Abstract

This paper develops new refinement rules for non-uniform Catmull-Clark surfaces that produce $G^1$ extraordinary points whose blending functions have a single local maximum. The method consists of designing an “eigen polyhedron” in $\mathbb{R}^2$ for each extraordinary point, and formulating refinement rules for which refinement of the eigen polyhedron reduces to a scale and translation. These refinement rules, when applied to a non-uniform Catmull-Clark control mesh in $\mathbb{R}^3$, yield a $G^1$ extraordinary point.

Keywords: Non-uniform, Catmull-Clark surfaces, NURBS

Concepts: Computing methodologies → Parametric curve and surface models;

1 Introduction

Several surface representations include both Catmull-Clark and NURBS surfaces as special cases [Sederberg et al. 1998; Müller et al. 2006; Sederberg et al. 2003a; Müller et al. 2010; Cashman et al. 2009; Cashman 2010; Kovacs et al. 2015]. A major aim of such surfaces is to facilitate the adoption of non-uniform subdivision surfaces by the CAD industry, where NURBS are widely used.

This paper solves a problem that vexes such surfaces: if knot intervals are different, the blending functions for extraordinary points can have two local maxima, as illustrated in Figure 1 (see also Figure 17 in [Kovacs et al. 2015]). Ugly blending functions unavoidably manifest themselves in real-world models, such as in Figure 2. The practical importance of this problem is elevated because such surfaces are gaining widespread commercial use, plus they play a central role in isogeometric analysis [Bazilevs et al. 2010].

The method in this paper creates $G^1$, wrinkle-free surfaces for any reasonable choice of knot intervals. Figure 1.f shows a blending function produced by our method.

The paper is laid out as follows. Sections 2 and 3 review refinement rules and prior art. Section 4 presents the notion of an eigen polyhedron for which refinement is simply a scale and a translation and Section 5 discusses how to design an eigen polyhedron that will specialize to NURBS in the valence four case, and to Catmull-Clark surfaces in the uniform case. Section 6 explains how to deduce refinement rules from a given eigen polyhedron, thus creating $G^1$ refinement. Section 7 concludes.

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Figures

Figure 1: Valence five extraordinary point blending function. The knot interval of red edges is 10, of black edges is 1.

Figure 2: Models using a blending function such as in Figure 1b-e.
We focus on the degree-three case, although the concepts should extend to other degrees. Our discussion assumes that all control-grid faces are four sided; if not initially, apply a single NURSS refinement [Sederberg et al. 1998]. We convey knot information by assigning a knot interval to each edge of the control grid [Sederberg et al. 1998; Sederberg et al. 2003b]. In Figure 3, \(d_i\) and \(e_i\) are knot intervals and can be any non-negative real numbers.

## 2 Refinement Rules

We define a CCNURBS to be a subdivision surface whose control mesh edges have knot intervals, and whose refinement equations specialize to Catmull-Clark surfaces when all knot intervals are the same and to NURBS when the valence is four and when, in Figure 3, \(d_i = d_i\), and \(e_i = e_i\), \(i = 0, 1, 2, 3\).

![Refinement Rules](image)

Refinement rules for CCNURBS amount to computing face, edge, and vertex points. Figure 3 shows labels for a valence \(n\) vertex \(V^k\) with neighboring face points \(F^k_i\) and edge points \(E^k_i\), \(i = 0, \ldots, n - 1\), with \(V^k, F^k, E^k \in \mathbb{R}^3\). The subscripts are modulo \(n\), the superscript \(k\) denotes refinement level, and \(d_i, d_i, e_i, e_i, \) and \(e_i\) are knot intervals. We will refer to edges of a control mesh that meet at an extraordinary point as spoke edges, and the image of a spoke edge on the limit surface as a spoke curve.

Defining a \((2n + 1) \times 3\) matrix

\[
P^k = [F^k_0, \ldots, F^k_{n-1}, E^k_0, \ldots, E^k_{n-1}, V^k]^T, \tag{1}
\]

refinement can be written \(P^{k+1} = MP^k\) where \(M^k\) is a \((2n + 1) \times (2n + 1)\) stochastic matrix whose elements \(M^k_{ij}\) are functions of knot intervals.

![Catmull-Clark subdivision rules](image)

For the CCNURBS formulation in [Sederberg et al. 1998], any edge can be assigned any knot interval. This freedom enables the creation of local creases and darts. However, even away from extraordinary points, the surfaces are only \(G^1\). Furthermore, the refinement matrices change at each iteration \((M^{k+1} \neq M^k)\), so eigenvalues and eigenvectors are not defined.

In this paper—as well as in [Sederberg et al. 2003a], [Cashman et al. 2009], and [Kovacs et al. 2015]—we require the knot intervals on opposing edges of every face to be identical, so in Figure 3, \(d_i = d_i\), and \(e_i = e_i\), \(i = 0, 1, \ldots\). This constraint causes no loss of design freedom, because local features can be added using T-junctions [Sederberg et al. 2003a]. Furthermore, two important advantages arise: the surfaces are \(C^2\) NURBS away from extraordinary points, and the refinement matrix is stationary—\(M^i = M, i = 1, \ldots, \infty\). In the remainder of this paper, we will thus simply write the refinement as

\[
P^{k+1} = MP^k. \tag{8}
\]
3 Prior Work

The ugly behavior illustrated in Figure 1 that often arises in CCNURBS limit surfaces has attracted research for over a decade. [Cashman et al. 2009] proposes a strategy of minimizing the difference in knot intervals for spoke edges at an extraordinary point. It does so by performing a preprocess of repeatedly splitting the largest knot interval at an extraordinary point if it is more than twice as large as the smallest knot interval at that extraordinary point, and then performing a local refinement to make all knot intervals the same. While this provides improvement in some cases, Figure 1.d shows that the strategy does not work universally.

[Kovacs et al. 2015] modifies the CCNURBS refinement rules in [Sederberg et al. 1998], yielding improved surfaces in some cases, although Figure 1.d shows that wrinkles remain.

Another method that has been explored for dealing with this problem is to modify the “patch” method in [Peters 2000] to handle differing knot intervals. This method produces one B´ezier patch (or subpatch) per extraordinary point and its nonuniform neighbors. While this assures smooth surfaces, the strategy does not work universally.

4 Eigen Polyhedra

In the examples in Figure 1, a single control point moved out of plane produces a limit surface that has two local maxima. We now look more closely at this curious phenomenon.

Figure 7 shows the spoke curves for the surface in Figure 1.b. Notice that the G2 extraordinary point lies between the two local maxima, and that the angle between two pairs of adjacent spoke curves appears to be zero. The reason this happens can be understood by examining the first neighborhood control grid faces after repeated refinement. Figure 8 shows how those faces become more narrow with each iteration, causing the angles between some edges to approach zero upon refinement. This observation suggests that the ugly behavior in Figure 1 and Figure 7 might be eliminated if refinement rules could be devised that avoid the collapsing of faces in Figure 8. This line of thinking led to the notion of eigen polyhedra, which are meshes that lie in the x–y plane. Denote by

\[ \hat{V}^k = [\hat{V}_0^k, \ldots, \hat{V}_{n-1}^k, \hat{V}_n^k, \hat{V}_n^k, \hat{V}_n^k]^T, \]

a \((2n + 1) \times 2\) matrix where \(\hat{V}_k^k, \hat{E}_k^k, \hat{F}_k^k \in \mathbb{R}^2\). We will use the expression “polyhedron \(\hat{P}^k\)” to mean the polyhedron in \(\mathbb{R}^2\) whose vertices are stored in a matrix \(\hat{P}^k\) and whose topology is illustrated in Figure 11.a.

Definition 1. Polyhedron \(\hat{P}^0\) is an eigen polyhedron of \(M\) if

\[ \hat{P}^1 = M\hat{P}^0 = \hat{P}^0 + \lambda \hat{P}^0 \]

where polyhedron \(\hat{P}^0\) has \(\hat{V}^0 = (0, 0)\), \(M\) is a \((2n + 1) \times (2n + 1)\) matrix whose rows sum to one, \(\lambda \in \mathbb{R}\), \(\hat{V}^0 \in \mathbb{R}^2\), and \(\mathbf{1}\) is a \((2n + 1) \times 1\) vector of 1’s.

In words, \(M\hat{P}^0\) produces a scale of \(\hat{P}^0\) by a factor of \(\lambda\), followed by a translation by \(\hat{V}^0\). As examples, we now describe eigen polyhedra for Catmull-Clark and for NURBS refinement matrices.

Catmull-Clark eigen polyhedron. A Catmull-Clark refinement matrix can be constructed from the refinement equations (2), (3), and (4). This \(M\) has an eigen polyhedron with vertices

\[ \hat{V}^0 = (0, 0) \]
\[ \hat{E}_i^0 = (\cos(\frac{2i}{n} \pi), \sin(\frac{2i}{n} \pi)), \]
\[ \hat{F}_i^0 = \gamma(\hat{E}_i^0 + \hat{E}_{i+1}^0) \]

where

\[ \gamma = \frac{4}{c_n + 1 + \sqrt{(c_n + 9)(c_n + 1)}}. \]
and \( c_n = \cos\left(\frac{2\pi}{3}\right) \). In this case,

\[
\lambda = \frac{1 + \gamma}{4\gamma} = \frac{5 + c_n + \sqrt{(c_n + 9)(c_n + 1)}}{16}
\]

and \( \hat{T}^0 = (0, 0) \).

![Figure 9: Catmull-Clark Eigen Polyhedra.](image)

**NURBS eigen polyhedron.** The NURBS refinement matrix, constructed from (5), (6), and (7), has an eigen polyhedron with vertices

\[
\hat{V}^0 = (0, 0)
\]

\[
\hat{E}_i^0 = \frac{2d_i + d_{i+2}}{3}(\cos\left(\frac{i}{2}\pi\right), \sin\left(\frac{i}{2}\pi\right)),
\]

\[
\hat{F}_i^0 = \hat{E}_i^0 + \hat{E}_{i+1}^0.
\]

In this case, \( \lambda = \frac{1}{3} \). Since \( \hat{V}^0 = (0, 0) \), \( \hat{T}^0 = \hat{V}^1 \) is obtained by substituting (16), (17) and (18) into (7) to get

\[
\hat{t}^0 = \left(\frac{d_0 - d_2}{6}, \frac{d_1 - d_3}{6}\right).
\]

![Figure 10: Examples of NURBS Eigen Polyhedra.](image)

**Properties of eigen polyhedra.** Since the rows of \( M \) each sum to 1, \( M \hat{t}^0 = \hat{t}^0 \). So, if \( \hat{P}^0 \) and \( M \) satisfy (10),

\[
\hat{P}^2 = M\hat{P}^1 = M(\lambda\hat{P}^0 + \hat{I}^0) = \lambda M\hat{P}^0 + \hat{I}^0
\]

\[
= \lambda^2\hat{P}^0 + (1 + \lambda)\hat{I}^0
\]

By induction,

\[
\hat{P}^k = \lambda^k\hat{P}^0 + (1 + \lambda^1 + \ldots + \lambda^{k-1})\hat{I}^0.
\]

From (9), the last row of \( \hat{P}^k \) is \( \hat{V}^k \). Since \( \hat{V}^0 = (0, 0) \),

\[
\hat{V}^k = (1 + \lambda^1 + \ldots + \lambda^{k-1})\hat{I}^0 = (1 + \lambda^1 + \ldots + \lambda^{k-1})\hat{V}^1
\]

and

\[
\hat{P}^k = \lambda^k\hat{P}^0 + \hat{I}^0.
\]

Denoting \( \hat{T}^k = \hat{V}^{k+1} - \hat{V}^k \), we can obtain from (22) by straightforward algebraic manipulation,

\[
\hat{P}^{k+1} = M\hat{P}^k = [\lambda(\hat{P}^k - \hat{V}^k\hat{I}^k) + \hat{I}\hat{V}^k] + \hat{I}\hat{t}^k.
\]

The geometric meaning of (23) is shown in Figure 11: If \( \hat{P}^0 \) is an eigen polyhedron of \( M \), then \( M\hat{P}^0 \) is equivalent to a scale by \( \lambda \) about point \( \hat{V}^k \) followed by a translation by \( \hat{T}^k \). Note that \( \hat{P}^k - \hat{V}^k\hat{I}^k, k = 1, 2, \ldots \), are eigen polyhedra of \( M \). It is straightforward to prove that \( \hat{T}^{k+1} = \lambda\hat{T}^{k} = \lambda\hat{T}^{0} \).

![Figure 11: Relationship between \( \hat{P}^k \) and \( \hat{P}^{k+1} \) in (23).](image)

Obviously, if polyhedron \( \hat{P}^0 \) is an eigen polyhedron of \( M \), polyhedron \( \hat{P}^0 \) will not exhibit the non-uniform scaling behavior in Figure 8 when it is repeatedly refined by \( M \).

If we subtract \( \hat{I}\hat{V}^1 \) from both sides of (10), we obtain

\[
M(\hat{P}^0 - \hat{I}\hat{V}^1) = \lambda(\hat{P}^0 - \hat{I}\hat{V}^1),
\]

so the two columns of \( \hat{P}^0 - \hat{I}\hat{V}^1 \) are eigenvectors of \( M \), each of whose eigenvalue is \( \lambda \). This suggests that \( M \) will have an eigen polyhedron if \( M \) has two identical eigen values. Unfortunately, other than the Catmull-Clark and NURBS special cases, generic CCNURBS refinement matrices do not have two identical eigen values [Kovacs et al. 2015].

Eigen polyhedra are similar to what [Ball and Storzy 1988] calls a natural configuration, and to the control grid of a characteristic map in [Reif 1995].

The main contribution of this paper is to use the notion of an eigen polyhedron to create CCNURBS refinement rules for which \( M \) has an eigen polyhedron, and thus has two identical eigenvalues.

In Section 5 we discuss how to design an eigen polyhedron \( \hat{P}^0 \) a priori, for any set of knot intervals and any valence—without first knowing \( M \). We then show in Section 6 how to construct a refinement matrix for which \( \hat{P}^0 \) is an eigen polyhedron. This procedure creates CCNURBS refinement rules that produce a \( G^1 \) surface and whose blending functions have a single local maximum.

## 5 Creating an Eigen Polyhedron

We now explain how to design an eigen polyhedron for a valence-
extraordinary point whose edges have knot interval values
We set \( \hat{\theta}_0 = (0, 0) \), and express the points \( \hat{E}_i^0 \) and \( \hat{F}_i^0 \), \( i = 0, \ldots, n - 1 \) as equations that are functions of \( n \) and \( \hat{d}_0, \ldots, \hat{d}_{n-1} \). We have some freedom in creating those equations, the only rigid requirements being that they must reduce to (11)–(13) when \( d_0 = \ldots = d_{n-1} \) (the Catmull-Clark case) and to (16)–(18) when \( n = 4 \) (the NURBS case).

We first consider the angles between spoke edges in the eigen polyhedron, \( \theta_i = \angle \hat{E}_i^0 \hat{V}_i^0 \hat{E}_{i+1}^0 \). Since all angles are the same in Catmull-Clark and NURBS eigen polyhedra, i.e.,

\[
\theta_i = \frac{2\pi}{n}, \quad i = 0, \ldots, n - 1,
\]

we propose using this equal-angle formula for all eigen polyhedra.

The length of the spoke edges for NURBS eigen polyhedra are functions of knot intervals. We similarly define the length \( l_i \) of spoke edge \( \hat{V}_j^0 \hat{E}_i^0 \) to be

\[
l_i = \frac{d_i + d_j^+ + d_j^-}{3}
\]

where

\[
d_i^+ = \sum_{j=i}^{i+n-1} d_j \cos(\theta_{i,j}), \quad \text{if } \cos(\theta_{i,j}) > 0
\]

\[
d_i^- = -\sum_{j=i}^{i+n-1} d_j \cos(\theta_{i,j}), \quad \text{if } \cos(\theta_{i,j}) < 0
\]

\[
\theta_{i,j} = \sum_{k=i}^{j-1} \theta_k, \quad i < j
\]

These lengths specialize to both NURBS and Catmull-Clark eigen polyhedron spoke-edge lengths.

We now define \( \hat{E}_i^0 = (l_i, 0) \) to lie on the x-axis, so \( \theta_{0,i} \) is the angle between spoke edge \( \hat{V}_0^0 \hat{E}_i^0 \) and the positive x-axis. Then,

\[
\hat{E}_i^0 = l_i(\cos(\theta_{0,i}), \sin(\theta_{0,i})).
\]

The points \( \hat{E}_i^0 \) are obtained using equation (13). The eigen polyhedron for the surface in Figure 1.d is shown in Figure 12.

6 Deriving \( M \) from an Eigen Polyhedron

Denoting a polyhedron created using the procedure in Section 5 by \( \hat{P}_d^0 \), we now discuss how to create a refinement matrix \( M \) for which \( \hat{P}_d^0 \) is an eigen polyhedron. We have three requirements:

1. \( M \) must satisfy (10)

2. \( M \) must specialize to Catmull-Clark refinement if the knot intervals are all equal

3. \( M \) must specialize to NURBS refinement if the valence is four and the knot intervals are as in Figure 5.

Since the rows of \( M \) express the face, edge, and vertex point computations, defining face, edge, and vertex point rules that satisfy these three requirements is equivalent to creating the desired \( M \).

Equation (10) involves \( \lambda \) and \( \hat{T}_0^0 \), so we begin by finding equations for \( \lambda \) and \( \hat{T}_0^0 \) that specialize to the NURBS and Catmull-Clark cases. Equation (15) for computing \( \lambda \) meets this requirement.

**Vertex point rule**

Observe from (21) that \( \hat{T}_0^0 = \hat{V}_1 \). This means that our equation for \( \hat{T}_0^0 \) must not only specialize to \( (0, 0) \) in the Catmull-Clark case and to (19) in the NURBS case, but that it also will serve as our vertex point equation. These requirements are met by using (27) as the vertex point equation. (This is a minor modification of the vertex point rule from [Sederberg et al. 1998].)

\[
\hat{V}_h^{k+1} = \frac{n - 3}{n} \hat{V}_k^0 + \frac{3}{n} \sum_{i=1}^{n} (m_i H_i^h + f_i G_i^h) / \sum_{i=1}^{n} (m_i + f_i)
\]

where

\[
H_i^h = g_i \hat{E}_i^0 + (1 - g_i) \hat{V}_k^0,
\]

\[
G_i^h = g_i (1 - g_{i+1}) \hat{E}_i^0 + g_{i+1} (1 - g_i) \hat{E}_{i+1}^0 + g_i g_{i+1} \hat{E}_i^0 + (1 - g_i) (1 - g_{i+1}) \hat{V}_k^0,
\]

\[
g_i = \frac{d_{i-2} + d_{i+2} + d_i}{d_{i-2} + d_{i+2} + 4d_i},
\]

\[
f_i = \prod_{j=1, j \neq i, i+1} d_j^+.
\]

Since \( \hat{V}_0^0 = (0, 0) \),

\[
\hat{T}_0^0 = \hat{V}_1 = \frac{3}{n} \sum_{i=1}^{n} (m_i H_i^0 + f_i G_i^0) / \sum_{i=1}^{n} (m_i + f_i)
\]

Labels are illustrated in Figure 13.
Face point rule

From the definition of an eigen polyhedron (10), we have

$$\hat{V}^1 = \hat{V}^1 + \lambda \hat{E}_0^0. \tag{29}$$

At this stage of the process we know $\hat{V}^1$, $\lambda$, and $\hat{E}_0^0$, so the Cartesian coordinates of $\hat{F}_i^1$ can be computed.

To devise a face point rule, we create an equation for $\hat{F}_i^1$ in terms of the four vertices of its generating face (see Figure 14.a). A reasonable way to do this is using a bi-linear equation:

$$F_i^1 = (1 - \alpha_{i,1})(1 - \alpha_{i,2})\hat{V}^0 + \alpha_{i,1}(1 - \alpha_{i,2})\hat{E}^0_i + (1 - \alpha_{i,1})\alpha_{i,2}E^0_{i+1} + \alpha_{i,1}\alpha_{i,2}\hat{F}_0^1. \tag{30}$$

The two bilinear equations in (30) can be solved using the following method from [Floater 2015]. Denote $v_1 = \hat{F}_i^1 - \hat{V}^0$, $v_2 = \hat{F}_i^1 - \hat{E}^0_i$, $v_3 = \hat{F}_i^1 - \hat{E}_0^0$, $v_4 = \hat{F}_i^1 - E^0_{i+1}$. And let $S_i = \frac{1}{2}v_i \times v_{i+1}$, $T_i = \frac{1}{2}v_{i-1} \times v_i$. Then

$$\alpha_{i,1} = \frac{2S_4}{2S_4 + T_1 + T_2 + \sqrt{D}} \tag{31}$$
$$\alpha_{i,2} = \frac{2S_1}{2S_1 + T_1 - T_2 + \sqrt{D}} \tag{32}$$

where $D = T_1^2 + T_2^2 + 2S_1S_3 + 2S_3S_4$. Solving this for numerical values of $\alpha_{i,1}$ and $\alpha_{i,2}$, the face point rule is (30). It can be shown that (30) satisfies (10) and specializes to NURBS and Catmull-Clark.

Edge point rule

From the definition of an eigen polyhedron, we have

$$\hat{E}^1 = \hat{V}^1 + \lambda(\hat{E}^0 - \hat{V}^0) = \hat{V}^1 + \lambda \hat{E}^0_i. \tag{33}$$

Since we know $\hat{V}^1$, $\lambda$, and $\hat{E}^0_i$, the Cartesian coordinates of $\hat{F}_i^1$ can be computed.

Edge points are a linear combination of the six vertices of the two adjacent faces for the associated edge (see Figure 14.b). Inspired by the non-uniform B-spline refinement rules, we first denote

$$P_{i,1} = (1 - \alpha_{i-1,1})\hat{V}^0 + \alpha_{i-1,1}\hat{E}^0_{i-1}; \tag{34}$$
$$P_{i,2} = (1 - \alpha_{i,2})\hat{V}^0 + \alpha_{i,2}\hat{E}^0_{i+1}; \tag{35}$$
$$P_{i,3} = (1 - \alpha_{i-1,1})\hat{E}^0_i + \alpha_{i-1,1}\hat{F}_0^1; \tag{36}$$
$$P_{i,4} = (1 - \alpha_{i,2})\hat{E}^0_i + \alpha_{i,2}\hat{F}_0^1. \tag{37}$$

then the edge point is computed via the following equation

$$\hat{E}_i^1 = (1 - \beta_{i,1})(1 - \beta_{i,2})\frac{1}{2}P_{i,1} + \frac{1}{2}\hat{V}^0 + \frac{1}{2}(1 - \beta_{i,1})\frac{1}{2}P_{i,2} + \frac{1}{2}\hat{E}^0_i + \frac{1}{2}(1 - \beta_{i,2})\frac{1}{2}P_{i,4} + \frac{1}{2}\hat{F}_0^1. \tag{38}$$

It is easy to see that $\hat{E}_i^1$ is also a bi-linear combination of four points $\frac{1}{2}P_{i,1} + \frac{1}{2}\hat{V}^0$, $\frac{1}{2}P_{i,2} + \frac{1}{2}\hat{E}^0_i$, and $\frac{1}{2}P_{i,4} + \frac{1}{2}\hat{F}_0^1$ with coefficients $\beta_{i,1}$ and $\beta_{i,2}$. Thus, we can solve the coefficients using the same method as above. After solving the $\beta_{i,1}$ and $\beta_{i,2}$, the edge point rule is defined via equation (38). It can be shown that (38) satisfies (10) and specializes to NURBS and Catmull-Clark.

6.1 $M$ in the Overall Mesh Refinement Process

While the $M$ we have just described was created in reference to a special planar polyhedron, applying $M$ to arbitrary control meshes in $\mathbb{R}^3$ yields excellent results. We now describe how $M$ fits into the overall mesh refinement process. The creation of $M$ assumes that extraordinary points are separated by more than one face. If not, perform an initial refinement using any CCNURBS formulation.

In Figure 15, the gray mesh is obtained after a single CCNURBS refinement, so the two extraordinary points are not adjacent. Refining the gray mesh yields the mesh whose vertices are either red or green. In Figure 15, the red control points are obtained by applying the appropriate $M$ to the 1-neighborhood of each respective extraordinary point. In general, each extraordinary point has its own refinement matrix, since knot intervals will generally not be the same for each extraordinary point.

The green vertices are obtained with conventional NURBS refinement rules using face, edge, and vertex point equations (5)–(7).

7 Results and Discussion

Figure 1.f and Figures 16—18 show several examples of blending functions produced by our method. Notice that the angles between spoke edges are non-zero using our method.

Figure 16: Valence 6, with knot intervals 1, 10, 10, 1, 1, 10.
Improved blending functions lead to improved models. Figures 20 and 21 show the result of applying our refinement method to the model in Figure 2.a and Figure 22 shows our method applied to the guitar neck model from Figure 2.b.

We did not study valences greater than eight because high valence extraordinary points are rarely used in practice. However, the study of how well the method works for higher valences is of mathematical interest and would be worth exploring.

In the examples shown, the extraordinary point blending functions have a single local maximum and are $G^1$. A good problem for future study is to prove whether this is true for any choice of knot intervals and valance. We do not have an analytical proof of this, but we did test a million different extraordinary points with randomly generated knot intervals $\in [10^{-6}, 1]$ and valences of $n = 3, 5, 6, 7$ and 8 and found that, in every case, the blending function had a single local maximum and was $G^1$.

To verify the existence of a single local maximum, we performed five levels of refinement on each test case and confirmed that the resulting control mesh had a single vertex whose $z$-coordinate was larger than all neighbors. We believe that five refinements is sufficient because we have observed that the twin peak phenomenon usually manifests itself in the control grid of the second refinement.

An extraordinary point is tangent continuous if the characteristic ring is regular and injective [Peters and Reif 2008]. To verify regularity and injectivity, we need to examine the characteristic map defined by three rings of control points. To determine those $12n + 1$ control points, form the $(12n+1) \times (12n+1)$ refinement matrix for those points, and compute the second and third eigenvectors. The
In all million test cases, the first ring of control points for the characteristic map is a translation of the eigen polyhedron. In other words, the value of \( \lambda \) used in creating the eigen polyhedron always turned out to be the second and third eigenvalue of \( M \). We have no mathematical proof that this will always be the case, and suggest that this is an interesting problem for future research. There exists some related literature on this general topic under the name “inverse eigenvalue problem.”

In Section 5, equation (25) defines the \( \theta_i \) for our eigen polyhedra to all be equal. We experimented with using different values of \( \theta_i \) for cases where \( n \neq 4 \) and the \( d_i \) are not equal and found that minor changes in surface quality can occur. For example, if all knot intervals are the same except for one knot interval of zero, we observed slightly improved surface quality when angles next to the zero-knot-interval edge are the same except for one knot interval of zero, we minor changes in surface quality can occur. For example, if all knot intervals are the same except for one knot interval of zero, we observed slightly improved surface quality when angles next to the zero-knot-interval edge are the same except for one knot interval of zero, we minor changes in surface quality can occur.

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Another topic that invites future research is zero knot intervals. The algorithm requires a modification to handle a zero knot interval: splitting a zero knot interval produces two zero knot intervals, so one should be removed. This is more complicated if several spoke edges have zero knot intervals because two adjacent zero knot intervals create a crease so there is not a unique tangent plane. In such cases, our method cannot be applied to the entire one-neighborhood, but could be applied piecewise to each domain bounded by creased spoke edges. There are numerous such cases to consider, and more theory to work out.

A closed-form equation for the Cartesian coordinates of the extraordinary point, as well as for tangent vectors for the spoke curves, can be developed from the eigen vectors. This would be helpful in performing exact evaluation of extraordinary limit points, and in developing an improved patching solution for non-uniform Catmull-Clark surfaces.

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References


