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SOME PROPERTIES FOR ANALYSIS-SUITABLE T-SPLINES*

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Abstract

Analysis-suitable T-splines (AS T-splines) are a mildly topological restricted subset of T-splines which are linear independent regardless of knot values [1–3]. The present paper provides some more iso-geometric analysis (IGA) oriented properties for AS T-splines and generalizes them to arbitrary topology AS T-splines. First, we prove that the blending functions for analysis-suitable T-splines are locally linear independent, which is the key property for localized multi-resolution and linear independence for non-tensor-product domain. And then, we prove that the number of T-spline control points contribute each Bézier element is optimal, which is very important to obtain a bound for the number of non zero entries in the mass and stiffness matrices for IGA with T-splines. Moreover, it is found that the elegant labeling tool for B-splines, blossom, can also be applied for analysis-suitable T-splines.

Mathematics subject classification: 65D07.
Key words: T-splines, Linear independence, iso-geometric analysis, Analysis-suitable T-splines.

1. Introduction

T-splines [4, 5] have been used to solve many limitations inherent in the industry standard NURBS representation, such as watertightness [4, 6], trimmed NURBS conversion [7] and local refinement [5, 8]. Thus, T-splines have proved to be an important technology across several disciplines including industrial, architectural and engineering design, manufacturing and engineering analysis. Especially, these capabilities make T-splines attractive for both geometric modeling and iso-geometric analysis (IGA), which use the smooth spline basis that defines the geometry as the basis for analysis. IGA was introduced in [9] and described in detail in [10]. Traditional design-through-analysis procedures such as geometry clean-up, defeaturing, and mesh generation are simplified or eliminated entirely. Additionally, the higher-order smoothness provides substantial gains to analysis in terms of accuracy and robustness of finite element solutions [11–13]. The use of T-splines as a basis for IGA has gained widespread attention [8, 14–16].

However, [17] discovered an example of a T-spline with *linear dependent* blending functions, which means that not all T-splines are suitable as a basis for IGA. Thus, an important development in the evolution of IGA was the advent of analysis-suitable T-splines (AS T-splines), a mildly topological restricted subset of T-splines. AS T-splines are optimized to meet the needs both for design and analysis [1, 8]. Such T-splines inherit all the good properties from T-splines, such as watertightness, NURBS compatible, convex hull, and affine invariant. Unlike the general T-splines, such T-splines are guaranteed to be linearly independent [1], the

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polynomial blending functions for such T-splines sum identically to one for an admissible T-mesh [18,19] and the T-spline space can be characterized in terms of piecewise polynomials [18]. Furthermore, algorithms have been developed whereby local refinement of such T-splines is well contained [8].

The present paper studies several more IGA-oriented properties for analysis-suitable T-splines, including local linear independence, the number of control points which contribute one Bézier element, and blossom. We also generalize them to arbitrary topology AS T-splines. Linear independence in [1] is the traditional *global* linear independence, which means the blending functions are linear independent in the whole domain. Local linear independent means if a T-spline is zero in a bounded domain, then all the coefficients for the blending functions which are not zero in the domain must be zeros. It is obvious that local linear independence is a stricter requirement than global linear independence. Some numerical methods, such as *localized multi-resolution*, *linear independence for non-tensor-product domain*, rely on locally linear independence. The number of control points which contribute one Bézier element has a clear impact on the use of T-splines in iso-geometric analysis applications. Because it is possible to obtain a bound for the number of non zero entries in the mass and stiffness matrices in applications to PDEs. Blossom, introduced by Dr. Lyle Ramshaw, can be thought of as an alternative method of labeling the control points for a B-Spline curve or surface. Blossom provides a clear and insight way to understand the algorithm for B-splines [20–22]. But it is much more complex to derive the blossom formula for T-spline than those for B-splines, which are discussed in Section 5.

T-splines and the other local refinable splines, including polynomial splines over hierarchical T-meshes (PHT) [23–26], hierarchical B-splines [27,28] and LR B-splines [29], are not local linear independent. Examples for the hierarchical B-splines and LR B-splines which are linear independent but not local linear independent (also the other two properties) are straightforward. We only provide examples for T-splines and PHT. For example, in Figure 1.1a, the corresponding T-splines are always linear independent but it is not local linear independent in the grey domain because they are 21 blending functions are not zero in the domain. And PHT defined over the T-mesh in Figure 1.1b. is also not local linear independent because they are 24 basis functions (each vertex corresponds four basis functions) are not zero in the grey domain. In both examples, the corresponding vertices for these basis functions are marked with red.

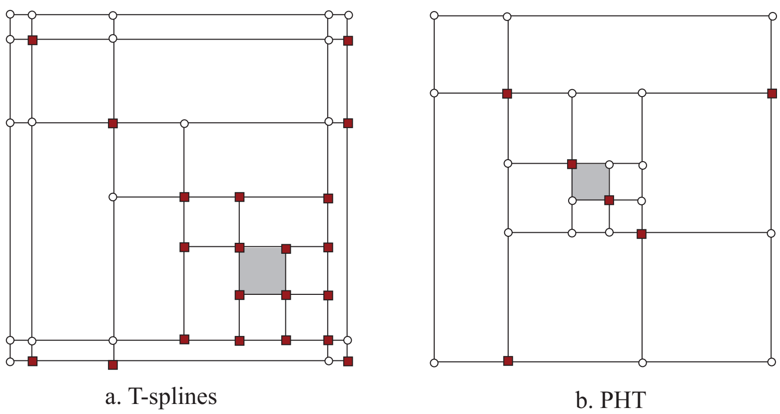


Fig. 1.1. Both T-splines and PHT are not always local linear independent even they are global linear independent.

In the summary, the main contribution of the present paper includes:

- We show that the blending functions for analysis-suitable T-splines are local linear independent;
- We prove that the number of control points which contribute one Bézier element is $(d_1 + 1) \times (d_2 + 1)$ for bi-degree (d_1, d_2) analysis-suitable T-splines;
- We discover blossom formula for analysis-suitable T-splines;
- We generalize the linear independence for arbitrary topological analysis-suitable T-splines;

The rest of the paper is structured as follows. Pertinent background on T-splines and analysis-suitable T-splines is reviewed in Section 2. Section 3 proves that any analysis-suitable T-splines have local linearly independent blending functions. Section 4 shows that the number of control points which contribute one Bézier element is optimal and Section 5 discusses the blossom for analysis-suitable T-splines. Section 6 generalizes the linear independence for arbitrary topological analysis-suitable T-splines. The last section discusses our conclusion and future work.

2. T-splines

In the section, we prepare some basic notations and preliminary results for arbitrary degree T-splines [14, 30].

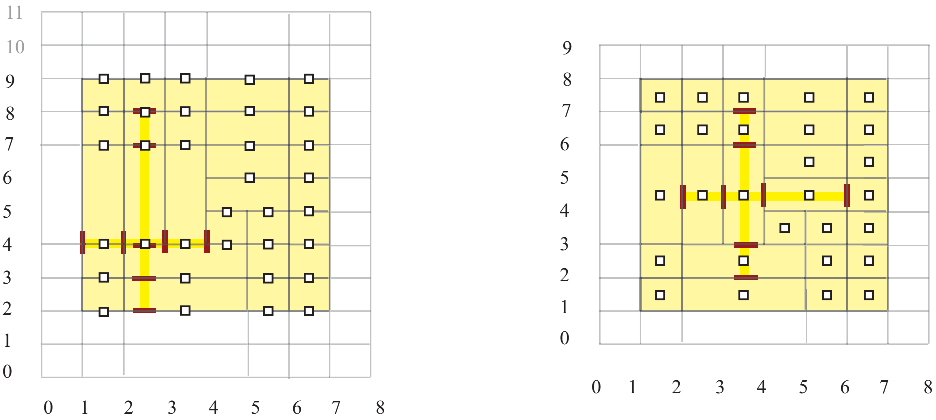
2.1. Index T-mesh

Similar as the approach of [14], we define a T-spline based on a T-mesh in the index domain which is referred as an *index T-mesh* in the paper. A T-mesh T for a bi-degree (d_1, d_2) T-spline is a connection of all the elements of a rectangular partition of the index domain $[0, c + d_1] \times [0, r + d_2]$, where all rectangle corners (or vertices) have integer coordinates. Three types of elements are

- Vertex: vertex of a rectangle, denoted as (δ_i, τ_i) or $\{\delta_i\} \times \{\tau_i\}$.
- Edge: a line segment connecting two vertices in the T-mesh and no other vertices lying in the interior, denoted as $[\delta_j, \delta_k] \times \{\tau_i\}$ or $\{\delta_i\} \times [\tau_j, \tau_k]$ for a horizontal or vertical edge.
- Face: a rectangle where no other edges and vertices in the interior, denoted as $[\delta_i, \delta_j] \times [\tau_k, \tau_l]$ or $(\delta_i, \delta_j) \times (\tau_k, \tau_l)$, where the second one is for an open face.

The valence of a vertex is the number of edges such that the vertex is an endpoint. For the interior vertices, we only allow valence three (called T-junctions) or four vertices. We adopt the notations $\vdash$, $\dashv$, $\perp$ and $\top$ to indicate the four possible orientations for the T-junctions. Denote the *active region* as a rectangle region $[p, c + d_1 - p] \times [q, r + d_2 - q]$, here p and q are the maximal integers equal or less than $\frac{d_1+1}{2}$ and $\frac{d_2+1}{2}$ respectively. As we will see below, the active region carries the anchors that will be associated with the blending functions while the other indices will be needed for the definition of the blending function when the anchor is close to the boundary.

An *anchor* is a point in the index T-mesh which corresponds one blending function. The anchor corresponds to the vertices, edges or faces depending on the degrees. If both d_1 and



a. bi-degree (2,3) b. bi-degree (2,2)

Fig. 2.1. The anchors and the local index vector for one blending function

d_2 are odd, an anchor corresponds a vertex in the active region of the T-mesh, if both d_1 and d_2 are even, then an anchor corresponds the barycenter of a face in the active region of the T-mesh. And the index coordinate for the anchor is the index coordinate of left-bottom vertex of the associated face. If d_1 is even and d_2 is odd or d_1 is odd and d_2 is even, an anchor is the middle point of a horizontal edge or a vertical edge in the active region of the T-mesh.

Definition 2.1. *A T-mesh is admissible if for any vertex (i,j), if $0 \le i \le d_1$ or $c \le i \le c+d_1$, then the vertex cannot be $\perp$, $\top$ and if $0 \le j \le d_2$ or $r \le j \le r+d_2$, then the vertex cannot be $\vdash$ and $\dashv$.*

2.2. Element T-mesh

For the i-th anchor $\mathbf{A}_i$, we define a local index vector $\vec{\delta_i} \times \vec{\tau_i}$ which is used to define the blending function $T_i(s,t)$. The values of $\vec{\delta_i} = [\delta_i^0, \ldots, \delta_i^{d_1+1}]$ and $\vec{\tau_i} = [\tau_i^0, \ldots, \tau_i^{d_2+1}]$ are

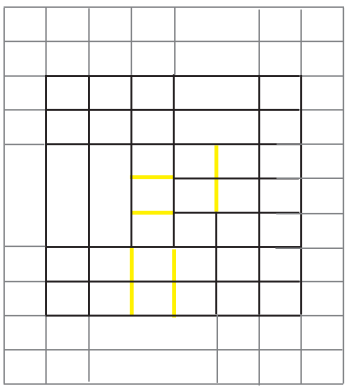


Fig. 2.2. The T-mesh of T-mesh in Figure 2.1, where the yellow edges are the new edges which are not in the origin T-mesh.

determined as follows. From the i-th anchor in the T-mesh, we shoot a ray in the s and t direction traversing the T-mesh and collect a total of d_1+2 and d_2+2 knot indices to form $\vec{\delta_i}$ and $\vec{\tau_i}$, as shown in Figure 2.1.

Each local index vector $\vec{\delta_i}\times\vec{\tau_i}$ defines a tensor-product mesh $[\delta_i^0,\ldots,\delta_i^{d_1+1}]\times[\tau_i^0,\ldots,\tau_i^{d_2+1}]$. And the element T-mesh, denoted as T_{elem} is a minimal T-mesh which contains these tensor-product meshes for all the anchors in a T-mesh T. For example, the element T-mesh for a bi-degree (2,3) T-spline defined on the T-mesh in Figure 2.1 a. is shown in Figure 2.2.

2.3. T-splines

The two indices correspond two global knot vectors $\vec{\mathbf{s}}=[s_0,\ldots,s_{c+d_1}]$ and $\vec{\mathbf{t}}=[t_0,\ldots,t_{r+d_2}]$. Each edge is assigned with a *knot interval* which is the associated parametric length of the edge. The validity rules for the knot configuration require that the sums of the knot intervals on opposite sides of a face must be equal [4]. For the i-th anchor, we can define two B-spline functions $B[\vec{\mathbf{s}}_i](s)$ and $B[\vec{\mathbf{t}}_i](t)$ according to the two knot vectors $\vec{\mathbf{s}}_i=[s_{\sigma_i^0},s_{\sigma_i^1},\ldots,s_{\sigma_i^{d_1+1}}]$ and $\vec{\mathbf{t}}_i=[t_{\tau_i^0},t_{\tau_i^1},\ldots,t_{\tau_i^{d_2+1}}]$ using the formula (8.16) in [31]. Then the blending function associated with the i-th anchor is defined as $T_i(s,t)=B[\vec{\mathbf{s}}_i](s)B[\vec{\mathbf{t}}_i](t)$.

A T-spline space is finally given as the span of all these blending functions and a T-spline surface is defined as

$$\mathbf{T}(s,t)=\sum_{i=1}^{n_A}\mathbf{C}_iT_i(s,t), \tag{2.1}$$

where $\mathbf{C}_i=(\omega_ix_i,\omega_iy_i,\omega_iz_i,\omega_i)\in\mathbb{P}^3$ are homogeneous control points, $\omega_i\in\mathbb{R}$ are weights, $T_i(s,t)$ are blending functions, and n_A is the number of control points or anchors.

2.4. Extension and Analysis-suitable T-splines

Analysis-suitable T-splines are defined in terms of T-junction extension, which is a line segment associated with each T-junction. For example, for a i-th T-junction (δ_i,τ_i) of type $\vdash$ or $\dashv$, the extension for the T-junction is the line segment $[\underline{i},\overline{i}]\times\{\tau_i\}$. $\underline{i}$ and $\overline{i}$ are determined such that the edges $[\underline{i},\delta_i)\times\{\tau_i\}$ have $[\frac{d_1+1}{2}]$ intersections with the T-mesh and the edges $(\delta_i,\overline{i}]\times\{\tau_i\}$ have $[\frac{d_1}{2}]$ intersections with the T-mesh for T-junction of type $\vdash$. Here $[d]$ means the maximal integer less or equal d. For T-junction of type $\dashv$, we can similarly define the extension except the number of intersections are exchanged. Also, we can define the extensions for the other kinds of T-junctions $\perp$, $\top$, where use degree d_2 instead of d_1. All these extension examples are illustrated in Figure 2.3.

Definition 2.2. *For a bi-degree (d_1,d_2) T-spline, a T-mesh is called analysis-suitable if the extensions for all the T-junctions $\vdash$ and $\dashv$, don't intersect the extensions for all the T-junctions $\perp$ and $\top$. A T-spline defined on an analysis-suitable T-mesh is called an analysis-suitable T-spline, for short AS T-spline.*

Analysis-suitable T-splines are optimized to meet the needs of both design and analysis. The following properties have been discovered for AS T-splines according to the requirement from IGA.

- The blending functions are linearly independent for all the knots [1,2,19].
- The basis constitutes a partition of unity [18].

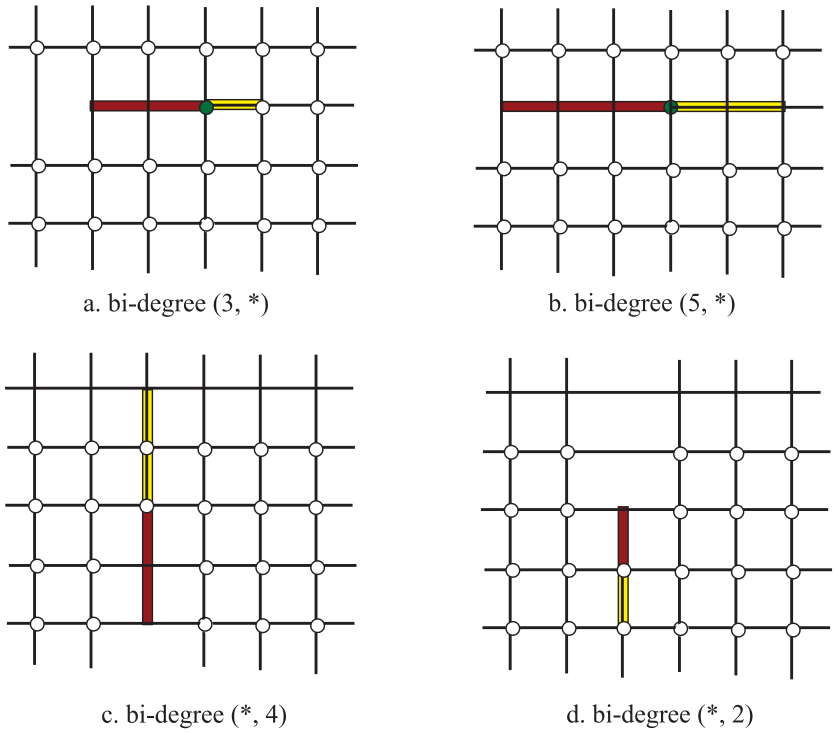


Fig. 2.3. The extensions for four different kinds of T-junctions.

- AS T-splines obey the convex hull property.
- They can be locally refined [4,8].
- A dual basis can be constructed [2,19].
- Spline space can be characterized in terms of piecewise polynomials [18].

3. Local Linear Independent for AS T-splines

In this section, we prove that the AS T-spline basis functions are local linear independent, i.e, if an AS T-spline surface is zero in some Bézier elements, then all the coefficient for basis functions which are not zero in these elements are zeros.

3.1. Dual basis for AS T-splines

Given a knot vector $(s_1, s_2, \ldots, s_m)$, let $B[s_k, \ldots, s_{k+d+1}](\xi)$ be a degree d B-spline basis functions associated with the knot vector $[s_k, \ldots, s_{k+d+1}]$. Following [32], we can define suitable functionals $\lambda[s_k, \ldots, s_{k+d+1}]$ which are dual to the B-splines basis $B[s_k, \ldots, s_{k+d+1}](\xi)$. These functions have the following property

$$\lambda[s_k, \ldots, s_{k+d+1}](B[s_j, \ldots, s_{j+d+1}](\xi)) = \delta_{k,j} = \begin{cases} 1, & k = j; \\ 0, & k \neq j. \end{cases}$$

Lemma 3.1. *Suppose*

$$f(\xi) = \sum_{j=1}^{n} c_j B[s_j, \ldots, s_{j+d+1}](\xi).$$

Then $c_j = \lambda[s_j, \ldots, s_{j+d+1}](f(\xi))$.

Proof. Note that

$$\begin{aligned}
\lambda[s_j, \ldots, s_{j+d+1}](f(\xi)) &= \lambda[s_j, \ldots, s_{j+d+1}]\left(\sum_{i=1}^{n} c_i B[s_i, \ldots, s_{i+d+1}](\xi)\right) \\
&= \sum_{i=1}^{n} c_i \lambda[s_j, \ldots, s_{j+d+1}](B[s_i, \ldots, s_{i+d+1}](\xi)) \\
&= \sum_{i=1}^{n} c_i \delta_{i,j} = c_j.
\end{aligned} \tag{3.1}$$

The proof is then complete. □

Lemma 3.2. *Suppose*

$$f(\xi) = \sum_{j=1}^{n} c_j B[s_j, \ldots, s_{j+d+1}](\xi).$$

If there exists an integer k*, such that* $f(\xi) = 0$ *for all* $s_k \le \xi < s_{k+1}$*, then for any* $j = k-d, k-d+1, \ldots, k$*,*

$$\lambda[s_j, \ldots, s_{j+d+1}](f(\xi)) = 0. \tag{3.2}$$

Proof. As B-splines are local linear independent, we have $c_j = 0$ for $j = k-d, k-d+1, \ldots, k$. According to Lemma 3.1, $c_j = \lambda[s_j, \ldots, s_{j+d+1}](f(\xi))$, we obtain (3.2). □

Note that [2,19] provided a similar way to define some functionals Λ^i which are dual to the AS T-splines basis $T_j(s,t)$, i.e.,

$$\Lambda^i(T_j(s,t)) = \delta_{i,j}.$$

The main results are including as the following theorem. Readers may refer to [2,19] for more details of the proof.

Theorem 3.1. *For bi-degree* (d_1, d_2) *analysis-suitable* T*-splines, suppose the local knot vectors for the* i*-th basis function are* $\vec{\mathbf{s}}_i = [s_{\sigma_i^0}, \ldots, s_{\sigma_i^{d_1+1}}]$ *and* $\vec{\mathbf{t}}_i = [t_{\tau_i^0}, \ldots, t_{\tau_i^{d_2+1}}]$*. Then*

$$\Lambda^i = \lambda[s_{\sigma_i^0}, \ldots, s_{\sigma_i^{d_1+1}}] \otimes \lambda[t_{\tau_i^0}, \ldots, t_{\tau_i^{d_2+1}}] \tag{3.3}$$

are the dual basis for T*-spline basis functions.*

3.2. Local linear independent

Lemma 3.3. *For a Bézier element with left-bottom corner* (δ_l, τ_b) *and top-right corner* (δ_r, τ_t)*, if the* i*-th basis function with knot vectors* $\vec{\delta_i} \times \vec{\tau_i}$ *is non-zero in the Bézier element, then there must exist an integer* j *such that* $\delta_l = \sigma_i^j$ *and* $\delta_r = \sigma_i^{j+1}$ *or an integer* k *such that* $\tau_b = \tau_i^k$ *and* $\tau_t = \tau_i^{k+1}$*.*

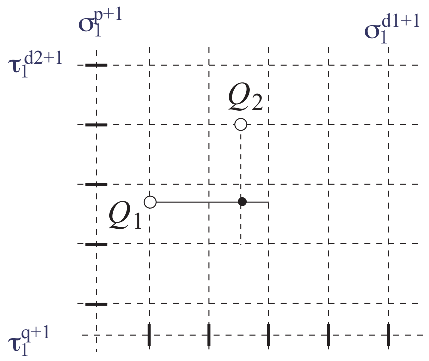


Fig. 3.1. Illustration for Lemma 3.3.

Proof. Suppose the lemma is false. Then there exists at least one cell C^* that violates the condition. Since C^* violates the condition, so there exists at least one corner of C^* that lies in the $[\sigma_i^t, \sigma_i^{t+1}] \times [\tau_i^l, \tau_i^{l+1}]$, $0 \le t \le d_1 + 1$ and $0 \le l \le d_2 + 1$. Without loss of generality, we assume $p + 1 \le t \le d_1 + 1$ and $q + 1 \le l \le d_2 + 1$, here $p = [\frac{d_1}{2}] + 1$ and $q = [\frac{d_2}{2}] + 1$. We have following three cases:

1. The corner is a vertex of the original T-mesh. This violates Lemma 3.2 in [19].
2. The corner is the result of the intersection of two perpendicular T-junction extensions. This violates the assumption that the T-mesh is analysis-suitable.
3. The corner is the result of the intersection of one T-junction extension and a T-mesh edge. Without loss of generality, we assume the edge is a horizontal edge and the T-junction extension is vertical and is associated with a T-junction Q_2. As the edge cannot intersect the vertical line σ_i^{p+1}, so it must terminate in a T-junction Q_1. Examining the extensions associated with Q_1 and Q_2, and we find that they must intersect. This violates the assumption that the T-mesh is analysis-suitable.

Thus, such cell does not exist. □

Theorem 3.2. *AS T-splines are local linear independent.*

Proof. For an arbitrary Bézier element with left-bottom corner (δ_l, τ_b) and top-right corner (δ_r, τ_t), assume a T-spline $f(s,t) = \sum_{i=1}^{n_T} c_i T_i(s,t)$ is zero in the element. According to Lemma 3.3, since $T_i(s,t)$ is non-zero in the Bézier element, so there exists an integer j such that $\delta_l = \sigma_i^j$ and $\delta_r = \sigma_i^{j+1}$ or exists an integer k such that $\tau_b = \tau_i^k$ and $\tau_t = \tau_i^{k+1}$. Denote $g_i(s) = \lambda[t_{\tau_i^0}, \ldots, t_{\tau_i^{d_2+1}}](f(s,t))$ and by Lemma 3.1 and Theorem 3.1:

$$c_i = \Lambda^i(f(s,t)) = \lambda[s_{\sigma_i^0}, \ldots, s_{\sigma_i^{d_1+1}}](g_i(s)). \tag{3.4}$$

If $\tau_b = \tau_i^k$ and $\tau_t = \tau_i^{k+1}$, according to Lemma 3.2, $f(s,t) = 0$ for $t_{\tau_i^k} \le t \le t_{\tau_i^{k+1}}$. Thus, $g_i(s) = 0$ which gives $c_i = 0$. Otherwise, $\delta_l = \sigma_i^j$ and $\delta_r = \sigma_i^{j+1}$ and $g_i(s) = 0$ for $\sigma_i^j \le s \le \sigma_i^{j+1}$. It follows from Lemma 3.2 that $\lambda[s_{\sigma_i^0}, \ldots, s_{\sigma_i^{d_1+1}}](g_i(s)) = 0$. Thus, $c_i = 0$, which completes the proof. □

4. Number of Control Points Contribute Each Bézier Patch

This section provides the results of the number of T-spline blending functions that are non zero in the domain of one Bézier patch. The main result states that AS T-splines have the optimal number, which is $(d_1+1)\times(d_2+1)$. Noticed that the result of this section is not associated with the real value of knot intervals, so we consider the problem in the index domain.

Definition 4.1. *For any face $(f_s^1, f_s^2)\times(f_t^1, f_t^2)$ in the element T-mesh, we said the i-th control point or anchor contributes the face if and only if*

$$(f_s^1, f_s^2)\times(f_t^1, f_t^2)\cap(\delta_i^0, \delta_i^{d_1+1})\times(\tau_i^0, \tau_i^{d_2+1})\neq\varnothing.$$

Lemma 4.1. *For a bi-degree (d_1, d_2) admissible AS T-spline, if any extensions have no common vertices, then the number of control points contribute any face in the element T-mesh is $(d_1+1)\times(d_2+1)$.*

Proof. Because the T-spline is analysis suitable, so the blending functions are local linear independent. Thus, the number of control points contribute each face in the element T-mesh is at most $(d_1+1)\times(d_2+1)$. Otherwise, the T-spline is not local linear independent in this face. On the other hand, according to the characterization of AS T-splines, for each Bézier patch, all the bi-degree (d_1, d_2) polynomials belong to the AS T-spline space, so the number of control points contribute the face is at least $(d_1+1)\times(d_2+1)$. Thus, the number of control points contribute the Bézier patch is exact $(d_1+1)\times(d_2+1)$. □

In order to generalize Lemma 4.1 to any AS T-splines, we develop the tool of the perturbed T-mesh. We first establish the result in the perturbed setting and then show that the result holds as the perturbed T-mesh converges to the original T-mesh. The detail description of perturbed T-mesh can be found in [18]. Here, we only provide a simple and intuitionist description.

Given a T-mesh T, the perturbed T-mesh, $\mathsf{T}[\delta]$, has same number of vertices, edges, faces and the same connection information as T, but perturbs some of the edges such that the extensions of the T-junctions in $\mathsf{T}[\delta]$ have no common vertices. For example, in Figure 4.1, the left T-mesh has pairs of overlapping extensions (V_1, V_2 and V_3, V_4). But if we perturb some edges and create a new T-mesh like the right one, the extensions of the T-junctions in the new T-mesh have no common vertices.

Because we perturb some edges, the perturbed T-mesh $\mathsf{T}[\delta]$ has more knot indices. So we can build a map π_s from the s-indices of $\mathsf{T}[\delta]$ to those of T, i.e., for any index i in $\mathsf{T}[\delta]$, $\pi_s(i)$ is the corresponding s-index in T. Similarly, we can define map π_t from the t-indices of $\mathsf{T}[\delta]$ to those of T. For example in the Figure 4.1, $\pi_s(4)=\pi_s(5)=4$ and $\pi_t(2)=\pi_t(3)=2$. A suitable perturbed T-mesh should satisfy that if $\pi_s(i)<\pi_s(j)$, then $i<j$ and if $\pi_t(i)<\pi_t(j)$, then $i<j$.

With the maps π_s and π_t, we can build a map π from the faces of element T-mesh T_{elem} to the faces in $\mathsf{T}_{elem}[\delta]$. For any face

$$f=(f_s^1, f_s^2)\times(f_t^1, f_t^2)\in\mathsf{T}_{elem},\quad \pi(f)=g=(g_s^1, g_s^2)\times(g_t^1, g_t^2)\in\mathsf{T}_{elem}[\delta]$$

if and only if

$$\pi_s(g_s^1)=f_s^1,\quad \pi_s(g_s^2)=f_s^2,\quad \pi_t(g_t^1)=f_t^1,\quad \pi_t(g_t^2)=f_t^2.$$

Lemma 4.2. *If T is analysis-suitable, then $\mathsf{T}[\delta]$ is analysis-suitable.*

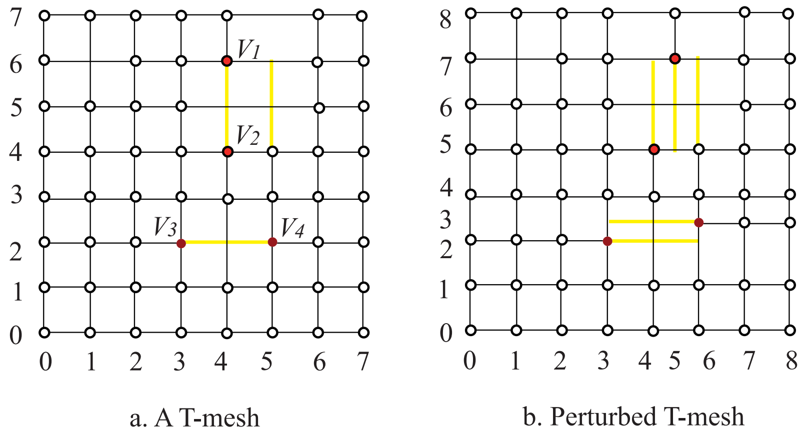


Fig. 4.1. A T-mesh extension of the same direction has common vertices (the extension of the those red T-junctions) and its corresponding perturbed T-mesh. The yellow lines are the new edges of corresponding element T-meshes.

Proof. See [18] for more details. □

Lemma 4.3. *A control point contributes a face f in T_{elem} if and only if the control point contributes the face $\pi(f)$ in $\mathsf{T}_{elem}[\delta]$.*

Proof. For any face $f = (f_s^1, f_s^2) \times (f_t^1, f_t^2)$ in T_{elem}, the i-th control point contributes the face if and only if

$$f \cap (\pi_s(\delta_i^0), \pi_s(\delta_i^{d_1+1})) \times (\pi_t(\tau_i^0), \pi_t(\tau_i^{d_2+1})) \neq \varnothing,$$

which holds if and only if

$$\pi(f) \cap (\delta_i^0, \delta_i^{d_1+1}) \times (\tau_i^0, \tau_i^{d_2+1}) \neq \varnothing.$$

Thus, it is equivalent to the corresponding control point(anchor) contributes the face $\pi(f)$ in $\mathsf{T}_{elem}[\delta]$. □

Theorem 4.1. *For a bi-degree (d_1, d_2) admissible AS T-spline, the number of control points contribute a face in the element T-mesh is $(d_1+1) \times (d_2+1)$.*

Proof. If extensions have no common vertices, we can get the result according to lemma 4.1. Otherwise, we generate a perturbed T-mesh $\mathsf{T}[\delta]$ which extensions have no common vertices. According to Lemma 4.3, the number of control points contribute the face $f \in \mathsf{T}_{elem}$ is same as the number of control points contribute the face $\pi(f)$ in $\mathsf{T}_{elem}[\delta]$. And because the T-mesh $\mathsf{T}[\delta]$ is analysis-suitable, so the number is $(d_1+1) \times (d_2+1)$ according to Lemma 4.1. In other words, the number of control points contribute a face in an admissible analysis-suitable element T-mesh is $(d_1+1) \times (d_2+1)$. □

5. Blossom

As we known, blossom provides a clear and insight way to understand the knot insertion, degree elevation and de Boor evaluation algorithm for B-splines. However, it is difficult to derive the blossom formula for general T-splines. In this section, we will show that AS T-splines have the similar blossom formula as B-splines.

Definition 5.1. *Two different index vectors* $\vec{\delta_i} = (\delta_i^0, \ldots, \delta_i^{d+1})$ *and* $\vec{\delta_j} = (\delta_j^0, \ldots, \delta_j^{d+1})$, $i \neq j$, *are called overlap if*

$$\forall k \in \{\delta_j^0, \ldots, \delta_j^{d+1}\} \cup \{i | \delta_i^0 \leq i \leq \delta_i^{d+1}\}, \quad k \in \{\delta_i^0, \ldots, \delta_i^{d+1}\}, \text{ and}$$
$$\forall k \in \{\delta_i^0, \ldots, \delta_i^{d+1}\} \cup \{i | \delta_j^0 \leq i \leq \delta_j^{d+1}\}, \quad k \in \{\delta_j^0, \ldots, \delta_j^{d+1}\}.$$

Lemma 5.1. *Given a knot vector* $\{s_k\}$ *and two overlap index vectors* $\vec{\delta_i}$, $\vec{\delta_j}$, *let* $\tilde{B}[s_{\vec{\delta_i}}]$ *be the blossom for B-spline with local knot vector* $s_{\vec{\delta_i}}$. *Then if* $\vec{\delta_i} = \vec{\delta_j}$, *then*

$$\tilde{B}[s_{\vec{\delta_i}}](s_{\delta_j^1}, \ldots, s_{\delta_j^d}) = 1.$$

Otherwise, $\tilde{B}[s_{\vec{\delta_i}}](s_{\delta_j^1}, \ldots, s_{\delta_j^d}) = 0$.

Proof. This can be precisely derived from the blossom of B-splines. □

Theorem 5.1. *Let* $\tilde{\boldsymbol{T}}(\tilde{s}_1, \ldots, \tilde{s}_{d_1}; \tilde{t}_1, \ldots, \tilde{t}_{d_2})$ *be the blossom of an analysis-suitable T-spline* $\boldsymbol{T}(s,t) = \sum_{i=0}^{n_A} C_i T_i(s,t)$. *Then the control points satisfy*

$$C_i = \tilde{\boldsymbol{T}}(s_{\delta_i^1}, \ldots, s_{\delta_i^{d_1}}; t_{\tau_i^1}, \ldots, t_{\tau_i^{d_2}}). \tag{5.1}$$

Proof. Without loss of generality, we prove the lemma for a patch which contributed by C_0, $\ldots$, C_n, here $n = (d_1+1)(d_2+1) - 1$. First, we prove that for any $0 \leq i, j \leq n$,

$$\tilde{T}_j(s_{\delta_i^1}, \ldots, s_{\delta_i^{d_1}}; t_{\tau_i^1}, \ldots, t_{\tau_i^{d_2}}) = \delta_{i,j}. \tag{5.2}$$

Actually, if $i = j$, the equation is obvious. Otherwise, according to paper [19], knot vector $\vec{\delta_i}$ and $\vec{\delta_j}$ are overlap or $\vec{\tau_i}$ and $\vec{\tau_j}$ are overlap because the T-spline is analysis-suitable. Without losing generality, we assume $\vec{\delta_i}$ and $\vec{\delta_j}$ are overlap. If $\vec{\delta_i} = \vec{\delta_j}$, then $\vec{\tau_i}$ and $\vec{\tau_j}$ are overlap but different, according to Lemma 5.1,

$$\begin{aligned}\tilde{T}_j(s_{\delta_i^1}, \ldots, s_{\delta_i^{d_1}}; t_{\tau_i^1}, \ldots, t_{\tau_i^{d_2}}) &= \tilde{B}[s_{\vec{\delta_i}}](s_{\delta_j^1}, \ldots, s_{\delta_j^{d_1}}) \times \tilde{B}[t_{\vec{\tau_i}}](t_{\tau_j^1}, \ldots, t_{\tau_j^{d_2}}) \\ &= \tilde{B}[s_{\vec{\delta_i}}](s_{\delta_j^1}, \ldots, s_{\delta_j^{d_1}}) \times 0 = 0.\end{aligned} \tag{5.3}$$

Otherwise, according to Lemma 5.1, $\tilde{B}[s_{\vec{\delta_i}}](s_{\delta_j^1}, \ldots, s_{\delta_j^{d_1}}) = 0$, so

$$\tilde{T}_j(s_{\delta_i^1}, \ldots, s_{\delta_i^{d_1}}; t_{\tau_i^1}, \ldots, t_{\tau_i^{d_2}}) = 0 \times \tilde{B}[t_{\vec{\tau_i}}](t_{\tau_j^1}, \ldots, t_{\tau_j^{d_2}}) = 0. \tag{5.4}$$

Thus, $\tilde{T}_j(s_{\delta_i^1}, \ldots, s_{\delta_i^{d_1}}; t_{\tau_i^1}, \ldots, t_{\tau_i^{d_2}}) = \delta_{i,j}$. With this equation, we have

$$\begin{aligned}\tilde{\mathbf{T}}(s_{\delta_i^1}, \ldots, s_{\delta_i^{d_1}}; t_{\tau_i^1}, \ldots, t_{\tau_i^{d_2}}) &= \sum_{j=1}^{n_A} C_j \tilde{T}_j(s_{\delta_i^1}, \ldots, s_{\delta_i^{d_1}}; t_{\tau_i^1}, \ldots, t_{\tau_i^{d_2}}) \\ &= \sum_{j=1}^{n_A} C_j \delta_{i,j} = C_i,\end{aligned} \tag{5.5}$$

which completes the proof. □

6. Arbitrary Topology Analysis-suitable T-splines

T-splines can be generalized to arbitrary topology, i.e., for which extraordinary vertices are allowed in the control grid, using subdivision surfaces [4,33] or patch-based methods [34]. The surface patches associated with the regular faces for both methods are identical, which are T-splines surfaces using the local mesh configuration (see Remark 6.1 as the description). However, the subdivision process generates elements near the extraordinary vertex which are comprised of an infinite sequence of piecewise polynomials. Thus, they are not backward compatible with NURBS which can be avoided in patch-based methods. In this section, we will show that under a simple condition, the T-spline blending functions are linear independent no matter how you construct the patches associated with extraordinary faces (subdivision or patch-based).

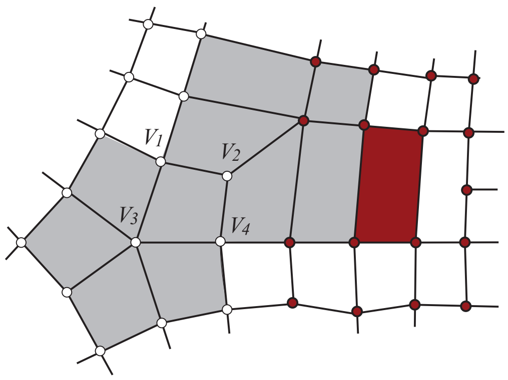


Fig. 6.1. Arbitrary topology analysis-suitable T-splines.

Definition 6.1. *An extraordinary face is a face which has at least one extraordinary point. The other faces are called regular faces.*

Definition 6.2. *An arbitrary topological T-mesh is analysis-suitable if*

- *Any extraordinary face at least has one vertex which is also a vertex of a regular face;*
- *No T-junctions extension intersects;*
- *Construction for regular patches is same as T-splines using the local mesh configuration.*

Remark 6.1. In the above conditions, condition one ensures that the linear independence property is independent from the definition of extraordinary face. For example in Figure 6.1, the face $V_1V_2V_3V_4$, V_2 and V_3 are extraordinary vertices but V_1 and V_4 are vertices of some regular faces. Condition two is required for analysis-suitable. Condition three states that the regular patches are constructed according to the T-spline rules with the local mesh configuration. Because the valences of the vertices for a regular face are all four. So we can figure out the local mesh configuration for any regular faces. For example, for the red face in Figure 6.1, the local mesh configuration (the blending functions for all the red points are not zero in this patch) is illustrated in Figure 6.1. Noticed that both the existing subdivision-based and patch-based methods satisfy this condition.

Theorem 6.1. *An analysis-suitable T-spline with extraordinary vertices has linear independent blending functions.*

Proof. Because the construction for regular patch is same as B-spline using the local mesh configuration and each control point at least contributes one regular patches according to condition one. If we consider the local linear independence for each regular patch, we can conclude the linear independence of all the blending functions for regular vertices and extraordinary vertices. □

Remark 6.2. In [34], the authors coupled collocated iso-geometric boundary element methods and unstructured analysis-suitable T-spline surfaces for linear elastostatic problems. However, the paper didn't provide the proof to assure that the blending functions for such unstructured T-splines are linearly independent. We can easily conclude that the blending functions defined in [34] are linear independent according to Theorem 6.1.

7. Conclusion

The present paper studies several new properties for analysis-suitable T-splines and shows that the blending functions for analysis-suitable T-splines are local linear independent. This property helps us to generalize the linear independence for arbitrary topological analysis-suitable T-splines and prove that the number of control points which contribute one Bézier element is $(d_1 + 1) \times (d_2 + 1)$ for bi-degree (d_1, d_2) analysis-suitable T-splines. We also show that we can label AS T-spline control points with the similar blossom formula as B-splines. All these issues are of great importance for IGA and also a foundation for the localized multi-resolution for analysis-suitable T-splines. Since the number of control points for each Bézier element is fixed and analysis-suitable T-splines have the same blossom formula as B-splines, so one of the main future work is how to construct an efficient evaluation algorithm for analysis-suitable T-splines .

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