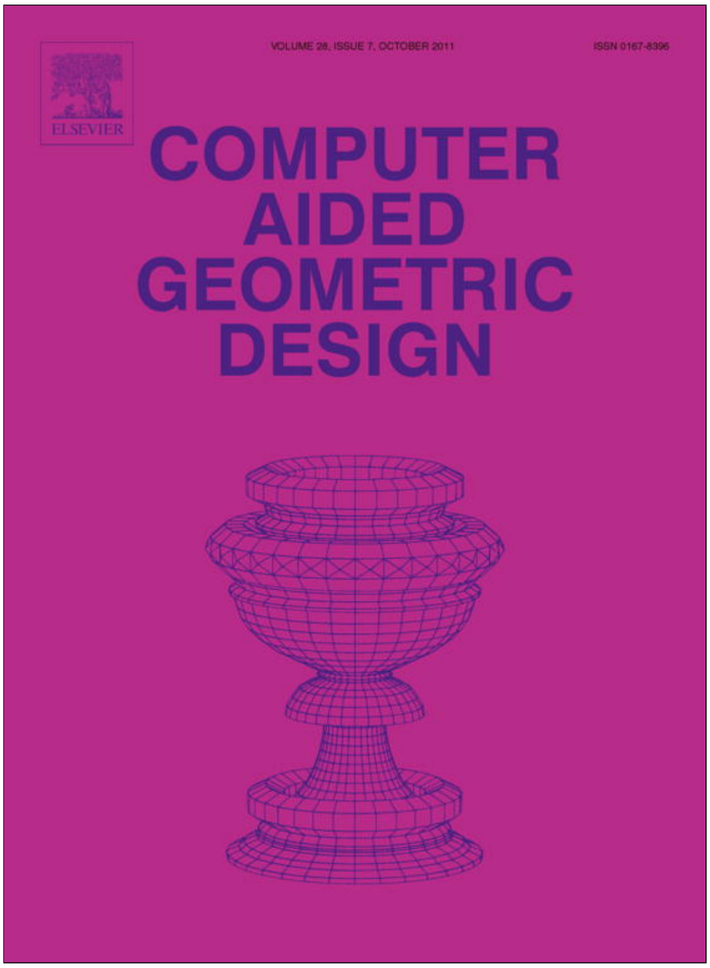
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On the instability in the dimension of splines spaces over T-meshes ☆

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ABSTRACT

The present paper reveals the instability in the dimension of the spline space $\mathcal{S}(d_1, d_2, d_1-1, d_2-1, \mathcal{T})$ over certain types of T-meshes $\mathcal{T}$, that is, the dimension is related to not only the topological information of $\mathcal{T}$ but also the geometry of $\mathcal{T}$. This insight suggests us to pay much attention to the structure of the T-meshes in modeling with splines over T-meshes.

1. Introduction

Spline functions are an important approximation tool in computational science, and they are widely used in many engineering fields. In order to better understand and apply spline functions, a major issue is to explore the algebraic structure, the dimension and the basis functions of the spline space. For the univariate case and tensor-product case, splines form linear spaces that have a very simple structure. However, it is much harder to understand the structure of the spline spaces defined over triangles, since the dimension of these spaces may depend not only on the number of triangles, the degree and smoothness of the splines, but also on the geometry of the triangulation (Morgan and Scott, 1977; Diener, 1990). It was shown that the dimension of the space of C^1 continuous piecewise polynomial functions of total degree 2 over the Morgan–Scott triangulation (Fig. 1) is 6 or 7, depending on the geometry of the triangulation. This makes the application of splines over triangles much more complicated. For more information of spline spaces over triangles, please refer to the book (Schumaker, 2007). In this paper, we will show that a similar situation exists for splines over T-meshes.

A T-mesh is a rectangular grid that allows T-junctions. The notion of splines over T-meshes was introduced in Deng et al. (2006), which have several merits. For example, the local refinement is very simple and tightly supported; The spline is a polynomial in each facet which has simple and efficient integration algorithm for numerical analysis. In Deng et al. (2006), a dimension formula was provided for splines over T-meshes when the order of smoothness is less than half of the degree of the spline functions. Applications of splines over T-meshes in geometric modeling were subsequently presented in Deng et al. (2008) and Li et al. (2007, 2010). However, in geometric modeling, the order of smoothness of spline surfaces is required to be as high as possible for a fixed degree. Thus we are expected to study spline functions of degree d with C^{d-1} continuity. Unfortunately, the dimension of such spline space over T-meshes is still unknown so far.

In this paper, we shall show that the dimension of the spline space of degree d with C^{d-1} continuity over some particular T-meshes is unstable, that is, the dimension is related with not only the topological information of the T-mesh but also the geometric information of the T-mesh, which was previously unknown and perhaps unexpected.

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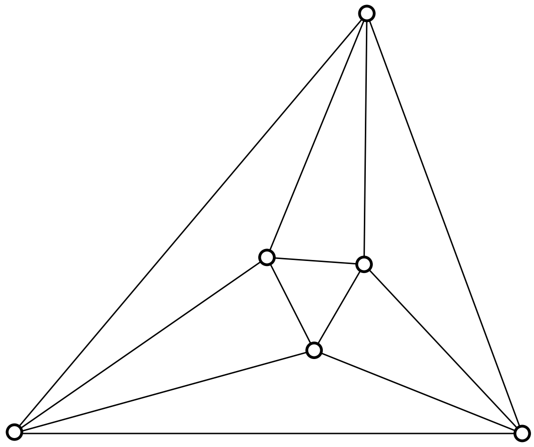


Fig. 1. Morgan–Scott triangulation.

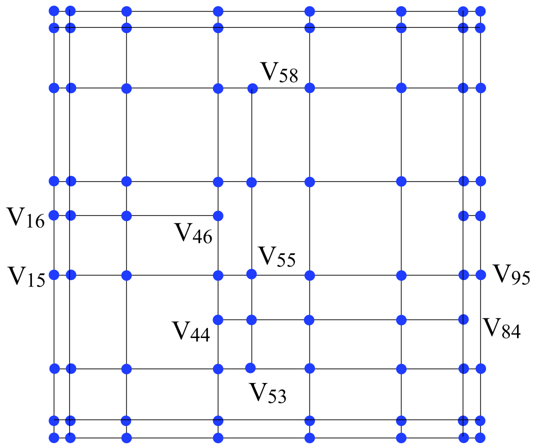


Fig. 2. A T-mesh.

The rest of the paper is structured as follows. Pertinent background on splines over T-meshes and the smoothing cofactor method for computing dimension of spline spaces are reviewed in Section 2. In Section 3, we provide several types of T-meshes over which the dimension of spline spaces is instability. We conclude the paper in Section 4 with a discussion of future work.

2. Preliminaries

In this section, we briefly review spline spaces over T-meshes and smoothing cofactor method which is one of the main tools to analyze the dimension of spline spaces.

2.1. Spline spaces over T-meshes

We begin with some terminology and notation. A *T-mesh* is almost, but not quite, a rectangular grid that allows T-junctions. A grid point in a T-mesh is also called a *vertex* of the T-mesh. If a vertex is on the boundary grid line, then it is called a *boundary vertex*. Otherwise, it is called an *interior vertex*. A valence 4 interior vertex is also called a *cross vertex*. The line segment connecting two adjacent vertices on a grid line is called an *edge* of the T-mesh. If an edge is on the boundary, then it is called a *boundary edge*; otherwise it is called an *interior edge*. An *edge segment* is a set of connected edges with two end points being T-junctions or boundary vertices. If all the vertices on an edge segment are interior vertices of the T-mesh, then the edge segment is called an *interior edge segment*. A *cross-cut* is an edge segment whose two endpoints lie on the boundary of the T-mesh. A *ray* is an edge segment with one endpoint lying on the boundary of the T-mesh.

For example, in Fig. 2, vertices V_{53}, V_{44} and V_{46} are interior vertices, and V_{15}, V_{16} and V_{95} are boundary vertices. Vertex V_{55} is a cross vertex and V_{44} is a T-junction. The edge segment $V_{15}V_{95}$ is a cross-cut, while $V_{16}V_{46}$ is a ray, and $V_{44}V_{84}$ and $V_{53}V_{58}$ are interior edge segments.

Given a T-mesh $\mathcal{T} \in \mathbb{R}^2$, let $\mathcal{F}$ denote all the rectangles in $\mathcal{T}$ and Ω the region occupied by all the rectangles in $\mathcal{T}$. A spline space over a T-mesh $\mathcal{T}$ is defined as in Deng et al. (2006)

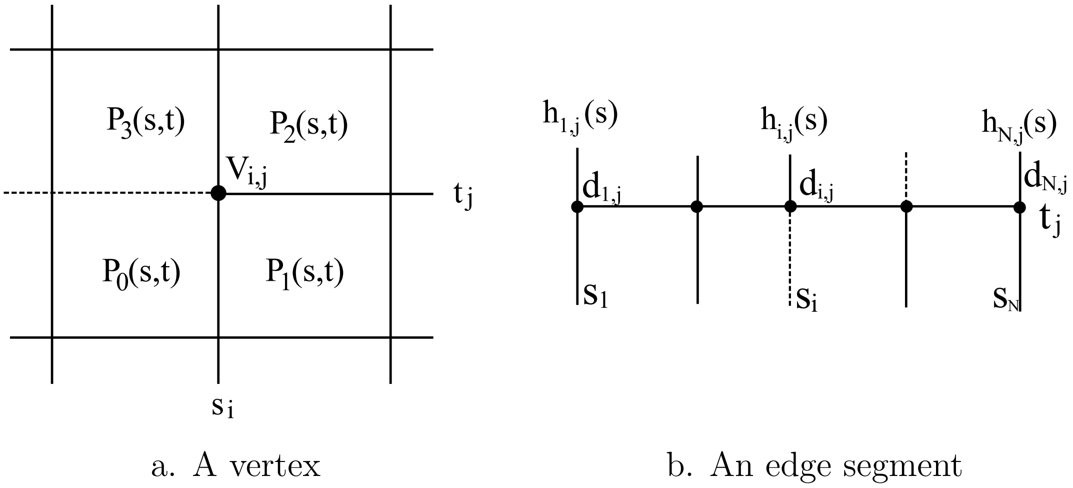


Fig. 3. Smoothing cofactors around a vertex and along an edge segment.

$$\mathcal{S}(d_1, d_2, \alpha, \beta, \mathcal{T}) := \{ f \in C^{\alpha,\beta}(\Omega) \mid f|_\phi \in \mathrm{P}_{d_1 d_2}, \forall \phi \in \mathcal{F} \},$$

where $\mathrm{P}_{d_1 d_2}$ is the function space of all the polynomials with bi-degree (d_1, d_2), and $C^{\alpha,\beta}(\Omega)$ is the space consisting of all the bivariate functions in s, t which are continuous in Ω with order α along s direction and with order β along t direction. It is obvious that $\mathcal{S}(d_1, d_2, \alpha, \beta, \mathcal{T})$ is a linear space, which is called the spline space over the given T-mesh $\mathcal{T}$.

To explore the algebraic structure of the spline space $\mathcal{S}(d_1, d_2, \alpha, \beta, \mathcal{T})$, a major issue is to study its dimension, denoted by $\dim \mathcal{S}(d_1, d_2, \alpha, \beta, \mathcal{T})$. In Deng et al. (2006) the authors presented the following result: If $d_1 \geqslant 2\alpha + 1$ and $d_2 \geqslant 2\beta + 1$, then

$$\dim \mathcal{S}(d_1, d_2, \alpha, \beta, \mathcal{T}) = F(d_1 + 1)(d_2 + 1) - E_h(d_1 + 1)(\beta + 1) - E_v(d_2 + 1)(\alpha + 1) + V(\alpha + 1)(\beta + 1),$$

where F is the number of cells, E_h is the number of horizontal interior edges, E_v is the number of vertical interior edges and V represents the number of interior vertices, respectively. An improved result about the dimension of splines over T-meshes can be found in Li et al. (2006).

In the present paper, we focus on the spline space $\mathcal{S}(d, d, d-1, d-1, \mathcal{T})$ for $d > 1$ which is denoted as $\mathcal{S}^d(\mathcal{T})$ for simplicity. The main tool to analyze the dimension of spline spaces used in this paper is smoothing cofactor method which was independently developed by Schumaker (1979) and Wang (2001).

2.2. Smoothing cofactor method

In this section, we will review the smoothing cofactor method (Wang, 2001) for computing the dimension of spline space over T-meshes which was proposed in Li et al. (2006).

Before providing the details, we first introduce some notations. For simplicity, we assume each knot line contains only one edge segment. Denote all the horizontal interior edge segments and vertical edge segments as $he_1, he_2, \ldots, he_n$ and $ve_1, ve_2, \ldots, ve_m$ respectively. Assume the horizontal edge segment he_i has n_i vertices whose indices are $h_{i,1}, \ldots, h_{i,n_i}$, and the vertical edge segment ve_j has m_j vertices whose indices are $v_{j,1}, \ldots, v_{j,m_j}$.

Referring to Fig. 3a, for an interior vertex $V_{i,j} = (s_i, t_j)$, suppose the surrounding bi-degree d patches are $P_i(s,t), i = 0, 1, 2, 3$ respectively. Notice that if the vertex $V_{i,j}$ is a T-junction such as in Fig. 3a, then $P_0(s,t)$ and $P_3(s,t)$ are identical.

As $P_0(s,t)$ and $P_1(s,t)$ are C^{d-1} continuous, there exists a degree d polynomial $p_{0,1}(t)$ such that

$$P_1(s,t) - P_0(s,t) = p_{0,1}(t)(s - s_i)^d. \tag{1}$$

Here $p_{0,1}(t)$ is the cofactor for the common edge of the patches $P_0(s,t)$ and $P_1(s,t)$.

Similarly, there exist degree d polynomials $p_{1,2}(s)$, $p_{2,3}(t)$ and $p_{3,0}(s)$ such that

$$P_2(s,t) - P_1(s,t) = p_{1,2}(s)(t - t_j)^d, \tag{2}$$

$$P_3(s,t) - P_2(s,t) = p_{2,3}(t)(s - s_i)^d, \tag{3}$$

$$P_0(s,t) - P_3(s,t) = p_{3,0}(s)(t - t_j)^d. \tag{4}$$

Thus,

$$\big(p_{0,1}(t) + p_{2,3}(t)\big)(s - s_i)^d = -\big(p_{1,2}(s) + p_{3,0}(s)\big)(t - t_j)^d. \tag{5}$$

Since $(s-s_i)^d$ and $(t-t_j)^d$ are prime to each other, there exists constant $d_{i,j}$ such that

$$p_{0,1}(t)+p_{2,3}(t)=d_{i,j}(t-t_j)^d, \qquad p_{1,2}(s)+p_{3,0}(s)=-d_{i,j}(s-s_i)^d. \tag{6}$$

From the above observation, we can attach a constant cofactor $d_{i,j}$ for each interior vertex $V_{i,j}$.

Remark 1. Similarly, we can attach a degree d polynomial to each boundary vertex. Thus, each boundary vertex has $d+1$ degree of freedoms.

However, all these cofactors are not free, and there are other constrains along each edge segment. Denote $G_{i,j}(t)=p_{0,1}(t)+p_{2,3}(t)$ and $H_{i,j}(s)=p_{1,2}(s)+p_{3,0}(s)$. For a horizontal interior edge segment he_j with n_j vertices $V_{i,j}, i=h_{j,1},\dots,h_{j,n_j}$, we have

$$\begin{cases} H_{1,j}(s)=-d_{h_{j,1},j}(s-s_{h_{j,1}})^d, \\ \dots \\ H_{i,j}(s)=-d_{h_{j,i},j}(s-s_{h_{j,i}})^d, \\ \dots \\ H_{n_j,j}(s)=-d_{h_{j,n_j},j}(s-s_{h_{j,n_j}})^d. \end{cases} \tag{7}$$

By the definition of $H_{i,j}(s)$, it is easy to see that $\sum_{i=1}^{n_j} H_{i,j}(s)=0$. Thus

$$\sum_{i=1}^{n_j} d_{h_{j,i},j}(s-s_{h_{j,i}})^d=0. \tag{8}$$

Similarly, for each vertical interior edge segment with m_i vertices,

$$\sum_{j=1}^{m_i} d_{i,v_{i,j}}(t-t_{v_{i,j}})^d=0. \tag{9}$$

The above equations are the constrains for C^{d-1} continuity along each interior edge segment.

Lemma 1. *Assume that s_i, $i=1,\dots,m$ are distinct, and t_j, $j=1,\dots,n$ are different. Then the dimensions of solution spaces of Eqs.* (8) *and* (9) *for $d_{i,j}$ are $(n_j-d-1)_+$ and $(m_i-d-1)_+$ respectively. Here $i_+=i$ for $i>0$ and $i_+=0$ for $i\leqslant 0$.*

Proof. Denote

$$\mathbf{d}_j=(d_{i,j}, i=1,\dots,n_j)^T,$$
$$A=\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ s_{h_{j,1}} & s_{h_{j,2}} & \dots & s_{h_{j,n_j-1}} & s_{h_{j,n_j}} \\ \dots & & \dots & & \dots \\ s^d_{h_{j,1}} & s^d_{h_{j,2}} & \dots & s^d_{h_{j,n_j-1}} & s^d_{h_{j,n_j}} \end{pmatrix}.$$

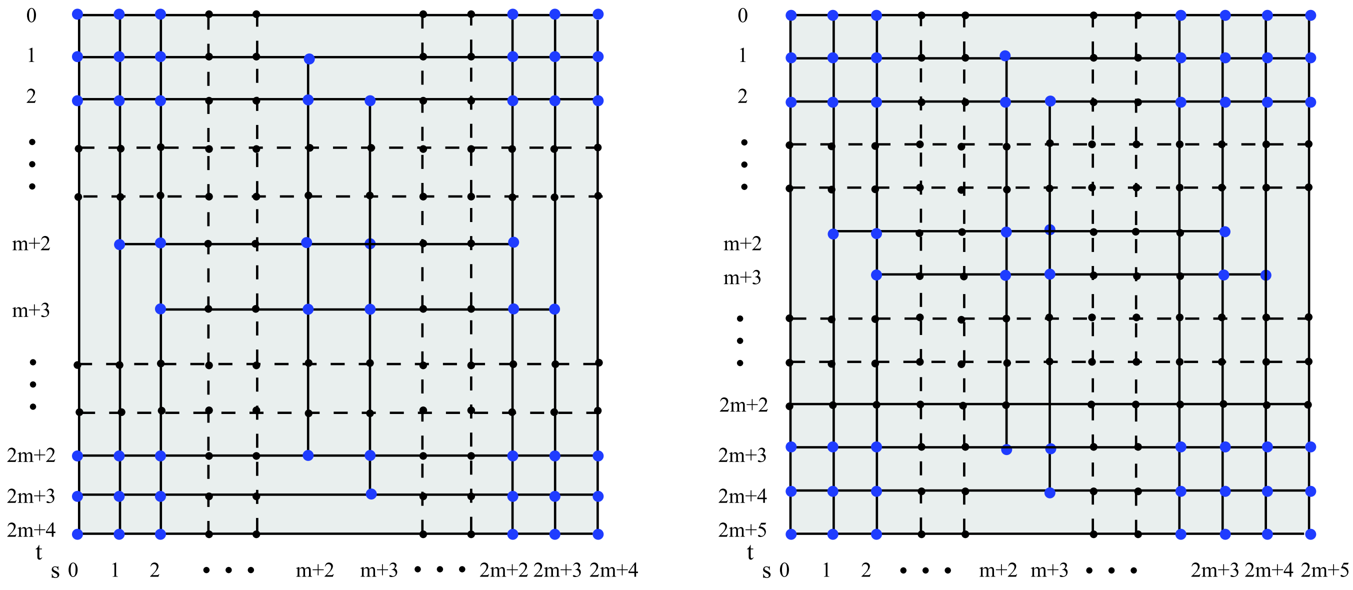
Then Eq. (8) is equivalent to the linear system

$$A\mathbf{d}_j=0. \tag{10}$$

It is easy to conclude that the rank of matrix A is $\min(d+1,n_j)$. The assertion follows. The conclusion for Eq. (9) is similar. □

Remark 2. We can derive similar constraints as Eqs. (8) and (9) for each cross-cut and each ray. Since each boundary vertex has $d+1$ degrees of freedom, the dimension of the solution space corresponding to a cross-cut is the number of interior vertices plus $d+1$, while the dimension of the solution space to a ray is the number of interior vertices.

For each edge segment, there is a linear constraints as (10). Putting all these constrains together, we can get a linear system of equations written in matrix form as $M\mathbf{x}=0$. Since each edge segment corresponds to $d+1$ rows of the matrix M, M is a matrix with $(d+1)(m+n)$ rows and n_V columns, here n_V is the number of interior vertices in the T-mesh. $\mathbf{x}$ is a column vector whose elements are cofactors of the vertices in the T-mesh. Referring to Li et al. (2006), we have the following lemma.



(a) $\mathcal{T}_{2m}$ (b) $\mathcal{T}_{2m+1}$

Fig. 4. Unstable T-meshes $\mathcal{T}_{2m}$ and $\mathcal{T}_{2m+1}$.

Lemma 2. *Given a T-mesh $\mathcal{T}$, and suppose it has c_h horizontal and c_v vertical cross-cuts respectively. Then*

$$\dim \mathcal{S}^d(\mathcal{T}) = (d+1)^2 + c_h(d+1) + c_v(d+1) + n_V - rank(M). \tag{11}$$

Thus, the key to compute the dimension of the spline space $\mathcal{S}^d(\mathcal{T})$ is to analyze the rank of matrix M. In the following section, we will show that the rank of the matrix M may depend on the geometry of $\mathcal{T}$ if $\mathcal{T}$ has some special structure.

3. Instability in the dimension of splines spaces over T-meshes

In this section, we provide several T-meshes $\mathcal{T}$ over which the dimension of spline space $\mathcal{S}^d(\mathcal{T})$ is unstable, that is, the dimension depends on not only the topological information of $\mathcal{T}$ but also the geometry of $\mathcal{T}$. The main result is the following theorem.

Theorem 3. *Let $s_{i,j} = s_i - s_j$, $t_{i,j} = t_i - t_j$, and denote*

$$\delta_{2m} = \frac{s_{m+2,1}s_{2m+3,m+3}t_{m+3,1}t_{2m+3,m+2}}{t_{m+2,1}s_{2m+3,m+3}s_{m+3,1}t_{2m+3,m+2}},$$

$$\delta_{2m+1} = \frac{s_{m+2,1}s_{2m+4,m+3}t_{m+3,1}t_{2m+4,m+2}}{t_{m+2,1}s_{2m+4,m+3}s_{m+3,1}t_{2m+4,m+2}}.$$

- *For the T-meshes $\mathcal{T}_{2m}$ as shown in Fig. 4a, if δ_{2m} equals to one, the dimension of the spline space $\mathcal{S}^{2m}(\mathcal{T}_{2m})$ is $(4m+2)^2 + 1$, otherwise the dimension is $(4m+2)^2$.*
- *For the T-meshes $\mathcal{T}_{2m+1}$ as shown in Fig. 4b, if δ_{2m+1} equals to one, the dimension of spline space $\mathcal{S}^{2m+1}(\mathcal{T}_{2m+1})$ is $(4m+4)^2 + 1$, otherwise the dimension is $(4m+4)^2$.*

Proof. We only provide the proof for the dimension formula for the spline space $\mathcal{S}^{2m+1}(\mathcal{T}_{2m+1})$.

In the T-mesh $\mathcal{T}_{2m+1}$, it has two horizontal interior edge segments t_{m+2}, t_{m+3} and two vertical interior edge segments t_{m+2}, t_{m+3}. In order to make the matrix M as simple as possible, we arrange the order of edge segments as t_{m+2}, s_{m+3}, t_{m+3} and s_{m+2}. The order of the vertices are arranged as $V_{m+3,m+2}$, $V_{1,m+2}, \ldots, V_{m+1,m+2}$, $V_{m+4,m+2}, \ldots, V_{2m+3,m+2}$, $V_{m+2,m+2}$, $V_{m+2,1}, \ldots, V_{m+2,m+1}$, $V_{m+2,m+4}, \ldots, V_{m+2,2m+3}$, $V_{m+2,m+3}$, $V_{2,m+3}, \ldots, V_{m+1,m+3}$, $V_{m+4,m+3}, \ldots, V_{2m+4,m+3}$, $V_{m+3,m+3}$, $V_{m+3,2}, \ldots, V_{m+3,m+1}$, $V_{m+3,m+4}, \ldots, V_{m+3,2m+4}$. Then we can get a sparse matrix M. For example, when $m = 1$, matrix M has the following form:

$$M=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
s_4 & s_1 & s_2 & s_5 & s_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
s_4^2 & s_1^2 & s_2^2 & s_5^2 & s_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
s_4^3 & s_1^3 & s_2^3 & s_5^3 & s_3^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_3 & t_1 & t_2 & t_5 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_3 & t_1^2 & t_2^2 & t_5^2 & t_4^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & t_3 & t_1^3 & t_2^3 & t_5^3 & t_4^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_3 & s_2 & s_5 & s_6 & s_4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_3^2 & s_2^2 & s_5^2 & s_6^2 & s_4^2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_3^3 & s_2^3 & s_5^3 & s_6^3 & s_4^3 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_4 & t_2 & t_5 & t_6\\
t_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_4^2 & t_2^2 & t_5^2 & t_6^2\\
t_3^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_4^3 & t_2^3 & t_5^3 & t_6^3
\end{pmatrix}.$$

By performing row reduction, we can transform matrix M into a simpler matrix $\hat{M}$ (which is almost triangular). For example, when $m=1$,

$$\hat{M}=\begin{pmatrix}
1 & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \star & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \star & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \star & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \star & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \star & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\delta_{2m+1} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 & 1
\end{pmatrix}.$$

Here $\star$ stands for possibly non-zero elements. Thus, if $\delta_{2m+1}=1$, the rank of matrix M is $4(2m+2)-1$ and the rank is $4(2m+2)$ if $\delta_{2m+1}\neq 1$. In other words, if $\delta_{2m+1}=1$, for example, $s_i=t_i, i=0,\dots,2m+5$, $\dim\mathcal{S}^{2m+1}(\mathcal{T}_{2m+1})=(4m+4)^2+1$, otherwise the dimension is $(4m+4)^2$. $\square$

We implemented the smoothing cofactor method to compute the dimension of spline spaces over T-meshes in Maple and verify the instability in the dimension of spline spaces $\mathcal{S}^d(\mathcal{T})$.

We conclude the section with two specific examples as shown in Fig. 5.

Example 1. For the T-mesh $\mathcal{T}_2$ as shown in Fig. 5, if $\frac{s_{3,1}s_{5,4}}{t_{3,1}t_{5,4}}=\frac{s_{4,1}s_{5,3}}{t_{4,1}t_{5,3}}$, the dimension of $\mathcal{S}(2,2,1,1,\mathcal{T}_2)$ is 37. Otherwise, the dimension is 36.

Example 2. For the T-mesh $\mathcal{T}_3$ as shown in Fig. 5, if $\frac{s_{3,1}s_{6,4}}{t_{3,1}t_{6,4}}=\frac{s_{4,1}s_{6,3}}{t_{4,1}t_{6,3}}$, the dimension f $\mathcal{S}(3,3,2,2,\mathcal{T}_3)$ is 65. Otherwise, the dimension is 64.

4. Conclusion and future work

The present paper discusses the dimension of spline spaces over T-meshes by the smoothing cofactor method. It is shown that the dimension of the spline space $\mathcal{S}(d,d,d-1,d-1,\mathcal{T})$ has a similar situation as spline space over triangles, which is unstable for certain types of T-meshes, that is, the dimension depends on not only the topological information of the T-mesh, but also the geometry of the T-mesh. And it is also similar as splines spaces over triangles, the degeneracies will disappear if the polynomial degree is large enough relative to the smoothness degree, for example when the order of smoothness is less than half of the degree of the spline functions (Deng et al., 2006). However, unlike spline space over triangles, the spline space over T-mesh has no geometry dependencies on vertex stars.

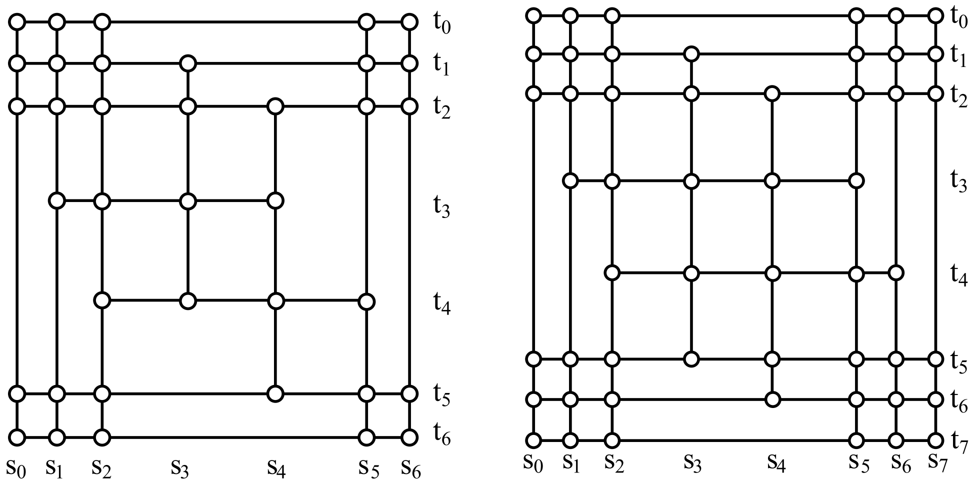


Fig. 5. Unstable T-meshes $\mathcal{T}_2$ and $\mathcal{T}_3$.

There are several interesting questions left as future works. For example, how to find out a class of T-meshes over which the dimension of spline spaces is stable. The other interesting question is how large should the polynomial degree relative to the smoothness to make the dimension of the spline space is stable.

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