} \tag{22}$$

Here $\kappa(y) := p_c(y) + \frac{\rho_2\phi(y)\theta(y)}{\sigma(y)} = p_c(y) + \rho_2\phi(y)\bar{\theta}(y)$ for $y \in \mathbb{R}$. We take the constant given by

$$\delta = \frac{1-\gamma}{1-(1-\rho_1^2)\gamma} \in (0,1). \tag{23}$$

Then it holds that $\frac{\delta(\delta-1)}{2}-\frac{\rho_1^2\gamma\delta^2}{2(\gamma-1)}=0$. This yields that the transformed solution $F(t,y)$ satisfies that on $(t,y)\in[0,T)\times\mathbb{R}$,

$$0=\frac{\partial F(t,y)}{\partial t}+\frac{1}{2}a^2(y)\frac{\partial^2 F(t,y)}{\partial y^2}+\zeta(y)\frac{\partial F(t,y)}{\partial y}+\frac{\psi(y)}{\delta}F(t,y), \tag{24}$$

while the terminal condition is given by $F(T,y)=1$ for all $y\in\mathbb{R}$. We next apply Proposition 2.3 of Becherer and Schweizer [1] to analyze the classical solution of Eq. (24). To this purpose, we first consider the following SDE given by

$$d\bar{Y}_t=\zeta(\bar{Y}_t)dt+a(\bar{Y}_t)\left(\rho_1 dW_t+\sqrt{1-\rho_1^2}d\hat{W}_t\right),\qquad \bar{Y}_0=Y_0\in\mathbb{R}, \tag{25}$$

where $\zeta(y)$ is given in (22). We next prove that SDE (25) admits a unique strong solution $\bar{Y}=(\bar{Y}_t)_{t\in[0,T]}$ under assumptions (**H1**) and (**H2**). Recall that the stochastic factor $Y=(Y_t)_{t\in[0,T]}$ satisfying SDE (1). We then define the change of measure by

$$\left.\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{G}_T}=\exp\left\{-\int_0^T\frac{\rho_1\gamma}{\gamma-1}\bar{\theta}(Y_s)d\left(\rho_1 dW_s+\sqrt{1-\rho_1^2}d\hat{W}_s\right)-\frac{1}{2}\int_0^T\left(\frac{\rho_1\gamma}{\gamma-1}\bar{\theta}(Y_s)\right)^2ds\right\}.$$

Notice that the assumption (**H2**) shows that $\bar{\theta}(y)$ is bounded, and hence the probability measure $\bar{\mathbb{P}}\sim\mathbb{P}$ is well defined. Moreover, under $\bar{\mathbb{P}}$, the process $\tilde{W}_t:=\rho_1 dW_t+\sqrt{1-\rho_1^2}d\hat{W}_t+\int_0^t\frac{\rho_1\gamma}{\gamma-1}\bar{\theta}(Y_s)ds$, $t\in[0,T]$, is a Brownian motion. Thus we can rewrite SDE (1) as

$$dY_t=\zeta(Y_t)dt+a(Y_t)d\tilde{W}_t,\qquad Y_0\in\mathbb{R}. \tag{26}$$

This implies that SDE (26) admits a unique solution under $\bar{\mathbb{P}}$. Since $\bar{\mathbb{P}}\sim\mathbb{P}$, we have that SDE (25) has a unique strong solution under $\mathbb{P}$. We next prove that the coefficient $\psi(y)$, $y\in\mathbb{R}$, given in (22), is in fact bounded from above. To this purpose, for $y\in\mathbb{R}$ fixed, let us consider the following function for $\ell\in\mathbb{R}_+$,

$$\bar{\psi}(y;\ell):=\gamma\left(r(y)-\frac{\bar{\theta}^2(y)}{2(\gamma-1)}+\kappa(y)\ell+\frac{\gamma-1}{2}(1-\rho_2^2)\phi^2(y)\ell^2\right)+\int_{\mathcal{U}}\{(1-\ell g(y,u))^\gamma-1\}\nu(y,du).$$

Notice that $\gamma\in(0,1)$ and the jump function $g(y,u)$ is positive. Then for all $(y,\ell)\in\mathbb{R}\times\mathbb{R}_+$,

$$\begin{aligned}\bar{\psi}(y;\ell)&\leq\gamma\frac{\kappa^2(y)+2(1-\gamma)(1-\rho_2^2)\phi^2(y)\left(r(y)+\frac{\theta^2(y)}{2(1-\gamma)\sigma^2(y)}\right)}{2(1-\gamma)(1-\rho_2^2)\phi^2(y)}\\&=\gamma\left(r(y)+\frac{\bar{p}_c^2(y)}{2(1-\gamma)(1-\rho_2^2)}+\frac{\bar{\theta}^2(y)}{2(1-\gamma)(1-\rho_2^2)}+\frac{\rho_2\bar{p}_c(y)\bar{\theta}(y)}{(1-\gamma)(1-\rho_2^2)}\right).\end{aligned}$$

Since $\rho_2\in(-1,1)$, under (**H2**) and the condition (17), there exists a constant $C\in\mathbb{R}$ such that $\bar{\psi}(y;\ell)\leq C$ for all $(y,\ell)\in\mathbb{R}\times\mathbb{R}_+$. Notice that $\psi(y)=\bar{\psi}(y,\ell^*(y))$ and $\ell^*(y)\in\mathbb{R}_+$ under the condition (17). Then Lemma 4.1 yields that $\psi(y)$, $y\in\mathbb{R}$, is bounded from above. Let $D_n=(-n,n)$ for $n\in\mathbb{N}$. Also notice that $\delta\in(0,1)$, and the terminal condition $F(T,y)=1$ for all $y\in\mathbb{R}$, which is smooth and bounded. Then the coefficients of PDE (24) with $(D_n)_{n\in\mathbb{N}}$ given above satisfy the local conditions in Proposition 2.3 of [1]. Hence PDE (24) admits a unique classical solution $F(t,y)$ on $(t,y)\in[0,T]\times\mathbb{R}$. Further $F(t,y)=\mathbb{E}[e^{\int_t^T\frac{\psi(\bar{Y}_s)}{\delta}ds}|\bar{Y}_t=y]$ for $(t,y)\in[0,T]\times\mathbb{R}$. Since $\frac{\psi(y)}{\delta}$, $y\in\mathbb{R}$, is bounded from above, $F(t,y)$ is bounded and positive for all $(t,y)\in[0,T]\times\mathbb{R}$. Using (20) and notice that $\delta\in(0,1)$, we have that $B(t,y)=F^\delta(t,y)$ is the unique bounded classical solution to Eq. (19). Thus we complete the proof of the theorem. □

We also have the following verification theorem:

Theorem 5.2. *Let the conditions of Theorem 5.1 hold. Then the value function is given by, for* $(t,x,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$,

$$V(t,x,y) = \frac{1}{\gamma} x^\gamma \left\{ \mathbb{E}\left[e^{\frac{1-(1-\rho_1^2)\gamma}{1-\gamma} \int_t^T \psi(\bar{Y}_s)ds} \Big| \bar{Y}_t = y \right] \right\}^{\frac{1-\gamma}{1-(1-\rho_1^2)\gamma}}, \tag{27}$$

where $\bar{Y} = (\bar{Y}_t)_{t\in[0,T]}$ *is the unique strong solution to SDE (25). Moreover, the optimal strategies corresponding to the fraction of wealth invested in the risky asset and the ratio of liabilities over surplus is given by, for* $t \in [0,T]$,

$$\pi_t^* = \pi^*\left(Y_t, \frac{\partial B(t,Y_t)}{\partial y} B^{-1}(t,Y_t) \right), \quad \text{and} \quad \ell_t^* = \ell^*(Y_t). \tag{28}$$

Here $(\pi^*(y,\varphi), \ell^*(y))$, $(y,\varphi) \in \mathbb{R}\times\mathbb{R}$, *is the optimal feedback functions which are given by (18) and* $Y = (Y_t)_{t\in[0,T]}$ *is the stochastic factor process described as (1).*

Proof. It can be observed that our transformed HJB PDE (24) admits a same structure to the HJB equation (3.11) considered in [19]. Under assumptions (**H1**) and (**H2**), it follows from Theorem 3.2 of [19] that, in order to prove the value function $V(t,x,y) = \frac{1}{\gamma}x^\gamma B(t,y) = \frac{1}{\gamma}x^\gamma F^\delta(t,y)$ for $(t,x,y) \in [0,T]\times\mathbb{R}_+\times\mathbb{R}$, it suffices to verify that $F(t,y)$ is a unique viscosity solution to Eq. (24). Theorem 5.1 has showed that $F(t,y) = \mathbb{E}\big[e^{\int_t^T \frac{\psi(\bar{Y}_s)}{\delta}ds} | \bar{Y}_t = y\big]$ is the unique classical solution of PDE (24) with the terminal condition $F(T,y) = 1$ for all $y \in \mathbb{R}$. Then it implies that $F(t,y)$ is also a unique viscosity solution to Eq. (24). Plugging the constant δ given by (23) into $V(t,x,y) = \frac{1}{\gamma}x^\gamma F^\delta(t,y)$, we get (27). On the other hand, from Lemma 4.2, it follows that $(\pi^*(y,\varphi), \ell^*(y))$ with $(y,\varphi) \in \mathbb{R}\times\mathbb{R}$, given by (18), is the optimal feedback functions. Also notice that $B(t,y) = F^\delta(t,y)$ with $\delta \in (0,1)$ given by (23) is the unique classical solution to our HJB equation (19). Then using (18), under (**H1**) and (**H2**), it follows that both of $\pi^*\big(y, \frac{\partial B(t,y)}{\partial y}/B(t,y)\big)$ and $\ell^*(y)$ are locally bounded. In terms of Definition 3.1, we have that the optimal strategy given by (28) is admissible. Thus we complete the proof of the verification theorem. □

6 Numerical Analysis

Recall Example 2.1 and choose uniformly elliptic Scott volatility here, i.e. $\sigma(y) = \sqrt{\vartheta_1(y)} = \sqrt{\varepsilon_1 + e^{\gamma_1 y}}$ and $\phi(y) = \sqrt{\vartheta_2(y)} = \sqrt{\varepsilon_2 + e^{\gamma_2 y}}$ for $y \in \mathbb{R}$. The condition (17) implies that for $y \in \mathbb{R}$, $j(y)g(y,1) \le p_c + \rho_2\theta\sqrt{\frac{\varepsilon_2+e^{\gamma_2 y}}{\varepsilon_1+e^{\gamma_1 y}}} =: \kappa(y)$. Recall $\Phi(\ell;y)$ defined by (15). Then, for $(\ell,y) \in \mathbb{R}_+\times\mathbb{R}$,

$$\Phi(\ell;y) = \kappa(y) + (\gamma-1)(1-\rho_2^2)\vartheta_1(y)\ell - j(y)\big(1-\ell g(y,1)\big)^{\gamma-1}g(y,1).$$

Consider the risk aversion parameter $\gamma = 0.5$. By solving $\Phi(\ell;y) = 0$ in ℓ to obtain the the following equation on the optimal ratio of liabilities over surplus given by

$$\begin{aligned} -\frac{1}{4}(1-\rho_2^2)^2\vartheta_1^2(y)g(y,1)\ell^3 + \left(\frac{1}{4}(1-\rho_2^2)^2\vartheta_1^2(y) + \kappa(y)(1-\rho_2^2)\vartheta_1(y)g(y,1) \right)\ell^2 \\ - \big(\kappa^2(y)g(y,1) + \kappa(y)(1-\rho_2^2)\vartheta_1(y)\big)\ell + \kappa^2(y) - j^2(y)g^2(y,1) = 0. \end{aligned} \tag{29}$$

Further we introduce the following coefficients, for $y \in \mathbb{R}$,

$$\begin{aligned} &\bar{A}(y) := -\frac{1}{4}(1-\rho_2^2)^2\vartheta_1^2(y)g(y,1), \quad \bar{B}(y) := \frac{1}{4}(1-\rho_2^2)^2\vartheta_1^2(y) + \kappa(y)(1-\rho_2^2)\vartheta_1(y)g(y,1), \\ &\bar{C}(y) := -\kappa^2(y)g(y,1) - \kappa(y)(1-\rho_2^2)\vartheta_1(y), \quad \bar{D}(y) := \kappa^2(y) - j^2(y)g^2(y,1). \end{aligned}$$

Then Eq. (29) can be rewritten as the following cubic equation in ℓ for $y \in \mathbb{R}$ fixed

$$\bar{A}(y)\ell^3 + \bar{B}(y)\ell^2 + \bar{C}(y)\ell + \bar{D}(y) = 0. \tag{30}$$

We next solve the above cubic equation by defining

$$\begin{aligned}\Delta(y) &:= 81\bar{A}^4(y)\bar{D}^2(y) - 54\bar{A}^3(y)\bar{B}(y)\bar{C}(y)\bar{D}(y) + 12\bar{A}^3(y)\bar{C}^3(y) + 12\bar{A}^2(y)\bar{B}^3(y)\bar{D}(y)\\ &\quad - 3\bar{A}^2(y)\bar{B}^2(y)\bar{C}^2(y).\end{aligned}$$

Then the solution which matches the optimal ratio of liabilities over surplus is given by

$$\begin{aligned}\ell^*(y) &= \frac{1}{6\bar{A}(y)}\left(-108\bar{A}^2(y)\bar{D}(y) + 36\bar{A}(y)\bar{B}(y)\bar{C}(y) - 8\bar{B}^3(y) + 12\sqrt{\Delta(y)}\right)^{\frac{1}{3}} \\ &\quad + \frac{1}{6\bar{A}(y)}\left(-108\bar{A}^2(y)\bar{D}(y) + 36\bar{A}(y)\bar{B}(y)\bar{C}(y) - 8\bar{B}^3(y) - 12\sqrt{\Delta(y)}\right)^{\frac{1}{3}} - \frac{\bar{B}(y)}{3\bar{A}(y)}.\end{aligned} \tag{31}$$

This follows from Eq. (18) that the optimal strategy for the risky asset is given by, for $(y,\varphi)\in\mathbb{R}\times\mathbb{R}$,

$$\begin{aligned}\pi^*(y,\varphi) &= \frac{\rho_2\phi(y)}{\sigma(y)}\ell^*(y) - \frac{\rho_1 a(y)}{(\gamma-1)\sigma(y)}\varphi - \frac{\theta(y)}{(\gamma-1)\sigma^2(y)}\\ &= \rho_2\sqrt{\frac{\varepsilon_2+e^{\gamma_2 y}}{\varepsilon_1+e^{\gamma_1 y}}}\ell^*(y) + \frac{2\rho_1 a}{\sqrt{\varepsilon_1+e^{\gamma_1 y}}}\varphi + \frac{2\theta}{(\varepsilon_1+e^{\gamma_1 y})}.\end{aligned} \tag{32}$$

Then we have the time-t optimal strategy of investment and risk control is given by

$$(\pi_t^*,\ell_t^*) = \left(\pi^*\left(Y_t, \frac{\partial B(t,Y_t)}{\partial y}B^{-1}(t,Y_t)\right), \ell^*(Y_t)\right), \qquad t\in[0,T].$$

Here $B(t,y) = F^\delta(t,y)$, $(t,y)\in[0,T]\times\mathbb{R}$, is the unique classical solution to PDE (19).

We next present a numerical analysis for the impact of market parameters on the optimal policy $(\pi^*(y,\varphi),\ell^*(y))$ given by (32) and (31) above. We first analyze the impact of the volatility level of the model on the optimal policy. Notice that in Example 2.1, the stochastic volatility appears in both of the risky asset price and the risk model. The volatility functions are given by $\sqrt{\vartheta_1(y)} = \sqrt{\varepsilon_1+e^{\gamma_1 y}}$ and $\sqrt{\vartheta_2(y)} = \sqrt{\varepsilon_2+e^{\gamma_2 y}}$ respectively. We take the parameters $\mu=0.6$, $r=0.1$, $p=0.7$, $c=0.1$, $a=0.01$, $\varepsilon_1=0.2$, $\varepsilon_2=0.3$, $\gamma_1=0.02$ and $\gamma_2=0.03$. The left top figure in Fig. 2 plots the the optimal policy with respect to the stochastic factor y. We notice that in the setting of market parameters in the left top figure in Fig. 2, the parameters γ_i, $i=1,2$, are positive. Hence the both of volatilities $\sqrt{\vartheta_1(y)}$ and $\sqrt{\vartheta_2(y)}$ are increasing in the stochastic factor y. This implies that when the stochastic factor y becomes larger, the volatility risk would be larger. Then it leads that the optimal strategy is decreasing w.r.t. the stochastic factor, which is also consistent with the observation presented in Benth, et al. [2].

Using the common parameter values as in the left top figure in Fig. 2, we analyze the sensitivity of the optimal strategy w.r.t. the correlation coefficient $\rho_1\in(-1,1)$ between the stochastic factor and asset return. In terms of (31) and (32) for the optimal strategy, only the optimal fraction of the risky asset $\pi^*(y,\varphi)$ depends on the correlation coefficient $\rho_1\in(-1,1)$ which is linear in ρ_1. The graph of optimal policies w.r.t. different values of ρ_1 is displayed in the right top figure in Fig. 2.

Further, we analyze the impact of the correlation coefficient ρ_2 between the insurer's risk and asset return on the optimal strategy. We plot the graph of optimal strategy w.r.t. different values of ρ_2 in the left bottom figure in Fig. 2. From the left bottom figure in Fig. 2 with fixed value of the stochastic factor, it can be seen that the optimal fraction strategy $\pi^*(y,\varphi)$ in the risky asset is increasing in ρ_2. While for the optimal liability ratio $\ell^*(y)$ with fixed value of the stochastic factor, it is decreasing first and then it is increasing in ρ_2. These observations are fully consistent with the findings in Zou and Cadenillas [22] which considers the case without stochastic factor.

Finally we analyze the sensitivity of the optimal strategy w.r.t. the parameter j of jumps' size. We use the common parameter values in the left top figure in Fig. 2 with additional parameter values $\rho_1=0.2$, $\rho_2=-0.1$, and $\varphi=5$. The right bottom figure in Fig. 2 shows that the optimal liability ratio $\ell^*(y)$ is decreasing w.r.t. the jumps' size j. It can be explained that when the

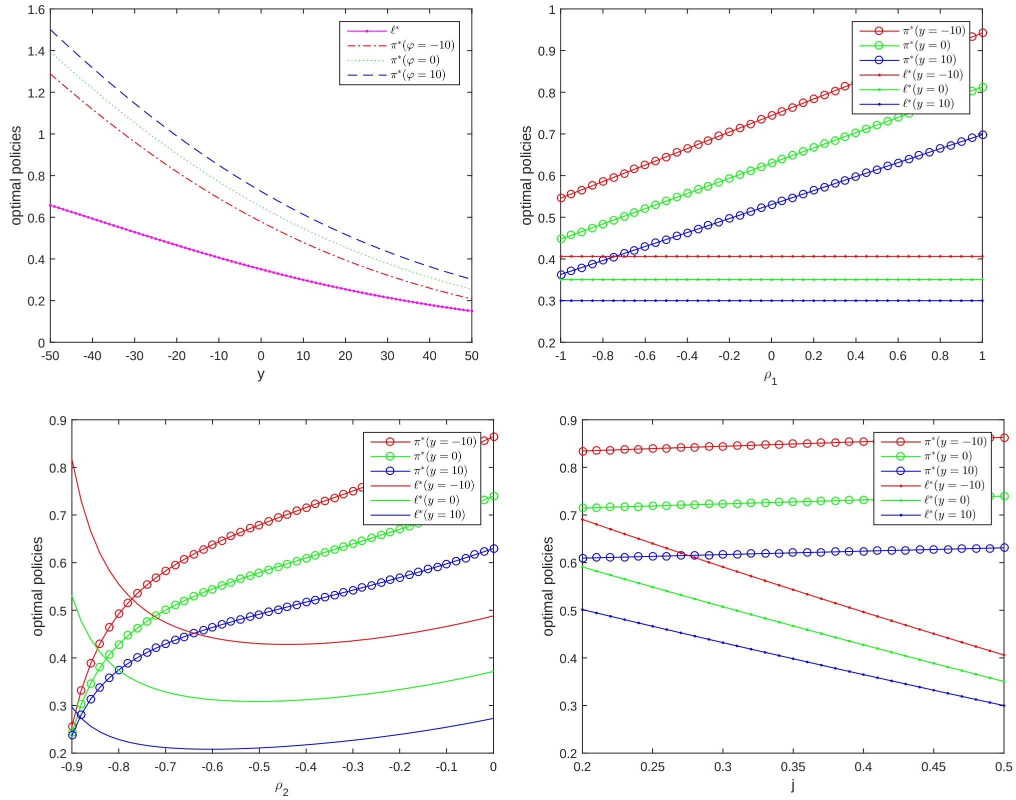

Figure 2: Left top panel: the dependence of the optimal policy on the stochastic factor y. Right top panel: the dependence of the optimal policy on the correlation coefficient ρ_1. Left bottom panel: the dependence of the optimal policy on the correlation coefficient ρ_2. Right bottom panel: the dependence of the optimal policy on the jumps' size j.

jumps' size is larger, the risk exposure to the investor in the risk process would be larger. Then it is natural to reduce the liability ratio for the insurer and in turn the insurer will increase the investment fraction of the risky asset since $\rho_2\sqrt{\frac{\varepsilon_2+e^{\gamma_2 y}}{\varepsilon_1+e^{\gamma_1 y}}}\ell^*$ increases for the negative ρ_2. This has been displayed in the right bottom figure in Fig. 2.

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