AS++ T-splines: Linear independence and Approximation

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Abstract

In this paper, we define analysis-suitable++ (AS++) T-splines which include analysis-suitable (AS) T-splines as a special case and maintain all the good mathematical properties as AS T-splines. We prove that AS++ T-splines are always linear independent regardless of the knot values and show that the classical construction of the dual basis for tensor-product B-splines and AS T-splines can be generalized to AS++ T-spline spaces. We also discuss how all of these issues pave the way to a mathematical theory for AS++ T-splines.

Keywords: T-splines, linear independence, isogeometric analysis, analysis-suitable T-splines

1. Introduction

T-splines [1, 2] have emerged as an important technology for computer aided geometric design (CAGD) [3, 4, 5] and iso-geometric analysis (IGA) [6, 7, 8]. Among all the basic properties, linear independence is one of the most priori ones regarding to IGA because the analysis community requires bases that are assured to be linearly independent. For general T-splines, [9] discovers an example of a T-spline with linearly dependent blending functions. Later, analysis-suitable T-splines (for short, AS T-splines) [10, 5, 11, 12, 13], a mildly topological restricted subset of T-splines, are introduced. The members of the class of T-splines are always linear independent for any knot values.

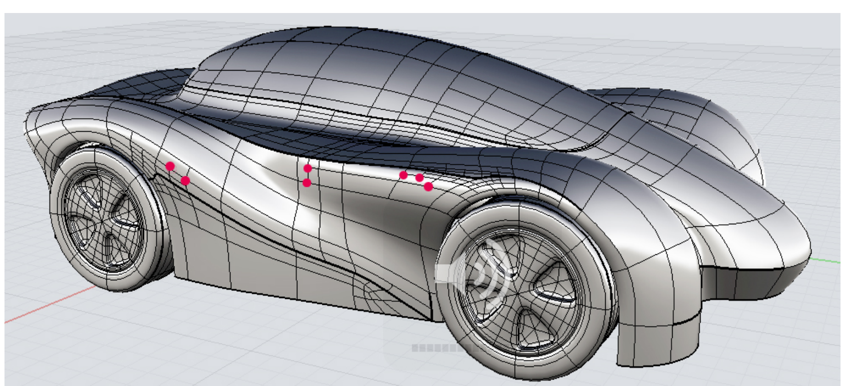

Figure 1: A car model from T-spline plugin.

However, linear independence is not required (although desirable) for most CAGD applications. In the commercial software: Autodesk T-spline plugin for Rhinocers [14], they still use the general T-

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splines with local refinement algorithm in [2]. Thus, the models created from the software in general are not AS T-splines. For example, the car model in the Figure 1 contains many T-junctions (some of them are marked with red) which don't satisfy the analysis-suitable constraints. In order to apply isogeometric analysis on the models, we need a preprocessing of conversion into AS T-splines. So a very natural question arises that can we improve the topological constraints for AS T-splines, such that they still maintain all the good mathematical properties as AS T-splines.

This paper gives a positive answer to this problem and identifies a class of T-splines that include AS T-splines as a special case, whose blending functions are guaranteed to be linearly independent. Like all general T-splines, the members of the class of T-splines provide watertight models, are NURBS compatible, obey the convex hull property, and are affine invariant—all the important properties for isogeometric analysis. Furthermore, such member of T-spline spaces are closed under all the existing local refinement algorithms, i.e., applying any existing local refinement algorithms in [2, 5] on the member of T-splines will produces the T-splines which are still in this class. This property and a new optimized local refinement algorithms with less propagation than those in [2, 5] will be given in the forthcoming paper [15]. The class of T-splines can also be characterized via piecewise polynomials [16]. For these reasons, we will refer to them as *AS++ T-splines*. In the present paper, we only focus on the linear independence and dual basis.

For the linear independence, all the existing methods in [10, 12, 13, 17] require that the T-mesh is an AS T-mesh. Thus we need to introduce new tools to prove the linear independence for AS++ T-splines. Given a set of blending functions $B_i(\xi), i = 1, 2, \dots, n$ and a set of linear functionals $\lambda_j(\cdot)$, we can form a *functional matrix* $\mathbb{D} = (d_{i,j})$, where $d_{i,j} = \lambda_i(B_j)$. The given linear functionals $\lambda_j(\cdot)$ are dual to these blending functions if and only if the functional matrix is an identical matrix. It is obvious that the existence of the dual basis can conclude the linear independence of the blending functions. It is easy to see that if there exist some functionals such that the functional matrix is an upper triangular matrix and the diagonal elements are all non zeros, then these blending functions are also linear independent. Based on this observation, we introduce *semi-dual basis*. A set of functionals $\lambda_j(\cdot)$ are said to be a set of semi-dual basis for the blending functions $B_i(\xi), i = 1, 2, \dots, n$, if there exists an order of the indices i_j such that $\lambda_{i_j}(B_{i_j}(\xi)) = 1$ and $\lambda_{i_j}(B_{i_k}(\xi)) = 0, j > k$. The existence of the semi-dual basis can guarantee that the blending functions are linearly independent. And then, we find out a more general class of T-meshes and show that the bi-cubic T-splines defined on such T-meshes have semi-dual basis, which improves the papers [10, 12]. Based on the semi-dual basis, we can also construct the dual basis for the T-splines. The existence of the dual basis provides the T-spline space with a rich mathematical structure and can be used to define a projector. This projector serves as a key ingredient for the analysis of the approximation properties for AS++ T-spline spaces. This paper is written specifically in terms of bi-cubic T-splines, although the concepts should extend to any degree.

The rest of the paper is structured as follows. In Section 2, we recall some basic notations for index T-meshes. Then we introduce the semi-dual T-meshes in Section 3. In Section 4, we prove that the bi-cubic AS++ T-meshes are semi-dual T-meshes and use it to prove the linear independence. The dual basis and approximation for AS++ T-spline spaces are also discussed in this section. The last section is the conclusion and future work.

2. Index T-meshes

An index T-mesh [8] T for a bi-cubic T-spline is a connection of all the elements of a rectangular partition of the index domain $[-1, c+2] \times [-1, r+2]$, where all rectangle corners (or vertices) have integer coordinates. Three types of elements are

- Vertex: vertex of a rectangle, denoted as (σ_i, τ_i) or $\{\sigma_i\} \times \{\tau_i\}$.
- Edge: a line segment connecting two vertices in the T-mesh and no other vertices lying in the interior. In the following, we will denote $[\sigma_j, \sigma_k] \times \{\tau_i\}$ for a horizontal edge or a set of connected horizontal edges. Similarly, we denote $\{\sigma_i\} \times [\tau_j, \tau_k]$ as a vertical edge or a set of connected vertical edges. Also both open and closed edges $[\sigma_j, \sigma_k] \times \{\tau_i\}$ and $(\sigma_j, \sigma_k) \times \{\tau_i\}$ are considered to be the same edges in the following proofs.
- Face: a rectangle where no other edges and vertices in the interior, denoted as $[\sigma_i, \sigma_j] \times [\tau_k, \tau_l]$ or $(\sigma_i, \sigma_j) \times (\tau_k, \tau_l)$, where the second one is for an open face.

The valence of a vertex is the number of edges which contain the vertex. For the interior vertices, we don't allow L-junctions or isolated vertices, while allow I-junctions, valence three (called T-junctions) and valence four vertices.

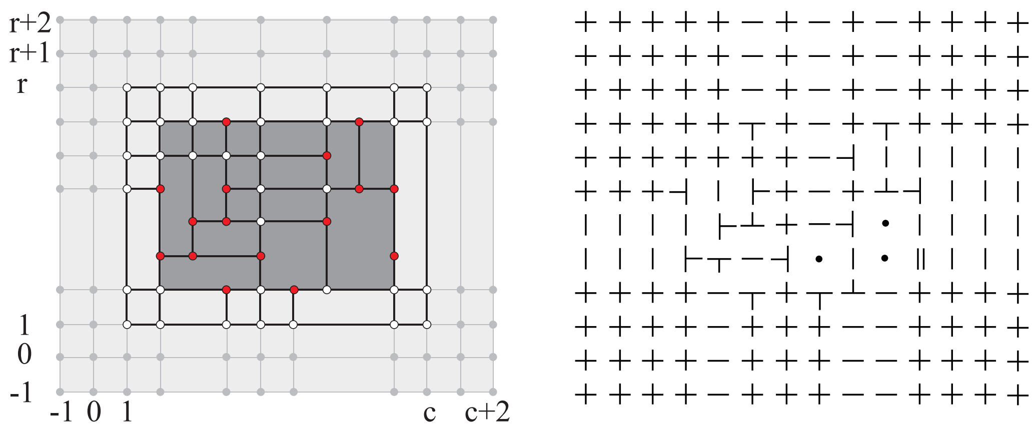


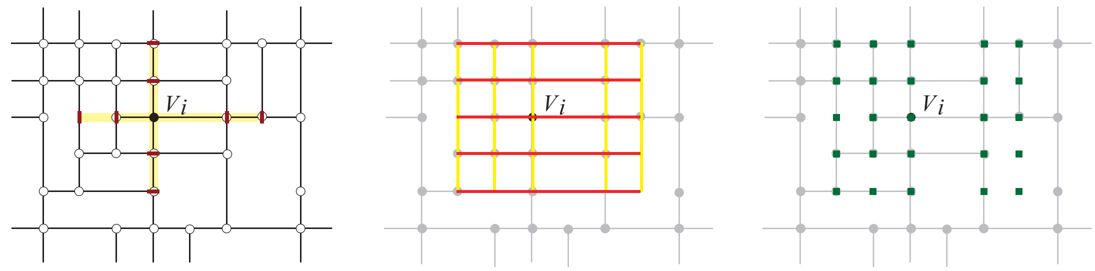
Figure 2: An example T-mesh and the associated symbolic T-mesh.

A symbolic T-mesh [10] is created from a T-mesh T by assigning a symbol in Table 1 to each vertex in a tensor product mesh formed from the index coordinates. The symbol is chosen to match the mesh topology of T. The symbolic T-mesh corresponding to the left T-mesh in the Figure 2 is shown on the right of Figure 2.

Table 1: Definition of possible symbols in a symbolic T-mesh

Symbol	Correspondence with T
$+$	Valence 4 vertex, corner vertex, or valence 3 boundary vertex in T
$\vdash$, $\dashv$, $\perp$, $\top$	Oriented valence three vertex in T
$\Vert$, $=$	Oriented I-junction with two incident edges
$\vert$, $-$	Vertical or horizontal edge in T
$\cdot$	No corresponding vertex or edge in T

For the i-th vertex $\mathbf{V}_i = (\sigma_i, \tau_i)$ in the rectangle $[1, c] \times [1, r]$, we define a local index vector $\vec{\sigma_i} \times \vec{\tau_i}$. From the vertex, we shoot a ray in both directions traversing the T-mesh and collect a set of knot indices $[\sigma_i^0, \dots, \sigma_i^4]$ and $[\tau_i^0, \dots, \tau_i^4]$ in both directions such that $\sigma_i^2 = \sigma_i$ and $\tau_i^2 = \tau_i$, as shown in



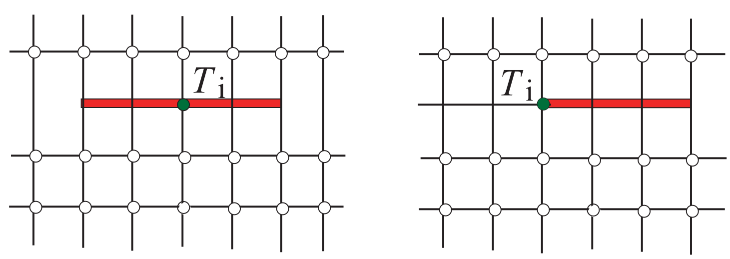
a. $\vec{\sigma_i} \times \vec{\tau_i}$ $\quad$ $SK(\mathbf{V}_i)$ $\quad$ $VK(\mathbf{V}_i)$

Figure 3: Define the local index vector $\vec{\sigma_i} \times \vec{\tau_i}$, the skeleton $SK(\mathbf{V}_i)$ and the vertex of the skeleton $VK(\mathbf{V}_i)$ for the T-mesh.

Figure 3 a. Let $hSK(\mathbf{V}_i)$ be the union of all the edges $[\sigma_i^0, \sigma_i^4] \times \{\tau_i^j\}$, $j = 0, 1, \ldots, 4$ and $vSK(\mathbf{V}_i)$ is the union of all the edges $\{\sigma_i^j\} \times [\tau_i^0, \tau_i^4]$, $j = 0, 1, \ldots, 4$. Denote $SK(\mathbf{V}_i) = hSK(\mathbf{V}_i) \cup vSK(\mathbf{V}_i)$ and $VK(\mathbf{V}_i) = \{(\sigma_i^j, \tau_i^k), j, k = 0, 1, \ldots, 4\}$. See Figure 3 as an illustration.

2.1. Extension and extended T-mesh

Extension is a very important concept for defining analysis-suitable T-splines. Referring to Figure 4, for a T-junction $\mathbf{T}_i$ of type of $\vdash$, the face extension $ext_a^f(\mathbf{T}_i)$ with an integer a is a line segment $[\sigma_i^{2-a}, \sigma_i^2] \times \{\tau_i^2\}$. If $\mathbf{T}_i$ is type of $\dashv$, then $ext_a^f(\mathbf{T}_i) = [\sigma_i^2, \sigma_i^{2+a}] \times \{\tau_i^2\}$. And if $\mathbf{T}_i$ is type of $\|$, then $ext_a^f(\mathbf{T}_i) = [\sigma_i^{2-a}, \sigma_i^{2+a}] \times \{\tau_i^2\}$. Similarly, we can define the face extension for T-junctions of type $\perp$ or $\top$ and $=$.



a. $\mathrm{ext}_2^f(\mathrm{T_i})$ $\quad$ b. $\mathrm{ext}_3^f(\mathrm{T_i})$

Figure 4: The face extension of a T-junction.

An extended T-mesh for a T-mesh T is a new T-mesh from the *extended T-mesh set* $ext(\mathsf{T})$, where

$$ext(\mathsf{T}) = \{\mathsf{T}_1 \| \mathsf{T}_1 = \bigcup_{\mathbf{T}_i \in \mathsf{T}} ext_{a_i}^f(\mathbf{T}_i) \bigcup \mathsf{T}, a_i \geq 0 \text{ is an integer associated with the T-junction } \mathbf{T}_i \ \}.$$

The edges in the extended T-mesh but not in the original T-mesh are called the *extended edges*. Two T-junction extensions maybe overlap. Thus, the *multiplicity* of an extended edge is the number of T-junction face extensions that contain the extended edge, which is two if the edge belongs to two overlapping extensions and is one for the other cases. For two extended T-meshes $\mathsf{T}_1, \mathsf{T}_2 \in ext(\mathsf{T})$, we said $\mathsf{T}_1 = \mathsf{T}_2$ if and only if all the extended edges are same and the multiplicities for the corresponding extended edges are also the same. Among all the extended T-meshes, there are two special extended T-meshes T_{ext} and T_{elem}, where

$$\mathsf{T}_{ext} = \bigcup_{\mathbf{T}_i \in \mathsf{T}} ext_2^f(\mathbf{T}_i) \bigcup \mathsf{T}, \qquad \mathsf{T}_{elem} = \bigcup_{\mathbf{V}_i \in \mathsf{T}} SK(\mathbf{V}_i) \bigcup \mathsf{T}.$$

Definition 2.1. *A T-mesh is called an analysis-suitable++ T-mesh (for short, AS++ T-mesh) if and only if:*

1. *For any two T-junctions $\mathbf{T}_i$, $\mathbf{T}_j$ which extensions are not parallel, denote $V = ext_2^f(\mathbf{T}_i) \cap ext_2^f(\mathbf{T}_j)$, then either $ext_2^f(\mathbf{T}_i) \cap ext_2^f(\mathbf{T}_j) = \varnothing$ (no V exists) or for any $\mathbf{V}_i$, $V \notin VK(\mathbf{V}_i)$;*
2. $\mathsf{T}_{ext} = \mathsf{T}_{elem}$.

The Lemma 3.2 (a) and (b) in [13] state that AS T-meshes satisfy the requirement of AS++ T-meshes, i.e., AS T-meshes are always AS++ T-meshes. On the contrary, AS++ T-meshes are not always AS T-meshes. For example, the T-mesh in Figure 5 a. is an AS++ T-mesh but is not an AS T-mesh because the extensions of two red T-junctions intersect. We can check that any two non-parallel face extensions don't intersect and $\mathsf{T}_{ext} = \mathsf{T}_{elem}$. The T-mesh in Figure 5 b. is also an AS++ T-mesh but is not an AS T-mesh. In this example, there are many T-junction intersections including two face extension intersection at the vertex V. But we can check that for any vertices $\mathbf{V}_i$, $V \notin VK(\mathbf{V}_i)$, which states that the T-mesh is an AS++ T-mesh.

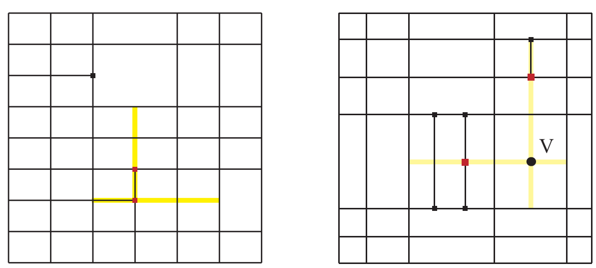
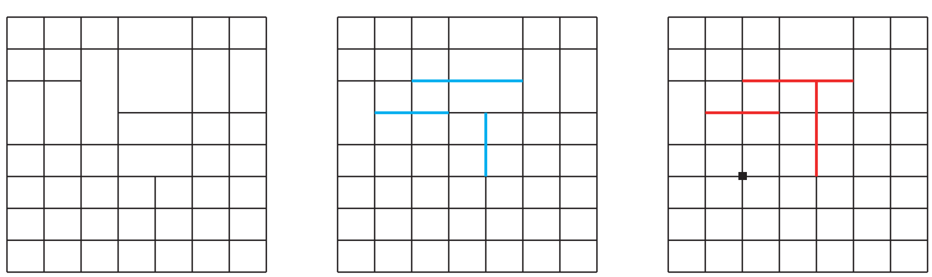


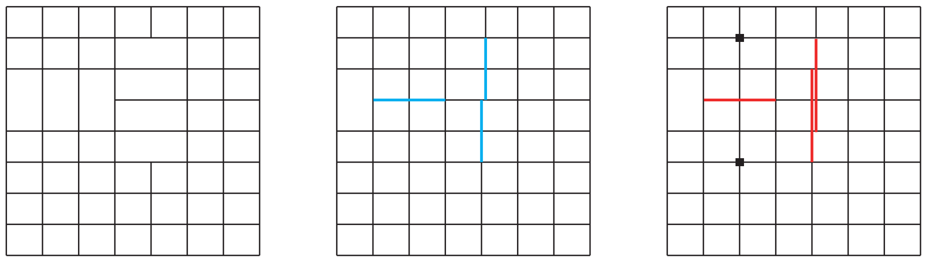
Figure 5: Two example AS++ T-meshes which are both not AS T-meshes. In this figures, the yellow edges are the extensions of some T-junctions which intersect. Thus, the two T-meshes are not AS T-meshes.



a.A T-mesh T b. The T-mesh T_{ext} c. The T-mesh T_{elem}

Figure 6: The T-mesh T in a is not an AS++ T-meshes because T_{ext} (b) is different from T_{elem} (c).

The T-mesh in the figure 6 a. is not an AS++ T-mesh because T_{ext} (Figure 6 b) is different from T_{elem} (Figure 6 c). And the T-mesh in the figure 7 a is also not an AS++ T-mesh because the



a.A T-mesh T b. The T-mesh T_{ext} c. The T-mesh T_{elem}

Figure 7: The T-mesh in a is also not an AS++ T-mesh because the multiplicities for some extended edges in T_{ext} (b) and T_{elem} (c) are different. The multiplicity for all the extended edges in T_{ext} is one but the multiplicity for two extended edges in T_{elem} is two.

multiplicities for some extended edges in T_{ext} (Figure 7 b) and T_{elem} (Figure 7 c) are different. In order to make the figure to be easily understood, we move one of the T-junctions a little bit such that we can count the multiplicities easier.

Remark 2.2. *Both conditions for the AS++ T-splines are related with the characterization of the T-spline space. As we have discussed for AS T-spline space in [11], the T-spline space has a rich connection with the extended spline space $\mathcal{T}_{ext}$ if $\mathsf{T}_{ext} = \mathsf{T}_{elem}$.*

3. Semi-Dual T-meshes

This section introduces the notion of semi-dual T-meshes.

Definition 3.1. *Two index vectors $\vec{\sigma_i} = (\sigma_i^0, \ldots, \sigma_i^4)$, $\vec{\sigma_j} = (\sigma_j^0, \ldots, \sigma_j^4)$, $i \neq j$, are called* ***overlap*** *[12, 13] if*

$$\forall k \in \{\sigma_i^0, \ldots, \sigma_i^4\} \cap \{k | \sigma_j^0 \leq k \leq \sigma_j^4\}, k \in \{\sigma_j^0, \ldots, \sigma_j^4\}$$
$$\forall k \in \{\sigma_j^0, \ldots, \sigma_j^4\} \cap \{k | \sigma_i^0 \leq k \leq \sigma_i^4\}, k \in \{\sigma_i^0, \ldots, \sigma_i^4\}$$

Definition 3.2. *An index vector $\vec{\sigma_i} = (\sigma_i^0, \ldots, \sigma_i^4)$ is said to* ***semi-intersect*** *another index vector $\vec{\sigma_j} = (\sigma_j^0, \ldots, \sigma_j^4)$, $i \neq j$, if $\exists k \in \{\sigma_i^0, \ldots, \sigma_i^4\} \cap \{k | \sigma_j^1 \leq k \leq \sigma_j^3\}, k \notin \{\sigma_j^1, \sigma_j^2, \sigma_j^3\}$. And an index vector $\vec{\sigma_i} = (\sigma_i^0, \ldots, \sigma_i^4)$ is said to* ***semi-cover*** *another index vector $\vec{\sigma_j} = (\sigma_j^0, \ldots, \sigma_j^4)$, if the interval $[\sigma_j^1, \sigma_j^3]$ is in the interval (σ_i^0, σ_i^4).*

Figure 8 gives several example index vectors to show the relations. The first two index vectors in Figure 8 a are overlapping. The next two pairs of index vectors are not overlapping because the indices for the red rectangles contradict the condition. In Figure 8 b, index vector $\vec{\sigma_i}$ semi-intersects $\vec{\sigma_j}$ because $\sigma_i^3 \in \{i | \sigma_j^1 \leq i \leq \sigma_j^3\}$ but $\sigma_i^3 \notin \{\sigma_j^1, \ldots, \sigma_j^3\}$. In Figure 8 c, index vector $\vec{\sigma_i}$ semi-covers $\vec{\sigma_j}$ because $[\sigma_j^1, \sigma_j^3] \subseteq (\sigma_i^0, \sigma_i^4)$.

Definition 3.3. *An index vector $\vec{\sigma_i} = (\sigma_i^0, \ldots, \sigma_i^4)$ is said to* ***contribute*** *to another index vector $\vec{\sigma_j} = (\sigma_j^0, \ldots, \sigma_j^4)$, if $\vec{\sigma_i}$ semi-intersects or semi-covers $\vec{\sigma_j}$. And a vertex $\mathbf{V}_i$ is said to* ***contribute*** *to another vertex $\mathbf{V}_j$ if and only if its local knot vector $\vec{\sigma_i}$ contributes to $\vec{\sigma_j}$ and the local knot vector $\vec{\tau_i}$ contributes to $\vec{\tau_j}$.*

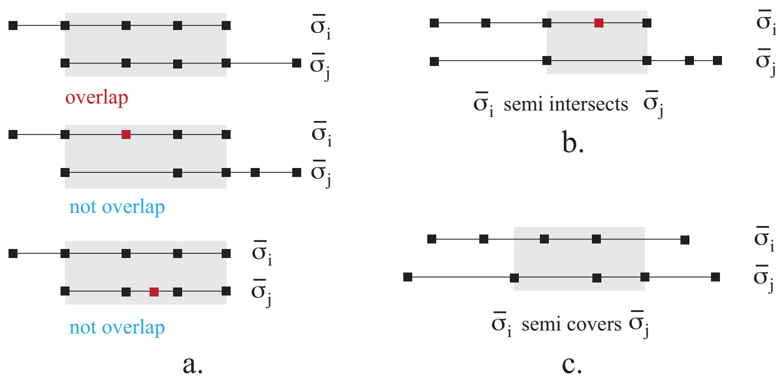


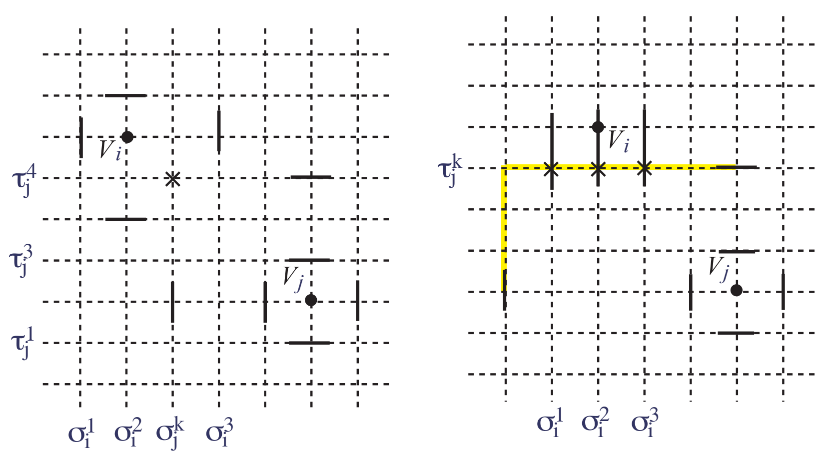
Figure 8: The relation between two index vectors: overlap, semi-intersect and semi-cover.

Definition 3.4. *For a T-mesh* T*, if there exist some indices* $(i_1, \ldots, i_m)$*,* $i_m = i_1$*, such that* $\mathbf{V}_{i_j}$ *contributes to* $\mathbf{V}_{i_{j+1}}$ *for* $j = 1, \ldots, m-1$*, then we called the T-mesh has* ***cyclic-contribution****.*

Definition 3.5. *For a T-mesh* T*, if it has no cyclic-contribution, then the T-mesh is called a* ***semi-dual T-mesh****, for short SD T-mesh.*

In the following, we will prove that a bi-cubic AS++ T-mesh is always a SD T-mesh.

Lemma 3.6. *In a bi-cubic AS++ T-mesh,* $i \neq j$*, if vertex* $\mathbf{V}_j$ *contributes to another vertex* $\mathbf{V}_i$*, then* $\vec{\sigma}_j$ *semi-covers* $\vec{\sigma}_i$ *and* $\vec{\tau}_j$ *semi-covers* $\vec{\tau}_i$*.*



(a) Index vector $\vec{\sigma}_j$ semi-intersects $\vec{\sigma}_i$ and index vector $\vec{\tau}_j$ semi-intersects $\vec{\tau}_i$

(b) Index vector $\vec{\sigma}_j$ semi-intersects $\vec{\sigma}_i$ and index vector $\vec{\tau}_j$ semi-intersects $\vec{\tau}_i$

Figure 9: Three impossible cases for one index contributes to another index vector.

Proof. We only need to prove that the following three cases are impossible.

- If $\vec{\sigma}_j$ semi-intersects $\vec{\sigma}_i$ and $\vec{\tau}_j$ semi-intersects $\vec{\tau}_i$;
  First, we consider four rectangles $(\sigma_i^k, \sigma_i^{k+1}) \times (\tau_i^l, \tau_i^{l+1})$, $k, l = 1, 2$. Because there are no other

symbols in the interior of edge $(\sigma_i^k, \sigma_i^{k+1}) \times \tau_i^2, k = 1, 2$ except of $'.'$ and $'-'$ and also there are no other symbols in the interior of edge $\sigma_i^2 \times (\tau_i^l, \tau_i^{l+1}), l = 1, 2$ except of $'.'$ and $'|'$, so in the interior of the four rectangles, there is no T-mesh vertices and also no symbols of $'|'$ and $'-'$. Now, suppose there exist k and l such that $\sigma_i^1 < \sigma_j^k < \sigma_i^3$ and $\tau_i^1 < \tau_j^l < \tau_i^3$. Then, we consider the symbol for the vertex (σ_j^k, τ_j^l). According to the analysis above, the symbol can only be symbol $'.'$. However, this contradicts the conditions of AS++ T-meshes because the symbol must be the intersection of two face extensions and the vertex belongs to $VK(\mathbf{V}_j)$.

- If $\vec{\sigma}_j$ semi-covers $\vec{\sigma}_i$ and $\vec{\tau}_j$ semi-intersects $\vec{\tau}_i$;
  Suppose there exists k such that $\tau_i^1 < \tau_j^k < \tau_i^3$, $\tau_j^k \neq \tau_i^2$ and $[\sigma_i^1, \sigma_i^3] \subset (\sigma_j^0, \sigma_j^4)$. Without loss of generalization, we assume $\sigma_i^2 > \sigma_j^2$, according to the first condition of the definition for AS++ T-meshes, the symbol at (σ_i^2, τ_j^k) cannot be $'.'$, so it can only be the symbol $'|'$. However, this contradicts the second condition of AS++ T-meshes because the interior of edge segments $(\sigma_j^1, \sigma_j^3) \times \{\tau_j^k\}$ has at lease three symbols of $'|'$, $\vdash$ or $\dashv$ (the symbol at (σ_i^1, τ_j^k), (σ_i^2, τ_j^k) and (σ_i^3, τ_j^k)).

- If $\vec{\sigma}_j$ semi-intersects $\vec{\sigma}_i$ and $\vec{\tau}_j$ semi-covers $\vec{\tau}_i$;
  This case is exact similar to the second case.

□

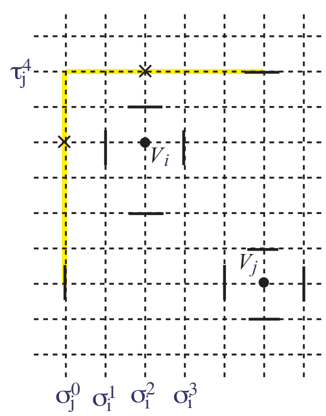


Figure 10: The only possible case for one index contributes to another index vector is the index vector $\vec{\sigma}_j$ semi-covers $\vec{\sigma}_i$ and $\vec{\tau}_j$ semi-covers $\vec{\tau}_i$.

Lemma 3.7. *In a bi-cubic AS++ T-mesh, $i \neq j$, if vertex $\mathbf{V}_j$ contributes to another vertex $\mathbf{V}_i$, then $\sigma_i^4 - \sigma_i^0 < \sigma_j^4 - \sigma_j^0$.*

Proof. According to Lemma 3.6, $\vec{\sigma}_j$ semi-covers $\vec{\sigma}_i$ and $\vec{\tau}_j$ semi-covers $\vec{\tau}_i$. Without loss of generalization, we assume $\sigma_j^1 > \sigma_i^1$ and $\sigma_j^3 > \sigma_i^3$ as illustrated in Figure 10. The other cases are similar. Then it is sufficient to prove that $\sigma_i^0 \geq \sigma_j^0$ and $\sigma_i^4 \leq \sigma_j^3$. Referring to Figure 10, we first prove that $\sigma_i^0 \geq \sigma_j^0$, or the symbol at vertex (σ_j^0, τ_i^2) cannot be $'-'$. Because the T-mesh is an AS++ T-mesh, so all the symbols on the edge $\{\sigma_j^0\} \times [\tau_i^2, \tau_j^4]$ cannot be $'.'$, which means there at least 4 symbols of $'+'$, $'-'$ on the edge $\{\sigma_j^0\} \times [\tau_i^2, \tau_j^4]$. This contradicts the second condition of AS++ T-meshes. Similarly, we can prove that $\sigma_i^4 \leq \sigma_j^3$. In conclusion, $\sigma_i^4 - \sigma_i^0 < \sigma_j^4 - \sigma_j^0$. □

Theorem 3.8. *A bi-cubic AS++ T-mesh is a semi-dual T-mesh.*

Proof. If the T-mesh has cyclic-contribution, then there exist indices $(i_1, \ldots, i_m)$, $i_m = i_1$, such that $\mathbf{V}_{i_j}$ contributes to $\mathbf{V}_{i_{j+1}}$ for $j = 1, \ldots, m-1$. According to Lemma 3.6 and Lemma 3.7, $\sigma^4_{i_{j+1}} - \sigma^0_{i_{j+1}} < \sigma^4_{i_j} - \sigma^0_{i_j}$. Thus, $\sigma^4_{i_1} - \sigma^0_{i_1} = \sigma^4_{i_m} - \sigma^0_{i_m} < \sigma^4_{i_1} - \sigma^0_{i_1}$, which is obvious not correct. Thus, a bi-cubic AS++ T-mesh has no cyclic-contributions, which completes the proof. □

4. Properties of AS++ T-splines

In this section we list some important properties satisfied by AS++ T-splines. We start with a brief introduction of T-spline spaces.

4.1. T-splines

Let T be an index T-mesh defined in Section 2, $\vec{\mathbf{s}}$, $\vec{\mathbf{t}}$ be two non-decreasing global knot vectors $[s_{-1}, s_0, \ldots, s_{c+2}]$ and $[t_{-1}, t_0, \ldots, t_{r+2}]$ in $[0, 1]$, i.e., $s_{-1} = \cdots = s_2 = t_{-1} = \cdots = t_2 = 0$, $s_{c-1} = \cdots = s_{c+2} = t_{r-1} = \cdots = t_{r+2} = 1$ and $0 < s_i, t_j < 1$ for $2 < i < c-1, 2 < j < r-1$. For the i-th vertex in the rectangle $[1, c] \times [1, r]$, we associated a blending function $T_i(s, t) = B[s_{\vec{\sigma_i}}](s)B[t_{\vec{\tau_i}}](t)$, where $B[s_{\vec{\sigma_i}}](s)$, $B[t_{\vec{\tau_i}}](t)$ are cubic B-spline basis functions defined in terms of knot vector $s_{\vec{\sigma_i}}$ and $t_{\vec{\tau_i}}$, where $s_{\vec{\sigma_i}} = [s_{\sigma^0_i}, s_{\sigma^1_i}, \ldots, s_{\sigma^4_i}]$, $t_{\vec{\tau_i}} = [t_{\tau^0_i}, t_{\tau^1_i}, \ldots, t_{\tau^4_i}]$.

A T-spline space $\mathbf{S}(\mathsf{T}, \vec{\mathbf{s}}, \vec{\mathbf{t}})$ defined on the T-mesh T with the knot vectors $\vec{\mathbf{s}}$ and $\vec{\mathbf{t}}$ is finally given as the span of all these blending functions and a T-spline surface is defined as

$$\mathbf{T}(s, t) = \sum_{i=1}^{n_A} \mathbf{T}_i T_i(s, t) \tag{1}$$

where $\mathbf{T}_i = (\omega_i x_i, \omega_i y_i, \omega_i z_i, \omega_i) \in \mathbb{P}^3$ are homogeneous control points, $\omega_i \in \mathbb{R}$ are weights, $T_i(s, t)$ are blending functions, and n_A is the number of vertices in the rectangle $[1, c] \times [1, r]$.

4.2. Dual Basis for univariate B-splines

We briefly introduce univariate B-splines with the aim of recalling a few results that we will need in the next sections. We only restrict to the case of cubic B-splines.

Given the integer $n > 0$, let a knot vector $[s_{-1}, s_0, s_1, \ldots, s_n, s_{n+1}, s_{n+2}]$ be given, with ordered knots $s_i \leq s_{i+1}$. For $1 \leq i \leq n$, we denote by $B_i(x) = B[s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2}](x)$ the cubic B-spline function associated with the knots $\{s_{i-2}, s_{i-1}, s_i, s_{i+1}, s_{i+2}\}$.

Following [18], we can define suitable functionals which are dual to the B-splines basis functions. Let

$$G_j(x) = g(\frac{2x - s_{j+2} - s_{j-2}}{s_{j+2} - s_{j-2}}),$$

where $g(x)$ is the transition function defined in [18],

$$g(x) = \begin{cases} 0, & x < -1; \\ \int_{-1}^{x} 2B[-1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 1](t)dt, & x \in [-1, 1); \\ 1, & x \geq 1. \end{cases} \tag{2}$$

and

$$\phi_j(x) = \frac{(x - s_{j-1})(x - s_j)(x - s_{j+1})}{6}.$$

We define the Schumaker's functional $\lambda[s_{j-2},\dots,s_{j+2}](.)$ associated with knot vector $\{s_{j-2},\dots,s_{j+2}\}$ as follows,

$$\lambda[s_{j-2},\dots,s_{j+2}](f)=\int_{s_{j-2}}^{s_{j+2}} f(x)D^4\psi_j(x)dx,$$

where $\psi_j(x)=G_j(x)\phi_j(x)$. $\lambda[s_{j-2},\dots,s_{j+2}](.)$ have the following property that

$$\lambda[s_{i-2},\dots,s_{i+2}](B_j(x))=\delta_{i,j}, \tag{3}$$

where δ_{ij} is the Kronecker delta function.

Lemma 4.1. *Given a function $f(x)$ in the B-spline space, suppose $f(x)=\sum_{j=1}^{n} c_j B[s_{j-2},\dots,s_{j+2}](x)$, then the coefficients $c_j=\lambda[s_{j-2},\dots,s_{j+2}](f(x))$.*

Proof. This can be directly derived from the equation (3). □

Lemma 4.2. *If two index vectors $\vec{\sigma_i}$ and $\vec{\sigma_j}$ are overlapping and not identical, then $\lambda[s_{\vec{\sigma_i}}](B[s_{\vec{\sigma_j}}])=0$.*

Proof. This can be directly derived from the definition of B-spline dual basis. You can also reach it from [12, 13]. □

According to the Theorem 4.41 in [18], we have the following lemma.

Lemma 4.3. *For any $f(x)\in L^2[s_j,s_{j+4}]$, there exists a constant C independent of the knots s_j, such that*

$$|\lambda[s_j,\dots,s_{j+4}](f)|\le Ch_j^{-\frac{1}{2}}||f||_{L^2[I_j]},$$

where $I_j=(s_j,s_{j+4})$ and $h_j=s_{j+4}-s_j$.

4.3. Linear independence of AS++ T-splines

For each vertex $\mathbf{V}_i$, denote $\Omega_i=(\sigma_i^0,\sigma_i^4)\times(\tau_i^0,\tau_i^4)$, and index set $\mathbb{I}_i=\{j|\Omega_j\cap\Omega_i\neq\emptyset\}$, then we define two new local knot index vectors $\vec{\alpha}_i=(\alpha_i^j), j=0,1,\dots,4$ and $\vec{\beta}_i=(\beta_i^j), j=0,1,\dots,4$, where $\alpha_i^j=\sigma_i^j$, $\beta_i^j=\tau_i^j$, $j=1,\dots,3$, and

$$\begin{aligned}
\alpha_i^0&=\max_k\left\{k\in\{\sigma_j^0,\dots,\sigma_j^4\}\cap[\sigma_i^0,\sigma_i^1), j\in\mathbb{I}_i\right\};\\
\alpha_i^4&=\min_k\left\{k\in\{\sigma_j^0,\dots,\sigma_j^4\}\cap(\sigma_i^3,\sigma_i^4], j\in\mathbb{I}_i\right\};\\
\beta_i^0&=\max_k\left\{k\in\{\tau_j^0,\dots,\tau_j^4\}\cap[\tau_i^0,\tau_i^1), j\in\mathbb{I}_i\right\};\\
\beta_i^4&=\min_k\left\{k\in\{\tau_j^0,\dots,\tau_j^4\}\cap(\tau_i^3,\tau_i^4], j\in\mathbb{I}_i\right\}.
\end{aligned}$$

Now we associate a suitable functionals $\lambda_i(\cdot)$ for the i-th vertex $\lambda_i(\cdot)=\lambda(s_{\vec{\alpha}_i})\otimes\lambda(t_{\vec{\beta}_i})(\cdot)$, here $\lambda(s_{\vec{\alpha}_i})$ and $\lambda(t_{\vec{\beta}_i})$ are the dual basis defined in Section 4.2 in terms of the knot vectors $s_{\vec{\alpha}_i}$ and $t_{\vec{\beta}_i}$.

Lemma 4.4. $\lambda(s_{\vec{\alpha}_i})[B[s_{\vec{\sigma}_i}](s)]=1$.

Proof. Without loss of generalization, we assume $\alpha_i^0\in(\sigma_i^0,\sigma_i^1)$ and $\alpha_i^4\in(\sigma_i^3,\sigma_i^4)$. Then the B-spline $B[s_{\vec{\sigma}_i}]$ can be written into the linear combination of the B-splines $B[s_{\sigma_i^0},s_{\alpha_i^0},s_{\sigma_i^1},\dots,s_{\sigma_i^3}]$, $B[s_{\vec{\alpha}_i}]$ and $B[s_{\sigma_i^1},\dots,s_{\alpha_i^4},s_{\sigma_i^4}]$. Because $\lambda[s_{\vec{\alpha}_i}](B[s_{\sigma_i^0},s_{\alpha_i^0},s_{\sigma_i^1},\dots,s_{\sigma_i^3}](s))=\lambda[s_{\vec{\alpha}_i}](B[s_{\sigma_i^1},\dots,s_{\alpha_i^4},s_{\sigma_i^4}](s))=0$, so $\lambda[s_{\vec{\alpha}_i}](B[s_{\vec{\sigma}_i}](s))=\lambda[s_{\vec{\alpha}_i}](B[s_{\vec{\alpha}_i}](s))=1$ □

Lemma 4.5. *For $i\neq j$, if $\vec{\sigma_j}$ doesn't contribute $\vec{\sigma_i}$, then $\lambda[s_{\vec{\alpha}_i}](B[s_{\vec{\sigma}_j}](s))=0$.*

Proof. If $\alpha_i^0 \in (\sigma_j^0, \sigma_j^4)$, then we insert the index α_i^0 into index knot vector $\vec{\sigma_j}$ and similarly if $\alpha_i^4 \in (\sigma_j^3, \sigma_j^4)$, then we insert the index α_i^4 into index knot vector $\vec{\sigma_j}$. After insertion, the new index vector is denoted as $\vec{\gamma}_j = (\gamma_j^0, \ldots, \gamma_j^k)$. And denote $\vec{\gamma}_j^l = (\gamma_j^l, \ldots, \gamma_j^{l+4})$, $l < 3$. Let $B[s_{\vec{\gamma}_j^l}]$ be the B-splines defined by the knot vector $s_{\vec{\gamma}_j^l}$, then $B[s_{\vec{\sigma}_j}]$ is a linear combination of B-splines $B[s_{\vec{\gamma}_j^l}]$. Because $\vec{\sigma_j}$ doesn't semi-intersect $\vec{\sigma_i}$, so $\vec{\alpha}_i$ and $\vec{\gamma}_j^l$ are overlapping. And because $\vec{\sigma_j}$ doesn't semi-cover $\vec{\sigma_i}$, so $\vec{\alpha}_i \neq \vec{\gamma}_j^l$. Thus, $\lambda[s_{\vec{\alpha}_i}](B[s_{\vec{\sigma}_j}](s)) = 0$. □

Lemma 4.6. *For $i \neq j$, if the vertex $\mathbf{V}_j$ doesn't contribute to vertex $\mathbf{V}_i$, then $\lambda_i(T_j) = 0$*

Proof. This can be directly derived from lemma 4.5. □

Lemma 4.7. *If a T-mesh is a SD T-mesh, then functionals $\lambda_i(\cdot)$ form a semi-dual basis for the blending functions.*

Proof. Because the T-mesh is a semi-dual T-mesh, we can arrange the vertex indices as $\mathbf{V}_{i_j}, j = 1, \ldots, n_A$ such that the vertex $\mathbf{V}_{i_j}$ doesn't contribute the vertex $\mathbf{V}_{i_k}$ if $k < j$. Now we apply the functionals $\{\lambda_{i_j}\}$ to the blending functions and we can get the functional matrix $\mathbb{D} = (d_{j,k})$, where $d_{j,k} = \lambda_{i_j}(T_{i_k}(s,t))$. According to Lemma 4.4, $d_{j,j} = \lambda_{i_j}(T_{i_j}(s,t)) = 1$. And according to Lemma 4.6, $\mathbb{D}$ is an upper triangular matrix because vertex $\mathbf{V}_{i_j}$ doesn't contribute to vertex $\mathbf{V}_{i_k}$ for $k < j$, which completes the proof. □

Lemma 4.8. *A bi-cubic semi-dual T-spline has linear independent blending functions.*

Proof. This is directly derived from Theorem 3.8 and Lemma 4.7. □

Theorem 4.9. *The blending functions for a bi-cubic AS++ T-spline are always linear independent.*

Proof. This is directly derived from Theorem 3.8 and Lemma 4.8. □

4.4. Dual basis and approximation

In a SD T-mesh and also an AS++ T-mesh, we can arrange the vertex indices as $\mathbf{V}_{i_j}, j = 1, \ldots, n_A$ such that the vector $\mathbf{V}_{i_j}$ doesn't contribute to the vertex $\mathbf{V}_{i_k}$ if $k < j$, i.e., $\lambda_{i_k}(T_{i_j}) = 0$.

Denote

$$\Lambda_{i_1}(.) = \lambda_{i_1}(.)$$

$$\Lambda_{i_k}(.) = \lambda_{i_k}(.) - \sum_{l=1}^{k-1} \lambda_{i_k}(T_{i_l})\Lambda_{i_l}(.).$$

Theorem 4.10. *$\Lambda_{i_k}(.)$ are the dual basis for the blending functions $T_{i_j}(s,t)$, i.e., $\Lambda_{i_k}(T_{i_j}) = \delta_{k,j}$.*

Proof. We prove the theorem via mathematical induction method. First for $k = 1$, $\Lambda_{i_1}(.) = \lambda_{i_1}(.)$. It is obvious that $\Lambda_{i_1}(T_{i_1}) = 1$. And for $j > 0$, because $\lambda_{i_1}(T_{i_j}) = 0$ for all $j > 0$ according to the assumption, $\Lambda_{i_k}(T_{i_j}) = \lambda_{i_1}(T_{i_j}) = 0$.

Suppose the theorem is correct for any $k < m$, now we prove that the theorem is also correct for $k = m$. The proof is divided into three parts.

First, we prove that if $j > m$, then $\Lambda_{i_m}(T_{i_j}) = 0$. Actually, $\lambda_{i_m}(T_{i_j}) = 0$ since $j > m$ and for all $l = 1, \ldots, m-1$, $\Lambda_{i_l}(T_{i_j}) = 0$ according to the induction assumption. Thus, for all $j > m$, $\Lambda_{i_m}(T_{i_j}) = 0$.

And then we prove that if $j < m$, $\Lambda_{i_m}(T_{i_j}) = 0$. Actually,

$$\begin{aligned}\Lambda_{i_m}(T_{i_j}) &= \lambda_{i_m}(T_{i_j}) - \sum_{l=0}^{m-1} \lambda_{i_m}(T_{i_l})\Lambda_{i_l}(T_{i_j}) \\ &= \lambda_{i_m}(T_{i_j}) - \sum_{l=0}^{m-1} \lambda_{i_m}(T_{i_l})\delta_{j,l} \\ &= \lambda_{i_m}(T_{i_j}) - \lambda_{i_m}(T_{i_j}) = 0.\end{aligned}$$

In the end, we prove that $\Lambda_{i_m}(T_{i_m}) = 1$. Actually,

$$\Lambda_{i_m}(T_{i_m}) = \lambda_{i_m}(T_{i_m}) - \sum_{l=0}^{m-1} \lambda_{i_m}(T_{i_l})\Lambda_{i_l}(T_{i_m}) = 1. \tag{4}$$

Thus, $\Lambda_{i_k}(.)$ form a set of dual basis for the semi-dual T-splines. □

An important consequence of existence of the dual basis (Theorem 4.10) is that we can build a projection operator $\Pi : L^2([0,1]^2) \to \mathbf{S}(\mathsf{T}, \vec{\mathbf{s}}, \vec{\mathbf{t}})$, defined by

$$\Pi(f)(s,t) = \sum_{i=1}^{n_A} \Lambda_i(f)T_i(s,t), \forall f \in L^2([0,1]^2) \tag{5}$$

It is straightforward to check that Π is a projection operator because $\Lambda_{i_j}(.)$ form a set of dual basis. The dual basis grants a very powerful tool to prove approximation properties for AS++ T-spline spaces, while approximation properties are a fundamental condition for a spline space to be used in the IGA problems.

Before giving the details, we introduce some additional notations. Let Q be a generic (open) element in the T-mesh T, and $h_s(Q)$, $h_t(Q)$ be the length in the s and t coordinate directions of the element Q. We denote $Q^i = (s_{\sigma_i^0}, s_{\sigma_i^4}) \times (t_{\tau_i^0}, t_{\tau_i^4})$ and $I(Q) = \{j | \Omega_j \bigcap Q \neq \emptyset\}$, and $\widetilde{Q}^i = \bigcup_{i \in I(Q)} Q^i$. Furthermore, we denote $h_{s,i} = s_{\alpha_i^4} - s_{\alpha_i^0}$ and $h_{t,i} = t_{\beta_i^4} - t_{\beta_i^0}$. According to Lemma 4.3 and the construction of dual basis for AS++ T-splines, we have the following lemmas.

Lemma 4.11. *For any $f \in L^2[0,1]$, there exists a constant C independent of the knot vectors, such that*

$$|\lambda_{i_j}(f)| \leq C(h_{s,i_j} h_{t,i_j})^{-\frac{1}{2}} ||f||_{L^2[0,1]}.$$

Proof. This can be directly derived from Lemma 4.3. □

Lemma 4.12. *Given an AS++ T-mesh T and two knot vectors $\vec{\mathbf{s}}, \vec{\mathbf{t}}$, assume that all the constant functions belongs to the AS++ T-spline space $\mathbf{S}(\mathsf{T}, \vec{\mathbf{s}}, \vec{\mathbf{t}})$, then there exists a constant C independent of T, $\vec{\mathbf{s}}$ and $\vec{\mathbf{t}}$ such that for any Q,*

$$||\Pi(f)||_{L^2(Q)} \leq C||f||_{L^2(\widetilde{Q})}. \tag{6}$$

Proof. Since all the constant functions belongs to the AS++ T-spline space $\mathbf{S}(\mathsf{T}, \vec{\mathbf{s}}, \vec{\mathbf{t}})$, thus there exists some weights ω_{i_j} such that

$$\sum_{j=1}^{n_A} \omega_{i_j} T_{i_j} = 1. \tag{7}$$

Let I be vector of $\{1, \ldots, 1\}$ with n_A elements, ω be the vector of $[\omega_{i_j}]$, λ be the vector of functional $\{\lambda_{i_j}(.)\}$, Λ be the vector of functional $\{\Lambda_{i_j}(.)\}$ and T be the vector of the blending functions $\{T_{i_j}(s,t)\}^T$.

Denote M the matrix $(m_{j,k})$ where $m_{j,k} = \lambda_{i_j}(T_{i_k})$. Then according to the construction of dual basis, we have $\lambda = \Lambda M$. According to the equation 7, $\omega T = 1$. Because Λ_{i_j} are dual basis for the blending functions $T_{i_j}(s,t)$, so $\omega = IM^{-1}$.

For any given Q, and any point $(s,t) \in Q$,

$$\begin{aligned}
|\Pi(f)(s,t)|^2 &= |\sum_{i=1}^{n_A} \Lambda_i(f) T_i(s,t)|^2 = |\Lambda(f)T|^2 \\
&= |\lambda(f) M^{-1} T|^2 \\
&= |\sum_{i=1}^{n_A} \lambda_i(f) \omega_i T_i(s,t)|^2 \\
&\leq \max |\lambda_i(f)|^2 \\
&\leq C \max_{i \in I(Q)} (h_{s,i} h_{t,i})^{-1} ||f||^2_{L^2(Q)} \\
&\leq C \max_{i \in I(Q)} (h_s(Q) h_t(Q))^{-1} ||f||^2_{L^2(Q)}.
\end{aligned}$$

Note that the constant C appearing above is independent of any other variable or parameter. Since the bound above holds for any $(s,t) \in Q$, integrating on the element Q and applying the above equation yields

$$||\Pi(f)||_{L^2(Q)} \leq h_s(Q) h_t(Q) ||\Pi(f)||_{L^\infty(Q)} \leq C ||f||_{L^2(\widetilde{Q})}.$$

□

Theorem 4.13. *Given an AS++ T-mesh* T *and two knot vectors* $\vec{\mathbf{s}}, \vec{\mathbf{t}}$*, assume that all the space of global bi-cubic polynomials are included in the AS++ T-spline space* $\mathbf{S}(\mathsf{T}, \vec{\mathbf{s}}, \vec{\mathbf{t}})$*, for any* Q*, denote* $\widehat{Q}$ *be the smallest rectangle containing* $\widetilde{Q}$ *and* h *to be the diameter of* $\widehat{Q}$*, then there exists a constant* C *independent of* T*,* $\vec{\mathbf{s}}$ *and* $\vec{\mathbf{t}}$ *such that for any* $r \in [0,4]$

$$||f - \Pi(f)||_{L^2(Q)} \leq C h^r ||f||_{H^r(\widehat{Q})}, \forall f \in H^r(\widehat{Q}), \tag{8}$$

where $H^r([0,1]^2)$ *indicates the Sobolev space of order* r*.*

Proof. Let p be any bicubic polynomials on $[0,1]^2$. Since $p \in \mathbf{S}(\mathsf{T}, \vec{\mathbf{s}}, \vec{\mathbf{t}})$, using all the above lemmas, it follows

$$\begin{aligned}
||f - \Pi(f)||_{L^2(Q)} &= ||f - p + p - \Pi(f)||_{L^2(Q)} \\
&\leq ||f - p||_{L^2(Q)} + ||\Pi(f - p)||_{L^2(Q)} \\
&\leq (1 + C) ||f - p||_{L^2(\widetilde{Q})} \\
&\leq (1 + C) ||f - p||_{L^2(\widehat{Q})}.
\end{aligned}$$

The result finally follows by standard polynomial approximation results. □

5. Conclusion

The paper generalizes the dual basis for analysis-suitable T-splines [12, 13] to semi-dual basis. Using semi-dual basis, we find out a more general class of T-splines and show that the blending functions for any such T-splines are linearly independent regardless of knot intervals. As we have found a new class of T-splines for which blending functions are linear independent, so it is very important to derive more properties for this class of T-splines, for example, the partition of unity, characterization, and refinement algorithm. All these issues will be discussed in the forthcoming papers. The other interesting topic is how to generalize the idea in this paper to arbitrary degrees, which will be left as future work.

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