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AS++ T-splines: arbitrary degree, nestedness and approximation

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Abstract
Bi-cubic analysis-suitable++ T-splines (AS++ T-splines) (Li in Comput Methods Appl Mech Eng 333:462–474, 2018) are T-splines defined on less restricted T-meshes than analysis-suitable T-splines (AS T-splines), which are both important tools in isogeometric analysis (IGA). In this paper, we generalize the bi-cubic AS++ T-splines to arbitrary degrees and describe some important mathematical properties. Specifically, we develop the conditions under which an AS++ T-spline space belongs to another AS++ T-spline space. This result provides the foundation for the optimized local refinement (Zhang in Comput Methods Appl Mech Eng 342:32–45, 2018) and also is one of the keys for AS++ T-spline approximation. In the end, the optimal approximation properties of the associated T-spline space are developed for arbitrary AS++ T-spline space with the assumption of existence of dual basis, which is automatically satisfied for bi-cubic AS++ T-spline spaces.

Mathematics Subject Classification 41A15 · 65D07 · 65M15

1 Introduction

T-spline [44,45] is an important technology in industrial design and engineering analysis, which can support local refinement [42,45] and create watertightness models [26,44,46] in the NURBS-compatible way. T-spline is also attractive in isogeometric analysis (IGA). IGA is introduced in [25], which adopts the computer-aided design (CAD) description as the basis for analysis with more efficient for each degree

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of freedom [4,8]. This has opened the door to the design through analysis [13,40]. Lots of technologies have been developed in recent years, such as different representations [27,28,32,35,36,49], efficient quadrature [5,9,20,22,24], handling trimmed geometry [1,2,14,23], approximation [12,21,39] and applications [34,37,38,47,50, 52]. Among all the design-through-analysis technologies, T-splines have gained widespread attention [6,7,18,19,33,42,48].

Although the whole class of T-splines is not suitable as a basis for IGA because of possible linear dependence [11], a mildly topological restricted subset of T-splines—analysis-suitable T-splines (AS T-splines), are optimized to meet the needs both for design and analysis [16,17,29,31,42,43]. Analysis-suitable++ T-splines (AS++ T-splines) are defined in [30,51] for bi-cubic case, which are T-splines defined on less restricted T-meshes, i.e., AS++ T-splines include AS T-splines as a special case. And meanwhile, bi-cubic AS++ T-splines maintain all the good properties as AS T-splines: linearly independence [30], NURBS compatibility, convex hull, watertightness and optimized local refinement [51]. Compared with AS T-splines, bi cubic AS++ T-splines can have less control points when approximate a same object. Thus, it is desirable to provide the basic mathematical properties for AS++ T-splines. However, the nestedness and approximation of bi-cubic AS++ T-splines remain unknown, and it is also a challenging task to generalize these results to arbitrary degrees.

In this paper, we continue to develop the theory of AS++ T-spline spaces. Precisely, we generalize the bi-cubic AS++ T-spline to arbitrary degrees and provide the refineability and nestedness of the spline space, i.e., the condition under which an AS++ T-spline space belongs to another AS++ T-spline space. This theoretical justification is the foundation of the local refinement algorithm for bi-cubic AS++ T-splines in [51] and one of the keys for the approximation properties. We then show that this condition, together with the assumption of the existence of dual basis, can be used to prove the AS++ T-spline spaces possesses the same optimal approximation properties as the tensor product B-spline spaces [3]. In summary, the paper provides the following main contributions.

– We generalize bi-cubic AS++ T-splines to arbitrary degree AS++ T-splines;
– We provide the conditions for two AS++ T-splines to be nested;
– We prove the optimal approximation properties for arbitrary degree AS++ T-splines with the existence of dual basis. We should mention that bi-cubic AS++ T-splines satisfy the assumption which has been proved in [30].
– We classify the admissible AS++ T-splines to be semi-standard which is one of the open problems posed in [45].

The rest paper is structured as follows. In Sect. 2, we recall some basic notations for T-meshes and T-splines. Then we introduce the AS++ T-meshes and AS++ T-splines in Sect. 3. Section 4 proves the conditions under which two arbitrary degree AS++ T-spline spaces are nested. And then, under the assumption of the existence of dual basis, several basic approximation results for arbitrary degree AS++ T-splines are proved in Sect. 5. The last section is the conclusion and future work.

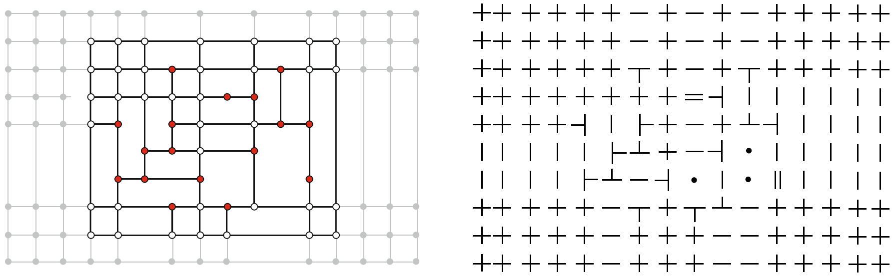


Fig. 1 A bi-degree (5, 1) T-mesh and the associated symbolic T-mesh

2 T-meshes and T-splines

This section reviews the basic concept for T-splines. Similar to the approach in [4], we define T-splines based on T-meshes in the index domain which are referred as *index T-meshes*.

2.1 Index T-mesh

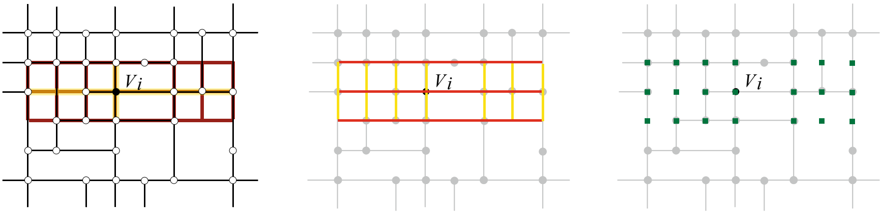
A T-mesh T for a T-spline surface of bi-degree (d_1, d_2) is a rectangular partition of the index domain $[0, c + d_1] \times [0, r + d_2]$, where c and r are the number of columns and rows in the mesh. We also define the subdomain $D = [p, c+d_1-p] \times [q, r+d_2-q]$, where $p = \left\lfloor \frac{d_1+1}{2} \right\rfloor$ and $q = \left\lfloor \frac{d_2+1}{2} \right\rfloor$. D contains the preimage of the control points. In a T-mesh, there are three types of topological elements: vertices, edges and faces. A vertex has integer coordinates, which is denoted as (σ_i, τ_i) or $\{\sigma_i\} \times \{\tau_i\}$. Edge is a line segment connecting two vertices in the T-mesh and no other vertices lying in its interior. We denote $[\sigma_j, \sigma_k] \times \{\tau_i\}$ as a horizontal edge or a set of connected horizontal edges. Similarly, we denote $\{\sigma_i\} \times [\tau_j, \tau_k]$ as a vertical edge or a set of connected vertical edges. A face is a rectangle bounded by edges with no other edges nor vertices in its interior, which is denoted as $[\sigma_i, \sigma_j] \times [\tau_k, \tau_l]$ or $(\sigma_i, \sigma_j) \times (\tau_k, \tau_l)$. The valence of a vertex is the number of edges which contain the vertex. For the interior vertices, we don't allow L-junctions, hanging vertices (a vertex of valence one) or isolated vertices, but we do allow valence-two vertices called I-junctions shared by two collinear edges, valence three (called T-junctions) and valence four vertices.

A symbolic T-mesh [31] is created from a T-mesh T by assigning a symbol in Table 1 to each vertex in a tensor product mesh formed from the index coordinates. The symbol is chosen to match the mesh topology of T. The symbolic T-mesh corresponding to the left T-mesh in Fig. 1 is shown on the right of Fig. 1.

An *anchor* is a point in D that is used in defining a blending function, as we will now explain. An anchor corresponds to a vertex if d_1 and d_2 are both odd, to an edge if one is even and the other odd, and to a face if both are even. Associated with anchor $\mathbf{A}_i$ are local index vector $\boldsymbol{\sigma}_i \times \boldsymbol{\tau}_i$ which are determined as follows. The values of $\boldsymbol{\sigma}_i = \left[\sigma_{a_i}^0, \ldots, \sigma_{a_i}^{d_1+1}\right]$ and $\boldsymbol{\tau}_i = \left[\tau_{a_i}^0, \ldots, \tau_{a_i}^{d_2+1}\right]$ are determined as follows. From the i-th anchor in the T-mesh, we shoot a ray in the s and t direction traversing the

Table 1 Definition of possible symbols in a symbolic T-mesh

Symbol	Correspondence with T
+	Valence 4 vertex, corner vertex, or valence 3boundary vertex in T
⊢, ⊣, ⊥, ⊤	Oriented valence three vertex in T
‖, =	Oriented I-junction with two incident edges
|, −	Vertical or horizontal edge in T
·	No corresponding vertex or edge in T



(a) $\boldsymbol{\sigma}_\mathbf{i} \times \boldsymbol{\tau}_\mathbf{i}$ **(b)** $SK(\mathbf{A}_i)$ **(c)** $VK(\mathbf{A}_i)$

Fig. 2 Define the local index vector $\boldsymbol{\sigma}_i \times \boldsymbol{\tau}_i$, the skeleton $SK(\mathbf{A}_i)$ and the vertex of the skeleton $VK(\mathbf{A}_i)$ for a bi-degree (5, 1) T-spline

T-mesh and collect a total of d_1+2 and d_2+2 knot indices to form $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$ such that the anchor is in the middle of the $\boldsymbol{\sigma}_i \times \boldsymbol{\tau}_i$, as shown in Fig. 2. For the anchor $\mathbf{A}_i$, let $hSK(\mathbf{A}_i)$ or $hSK(\boldsymbol{\sigma}_i \times \boldsymbol{\tau}_i)$ be the union of all the edge segments $\left[\sigma_{a_i}^0, \sigma_{a_i}^{d_1+1}\right] \times \left\{\tau_{a_i}^j\right\}$, $j = 0, 1, \ldots, d_2+1$ and $vSK(\mathbf{A}_i)$ be the union of all the edge segments $\left\{\sigma_{a_i}^j\right\} \times \left[\tau_{a_i}^0, \tau_{a_i}^{d_2+1}\right]$, $j = 0, 1, \ldots, d_1+1$. Denote $SK(\mathbf{A}_i) = hSK(\mathbf{A}_i) \cup vSK(\mathbf{A}_i)$ and $VK(\mathbf{A}_i) = \left\{\left(\sigma_{a_i}^j, \tau_{a_i}^k\right)\right\}$, $j = 0, 1, \ldots, d_1+1$, $k = 0, 1, \ldots, d_2+1$, as illustrated in Fig. 2. $vSK(\boldsymbol{\sigma}_i \times \boldsymbol{\tau}_i)$, $SK(\boldsymbol{\sigma}_i \times \boldsymbol{\tau}_i)$ and $VK(\boldsymbol{\sigma}_i \times \boldsymbol{\tau}_i)$ can be defined similarly. Denote $\mathcal{S}_T$ and $\mathcal{S}_A$ to be the set of all the T-junctions and anchors respectively.

2.2 T-splines

Let $\mathbf{s} = [s_0, s_1, \ldots, s_{c+d_1}]$ and $\mathbf{t} = [t_0, t_1, \ldots, t_{r+d_2}]$ be two global knot vectors , and $T_i(s,t) = B[\mathbf{s}_i](s)B[\mathbf{t}_i](t)$, where $B[\mathbf{s}_i](s)$ and $B[\mathbf{t}_i](t)$ are degree d_1 and d_2 B-spline functions in terms of the knot vectors $\mathbf{s}_i = \left[s_{\sigma_{a_i}^0}, s_{\sigma_{a_i}^1}, \ldots, s_{\sigma_{a_i}^{d_1+1}}\right]$ and $\mathbf{t}_i = \left[t_{\tau_{a_i}^0}, t_{\tau_{a_i}^1}, \ldots, t_{\tau_{a_i}^{d_2+1}}\right]$, respectively. A T-spline space $\mathbf{S}(\mathsf{T}, \mathbf{s}, \mathbf{t})$ defined on the T-mesh T with the knot vectors $\mathbf{s}$ and $\mathbf{t}$ is defined as

$$\mathbf{S}(\mathsf{T}, \mathbf{s}, \mathbf{t}) = span\{T_i(s,t)\}.$$

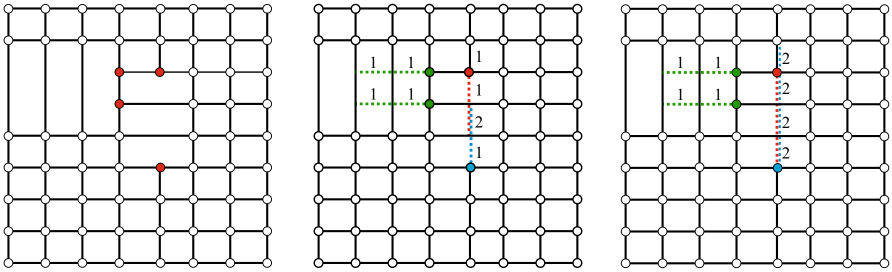


Fig. 3 A bi-cubic T-mesh (**a**) and two extended T-meshes (**b** and **c**). All the red vertices in **a**. are the T-junctions in the T-mesh and the vertices are shown in different color in **b** and **c** to show the different extended edges. The multiplicity is shown as the number closed to the corresponding extended edges. We can see that the multiplicity of some edges of the two extended T-meshes are different

3 AS++ T-splines

In this section, we generalize bi-cubic AS++ T-splines [30] to arbitrary degrees.

First, we introduce more notations. For a T-junction $\mathbf{T}_i$ (σ_i, τ_i) of type $\vdash$, the *n-bays face extension* $ext_n^f(\mathbf{T}_i)$ is the line segment $\left[\sigma_i^{p-n}, \sigma_i^p\right] \times \left\{\tau_i^q\right\}$ and if $\mathbf{T}_i$ is of type $\dashv$, then $ext_n^f(\mathbf{T}_i)$ is the line segment $\left[\sigma_i^p, \sigma_i^{n+p}\right] \times \left\{\tau_i^q\right\}$. Analogously, we can define the n-bays face extension for T-junctions of type $\perp$ or $\top$. For simplicity, $ext^f(\mathbf{T}_i)$ denotes $ext_{n_i}^f(\mathbf{T}_i)$ if $n_i = p$ for T-junction $\vdash$, $\dashv$ and $n_i = q$ for T-junction $\perp$, $\top$.

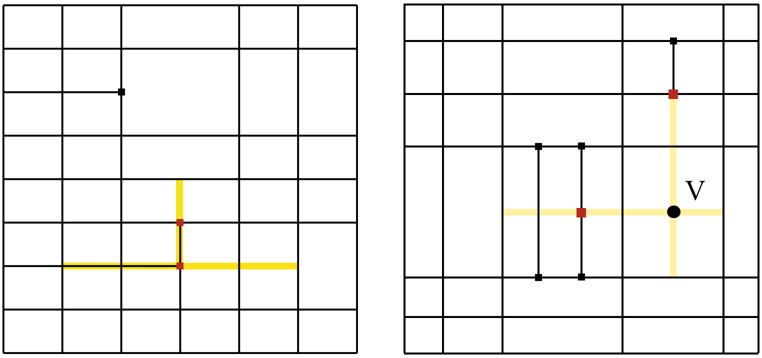
Two T-meshes $\mathsf{T}_1 \subseteq \mathsf{T}_2$ if and only if we can insert edges and vertices into T-mesh T_1 to create T-mesh T_2. For any T-mesh T, one can extend all the T-junctions such that they reach the boundary and then we can create the associated tensor-product mesh $\mathsf{B}(\mathsf{T})$. *The extended T-mesh set* is defined as

$$ext(\mathsf{T}) = \{\mathsf{T}_1 \| \text{For all T-meshes } \mathsf{T}_1, \mathsf{T} \subseteq \mathsf{T}_1 \subseteq \mathsf{B}(\mathsf{T})\}.$$

The edges belonging to some $ext_{n_i}^f(\mathbf{T}_i)$ are called *extended edges*. In an extended T-mesh, some edges could overlap because of the extended edges. This information is characterized via *multiplicity*, which is the number of overlapping edges. Figure 3 a. shows a simple example to discuss the multiplicity. The original T-mesh is shown in (a) and two extended T-meshes are shown in (b) and (c), where the extended edges are marked as dashed lines. The multiplicity of some edges are shown with numbers closed to the corresponding edges. In the figure, we perturb some extended edges in order to make the overlapping edges clearer.

Definition 1 Given two extended T-meshes $\mathsf{T}_1, \mathsf{T}_2 \in ext(\mathsf{T})$, the notation $\mathsf{T}^1 \preceq \mathsf{T}^2$ means that $\mathsf{T}^1 \subseteq \mathsf{T}^2$ and the multiplicity of each edge in T^1 is equal or less than that of the corresponding edge in T^2. $\mathsf{T}^1 = \mathsf{T}^2$ if and only if $\mathsf{T}^1 \preceq \mathsf{T}^2$ and $\mathsf{T}^2 \preceq \mathsf{T}^1$.

In all the extended T-meshes, we are interested in two special ones, T_{ext} and T_{elem}, where



(a) bi-degree $(4, 2)$ **(b)** bi-cubic

Fig. 4 Two example AS++ T-meshes which are both not AS T-meshes. In these T-meshes, the yellow edges are T-junctions extensions. We can see that the extensions cause intersection which means that both T-meshes are not AS T-meshes

$$\mathsf{T}_{ext} = \mathsf{T} \bigcup_{\mathbf{T}_i \in \mathcal{S}_T} ext^f(\mathbf{T}_i), \mathsf{T}_{elem} = \mathsf{T} \bigcup_{\mathbf{A}_i \in \mathcal{S}_A} SK(\mathbf{A}_i).$$

For the T-mesh T_{ext}, it is obvious that it can be written as $\mathsf{T} \cup ext^f_{n_i}(\mathbf{T}_i)$, where $n_i = p$ if $\mathbf{T}_i$ is of type $\vdash$, $\dashv$ and $n_i = q$ if $\mathbf{T}_i$ is of type $\perp$, $\top$. For the T-mesh T_{elem}, each $SK(\mathbf{A}_i)$ contains the edges from the origin edges with some T-junctions extensions. Thus, it can also be written in the form of $\mathsf{T} \bigcup_{\mathbf{T}_i \in \mathcal{S}_T} ext^f_{n_i}(\mathbf{T}_i)$.

Definition 2 A T-mesh is called an analysis-suitable++ T-mesh (AS++ T-mesh for short) if and only if

1. For any two T-junctions $\mathbf{T}_i$, $\mathbf{T}_j$, if their extensions are not parallel, and denote $V = ext^f(\mathbf{T}_i) \cap ext^f(\mathbf{T}_j)$, then either $ext^f(\mathbf{T}_i) \cap ext^f(\mathbf{T}_j) = \varnothing$ (no V exists) or for any $\mathbf{A}_k$, $V \notin VK(\mathbf{A}_k)$;
2. $\mathsf{T}_{ext} = \mathsf{T}_{elem}$.

A T-spline defined on an AS++ T-mesh is called an analysis-suitable++ T-spline (AS++ T-spline).

Lemma 3.2 (a) and (b) in [17] state that AS T-meshes satisfy the requirement of AS++ T-meshes, i.e., AS T-meshes are always AS++ T-meshes. On the contrary, AS++ T-meshes are not always AS T-meshes. For example, the bi-degree (4, 2) T-mesh in Fig. 4 a. is an AS++ T-mesh but is not an AS T-mesh because the extensions of two red T-junctions intersect. We can check that any two non-parallel face extensions don't intersect and $\mathsf{T}_{ext} = \mathsf{T}_{elem}$ for this example T-mesh. The bi-cubic T-mesh in Fig. 4b. is also an AS++ T-mesh but is not an AS T-mesh. In this example, there are several T-junction intersections including two face extension intersections at the vertex V. But we can check that for any vertex $\mathbf{A}_i$, $V \notin VK(\mathbf{A}_i)$, which states that the T-mesh is an AS++ T-mesh.

4 Refineability and nestedness

This section discusses the refineability and nestedness for AS++ T-spline spaces. The refineability and nestedness of AS++ T-splines are the foundation of the optimized local refinement algorithm [51] and also one of the keys to build the optimal approximation properties (Theorem 2 and 3). In this section, we always assume that two T-spline spaces have consistent knot vectors, i.e., the same knot indices correspond to the same knot values. Thus, we use $\mathbf{S}(\mathsf{T})$ to denote the T-spline space for simplicity and explore the refineability and nestedness. In other words, we establish the conditions under which $\mathbf{S}(\mathsf{T}^1) \subseteq \mathbf{S}(\mathsf{T}^2)$ for two given AS++ T-meshes T^1 and T^2.

We should mention that the approach in this section is totally different from that for AS T-spline spaces. For the approach in [10,29], the refineability and nestedness are developed based on the characterization, linear independence and dual basis for AS T-splines spaces. The basic idea is to build the relation between the AS T-spline space and the spline space over the extended T-mesh without multiple knots and overlapping vertices. And then the condition can be obtained for the spline space over extended T-mesh. In the end, the theorem can be generalized to general case with the help of dual basis. However, the present approach directly proves the theorem without the results of linear independence and characterization.

First, we give the following key lemma (Lemma 1), which provides the condition to characterize whether a B-spline basis function belongs to an AS++ T-spline space. Given two knot vectors $\mathbf{s}$ and $\mathbf{t}$ and an AS++ T-spline space $\mathbf{S}(\mathsf{T})$, the middle of the knot vectors $\mathbf{s}$ and $\mathbf{t}$ can define two local knot vector inferred from the T-mesh using the rules in Sect. 2.1. These two local vectors have the following three possible relations.

1. The two knot vectors are identical;
2. There is at least one knot in the $\mathbf{s}$ (or $\mathbf{t}$) which is not in the knot inferred from the T-mesh. We say the B-spline has *extra knots*;
3. There is at lest one knot not in the $\mathbf{s}$ (or $\mathbf{t}$) which is in the knot inferred from the T-mesh. We say the B-spline has *missing knots*;

In order to verify that the B-spline basis function belongs to the AS++ T-spline space, it is sufficient to show that the B-spline basis can be split into a set of blending functions in the AS++ T-spline space through the knot insertions. For the given B-spline basis, we first prove that it can only belong to case (1) and (3). If it belongs to case (3), then we can insert any missing knot into the B-spline to create two new B-splines. And then we prove that the two new B-spline basis functions can also belong to case (1) and (3). By using the recursive process, we can prove that the given B-spline basis function belongs to the AS++ T-spline space.

We illustrate the basic idea by using the bi-cubic T-spline in Fig. 5. The given B-spline basis function is $B[0, 1, 3, 4, 5] \times B[1, 3, 5, 6, 7]$, which corresponds to the vertex $V_1 = (3, 5)$. However, by comparing with the blending function for V_1, there is one missing knot 2 in the s-knot vector. So we insert $s = 2$ into the B-spline and get two B-spline basis functions $B[1, 2, 3, 4, 5] \times B[1, 3, 5, 6, 7]$ and $B[0, 1, 2, 3, 4] \times B[1, 3, 5, 6, 7]$, which correspond to V_1 and V_2. For the vertex V_2, it is the same as the blending function of V_2. For the vertex V_1, there is one missing knot 2 in the t-knot vector. So we insert $t = 2$ into the B-spline and get two B-spline basis

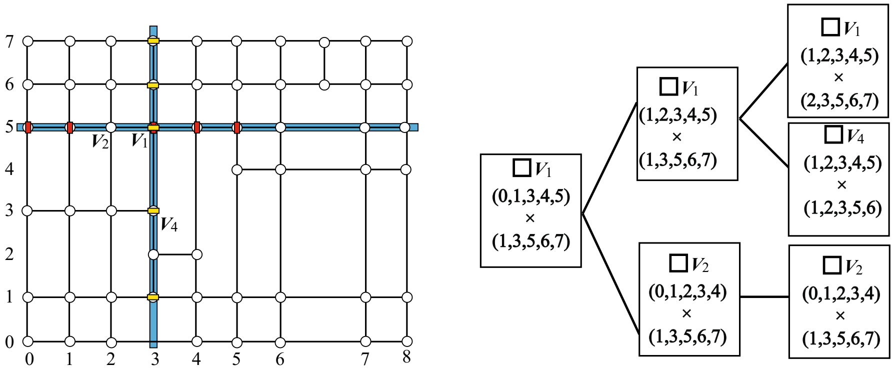
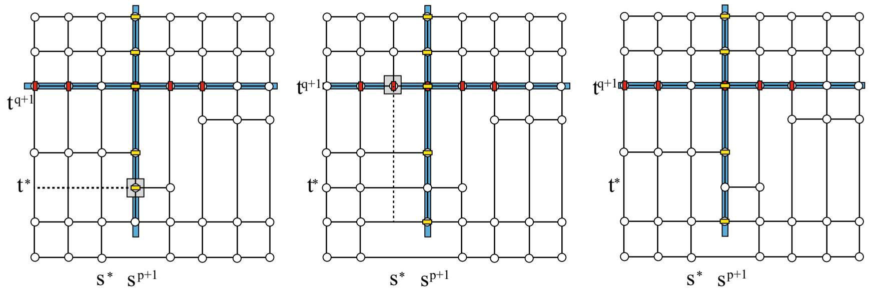


Fig. 5 Illustration of the basic idea of the proof



(a) Knot vector **s** has missing knot **(b)** Knot vector **t** has missing knot **(c)** Both knot vectors have missing knots

Fig. 6 Three possible cases of the missing knots in the B-spline basis function when d_1 and d_2 are both odd. The present figure is an example of $d_1 = d_2 = 3$

functions $B[1, 2, 3, 4, 5] \times B[2, 3, 5, 6, 7]$ and $B[1, 2, 3, 4, 5] \times B[1, 2, 3, 5, 6]$, which are identical with those blending functions for V_1 and V_4. Thus, we can conclude that the B-spline $B[0, 1, 3, 4, 5] \times B[1, 3, 5, 6, 7]$ belongs to the AS++ T-spline space.

Lemma 1 *Given an AS++ T-mesh* T *and two knot vectors* $\mathbf{s} = [s^0, \ldots, s^{d_1+1}]$ *and* $\mathbf{t} = [t^0, \ldots, t^{d_2+1}]$*. If* $SK(\mathbf{s} \times \mathbf{t}) \subseteq \mathsf{T}_{ext}$*, then* $B[\mathbf{s}]B[\mathbf{t}](s, t) \subseteq \mathbf{S}(\mathsf{T})$*, where* $B[\mathbf{s}]$ *and* $B[\mathbf{t}]$ *are degree* d_1 *and* d_2 *B-spline basis for knot vectors* $\mathbf{s}$ *and* $\mathbf{t}$.

Proof The lemma is proved via recursion by considering the degrees, which includes four possible cases.

1. Both d_1 and d_2 are odd, which are denoted as $d_1 = 2p + 1$ and $d_2 = 2q + 1$. In this case, as $SK(\mathbf{s} \times \mathbf{t}) \subseteq \mathsf{T}_{ext}$, so (s^{p+1}, t^{q+1}) corresponds to a vertex in the T-mesh. Otherwise, (s^{p+1}, t^{q+1}) lies on some extended edge and the extended edge violates the second AS++ T-mesh condition. Next we prove that the B-spline can not have extra knots. Otherwise, without loss of generality, we assume that the knot s^* is the extra knot. Then (s^*, t^{q+1}) lies on some extended edge and this extended edge violates the second AS++ T-mesh condition.

Thus, either the B-spline is identical with the blending function for the anchor (s^{p+1}, t^{q+1}) or there are some missing knots in the knot vector **s** or **t**. We use Fig. 6 to illustrate the three possible cases.

- There is some missing knot s^* in **s**.
  Without loss of generality, we assume $s^* \in (s^0, s^{p+1})$. In this case, we insert the knot s^* into the original B-spline and get two new B-splines with the new B-spline basis functions $B_1 = B\left[s^0, \ldots, s^*, \ldots, s^{d_1}\right] \times B\left[\mathbf{t}\right]$ and $B_2 = B\left[s^1, \ldots, s^*, \ldots, s^{d_1+1}\right] \times B[\mathbf{t}]$, respectively. It is obvious that B_2 corresponds to vertex $\left(s^{p+1}, t^{q+1}\right)$. And B_1 corresponds to vertex $\left(s^*, t^{q+1}\right)$ if $s^* \in \left(s^p, s^{p+1}\right)$ or corresponds to vertex $\left(s^p, t^{q+1}\right)$ if $s^* \in \left(s^0, s^p\right)$. If $\left(s^p, t^{q+1}\right)$ or $\left(s^*, t^{q+1}\right)$ is not a vertex in the AS++ T-mesh, then the symbol for $\left(s^p, t^{q+1}\right)$ or $\left(s^*, t^{q+1}\right)$ can only be $|$, which means that the symbol of $\left(s^{p+1}, t^{q+1}\right)$ must be $\vdash$. However, this violates the second condition of AS++ T-mesh because the extended T-mesh T_{ext} at least contains $p+2$-bays extension of the T-junction $\left(s^{p+1}, t^{q+1}\right)$.
  We then prove that both B-splines B_1 and B_2 cannot have extra knots. For the B-spline B_2, this is trivial because the situation is the same as the given B-spline. For the B-spline B_1, if it has extra knot, then it can only be in the knot vector **t**, which is supposed to be t^*. Then $\left(s^{p+1}, t^*\right)$ must be $\vdash$. However this violates the second condition of AS++ T-mesh because the T_{ext} contains the T-junction $\left(\sigma^{p+1}, \tau^{q+1}\right)$'s at least $p+2$-bays extension.
  Thus, after knot insertion, both B-spline basis functions B_1 and B_2 can also belong to case (1) and (3), which means that we can complete the proof via recursive process.
- There is some missing knot t^* in **t**.
  This case is the same as the first case except exchanging knot vectors **s** and **t**.
- There are missing knots s^* and t^* in the knot vectors **s** and **t**, respectively.
  Without loss of generality, we assume $s^* \in (s^0, s^{p+1})$ and $t^* \in (t^0, t^{q+1})$. In this case, if the symbol of (s^{p+1}, t^*) is of type $\vdash$, then we insert knot s^* into the knot vector **s** and use the same analysis as for the first case. Otherwise, we insert knot t^* into the knot vector **t** and use the same analysis as for the second case. In both cases, we can obtain that two new B-spline basis functions can only belong to case (1) and (3). Thus, we can complete the proof through recursive process.

2. Both d_1 and d_2 are even, which are denoted as $d_1 = 2p$ and $d_2 = 2q$.
   In this case, the knot vector $\mathbf{s} \times \mathbf{t}$ corresponds to the middle of face $\left[s^p, s^{p+1}\right] \times \left[t^q, t^{q+1}\right]$. We first prove that $\mathbf{s} \times \mathbf{t}$ cannot have extra knots. Otherwise, without loss of generality, assume that the knot is s^* which is not in the local knot vectors of the corresponding blending function. Then (s^*, t^q) is $\top$ or $\left(s^*, t^{q+1}\right)$ is $\perp$. In either case, as $SK(\mathbf{s} \times \mathbf{t}) \subseteq \mathsf{T}_{ext}$, so T_{ext} contains T-junction (s^*, t^q) or $\left(s^*, t^{q+1}\right)$'s at least $q+1$-bays extension, which violates the second AS++ T-mesh condition. So if the knot vectors are not identical with those of the corresponding blending functions, then they must contain some missing knots in the knot vector **s** or **t**. We use Fig. 7 to illustrate the three possible cases.

- There are some missing knot s^* in $\mathbf{s}$.
  Without loss of generality, we assume $s^* \in \left(s^0, s^{p+1}\right)$. Similarly, we insert the knot s^* into the original B-spline and get two new B-spline basis functions $B_1 = B\left[s^0, \ldots, s^*, \ldots, s^{d_1}\right] \times B[\mathbf{t}]$ and $B_2 = B\left[s^1, \ldots, s^*, \ldots, s^{d_1+1}\right] \times B[\mathbf{t}]$. We can prove that both B-splines B_1 and B_2 cannot have extra knots. If $s^* \in \left(s^0, s^p\right)$, it is trivial that the B-spline B_2 has no extra knots because all the knots are the same as the given B-spline except s^*. For the B-spline B_1, if it has extra knots, then the knots are from the knot vector $\mathbf{t}$, which is supposed to be t^*. Then the symbol of (s^p, t^*) must be $\vdash$. However this violates the second condition of AS++ T-mesh because the T_{ext} contains the T-junction (s^p, t^*)'s at least $p+1$-bays extension. If $s^* \in \left(s^p, s^{p+1}\right)$ and B-spline B_1 or B_2 has extra knot t^* in the vector $\mathbf{t}$, then the symbol of (s^*, t^*) is $\vdash$ or $\dashv$. However, both cases will violate the second condition of AS++ T-mesh because the extended T-mesh T_{ext} contains the T-junction (s^*, t^*)'s at least $p+1$-bays extension.
  Thus, both B-spline basis functions B_1 and B_2 can also belong to case (1) and (3), which means we can complete the proof via recursive process in this case.
- There are some missing knot t^* in $\mathbf{t}$.
  This case is the same as the first case except exchanging the two knot vectors $\mathbf{s}$ and $\mathbf{t}$.
- There are missing knots s^* and t^* in the knot vectors $\mathbf{s}$ and $\mathbf{t}$.
  Without loss of generality, we assume $s^* \in \left(s^0, s^{p+1}\right)$ and $t^* \in \left(t^0, t^{q+1}\right)$. If $s^* \in \left(s^0, s^p\right)$ and the symbol of (s^p, t^*) is of type $\vdash$, then we insert knot s^* into the knot vector $\mathbf{s}$ and use the same analysis as for the first case. Otherwise, we insert knot t^* into the knot vector $\mathbf{t}$ and use the same analysis as for the second case. If $s^* \in \left(s^p, s^{p+1}\right)$ and the symbol of (s^*, t^*) is of type $\vdash$ or $\dashv$, then we insert knot s^* into the knot vector $\mathbf{s}$ and use the same analysis as for the first case. Otherwise, we insert knot t^* into the knot vector $\mathbf{t}$ and use the same analysis as for the second case. In all these cases, we can obtain that two new B-spline basis functions can only belong to case (1) and (3). Thus, the proof can be finished through the recursive process.

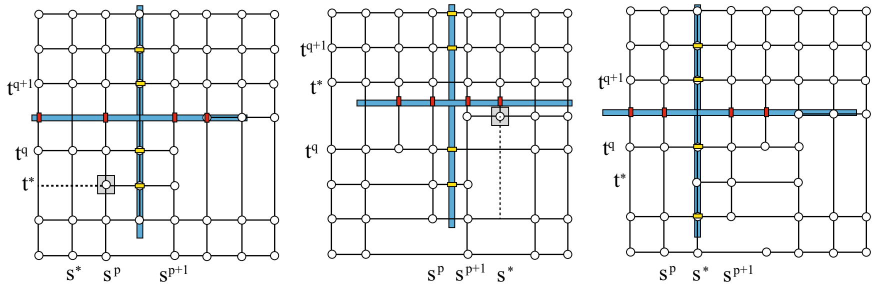
3. d_1 is even and d_2 is odd, which are denoted as $d_1 = 2p$ and $d_2 = 2q + 1$.
   The proof for this case is similar as the above two cases except using the even degree case for the vector $\mathbf{s}$ and using the odd one for the vector $\mathbf{t}$.
4. d_1 is odd and d_2 is even.
   This case is exactly the same as the case that d_1 is even and d_2 is odd.

In summary, for all possible bi-degrees, we can complete the proof via recursive process. □

Now, we are ready to provide the nestedness of two AS++ T-spline spaces.

Theorem 1 *Given two bi-degree (d_1, d_2) AS++ T-meshes, T^1 and T^2, if $\mathsf{T}^1 \subseteq \mathsf{T}^2$ and $\mathsf{T}^1_{ext} \preceq \mathsf{T}^2_{ext}$, then $\mathbf{S}\left(\mathsf{T}^1\right) \subseteq \mathbf{S}\left(\mathsf{T}^2\right)$.*

Proof It is sufficient to prove that for any blending function $T_i^1(s,t) \in \mathbf{S}(\mathsf{T}^1)$, it holds that $T_i^1(s,t) \in \mathbf{S}(\mathsf{T}^2)$. Suppose the i-th anchor in the T-mesh T^1 is $\mathbf{A}_i^1$, then $SK(\mathbf{A}_i^1) \subseteq \mathsf{T}^1_{ext}$ because the T-mesh T^1 is an AS++ T-mesh. Thus $SK(\mathbf{A}_i^1) \subseteq \mathsf{T}^1_{ext} \subseteq$



(a) Knot vector **s** has missing knot **(b)** Knot vector **t** has missing knot **(c)** Both knot vectors have missing knots

Fig. 7 Three possible cases of the missing knots in the B-spline basis function when d_1 and d_2 are both even. The present figure is an example of $d_1 = d_2 = 2$

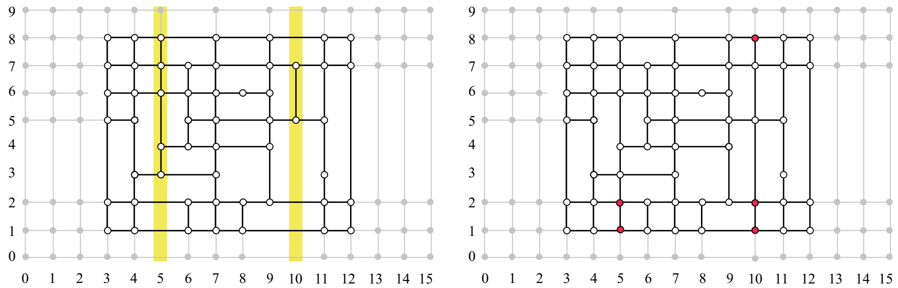


Fig. 8 An example T-mesh (**a**) of bi-degree (5, 1) is not an admissible T-mesh while the corresponding admissible T-mesh is shown in **b**

T^2_{ext}. According to Lemma 1, we can obtain $T_i^1(s,t) \in \mathbf{S}(\mathsf{T}^2)$ which completes the proof. □

Definition 3 A T-mesh is admissible if for any vertex (i, j), if $0 \le i \le d_1$ or $c \le i \le c + d_1$, then the vertex cannot be $\perp$ and $\top$, and if $0 \le j \le d_2$ or $r \le j \le r + d_2$, then the vertex cannot be $\vdash$ and $\dashv$.

Figure 8 shows two example bi-degree (5, 1) T-meshes. The T-mesh a is not an admissible T-mesh because the symbols in the yellow lines violate the assumption. For example, the symbols of (5, 3) and (10, 5) are both $\perp$. In order to convert the T-mesh into an admissible T-mesh, one possible solution is the T-mesh in Fig. 8b.

Proposition 1 *If a bi-degree (d_1, d_2) AS++ T-mesh is an admissible T-mesh, then the associated T-spline space contains the space of bi-degree (d_1, d_2) polynomials.*

Proof For bi-degree (d_1, d_2), let T_p be the tensor-product T-mesh for the knot vectors $\mathbf{s}_p = [s_0, s_1, \ldots, s_{2d_1+1}]$ and $\mathbf{t}_p = [t_0, t_1, \ldots, t_{2d_2+1}]$, where $s_0 = s_1 = \cdots = s_{d_1}$, $s_{d_1+1} = \cdots = s_{2d_1+1}$, $t_0 = t_1 = \cdots = t_{d_2}$, $t_{d_2+1} = \cdots = t_{2d_2+1}$. It is easy to see that the associated T-spline space $\mathbf{S}(\mathsf{T}_p, \mathbf{s}_p, \mathbf{t}_p)$ is the space of bi-degree (d_1, d_2) polynomials. For any other bi-degree (d_1, d_2) AS++ admissible T-mesh T, since the mesh is admissible, so $\mathsf{T}_p \preceq \mathsf{T}$. Thus, $\mathbf{S}(\mathsf{T}_p, \mathbf{s}_p, \mathbf{t}_p) \subseteq \mathbf{S}(\mathsf{T}, \mathbf{s}, \mathbf{t})$, i.e., the associated AS++ T-spline space contains the space of bi-degree (d_1, d_2) polynomials. □

Corollary 1 *If an AS++ T-mesh is an admissible T-mesh, then the associated T-spline space can reproduce constants. In other words, there exist weights ω_i, such that $\sum_{i=1}^{n_A} \omega_i T_i(s,t) = 1$.*

Proof This immediately follows from Proposition 1 that the constant function belongs to the AS++ T-spline if the T-mesh is admissible. And the weights can be computed from the extraction of the contributions from the basis functions in the space of bi-degree (d_1, d_2) polynomials, which means the weights are positive. □

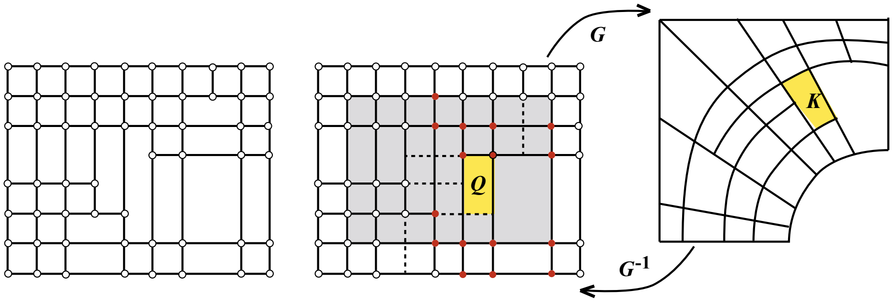
Remark 1 T-splines can be classified into three categories: standard, semi-standard and non-standard. If for any valid knot intervals, there exist a set of weights ω_i such that $\sum_{i=1}^{n} \omega_i T_i(s,t) \equiv 1$, then the T-spline is *semi-standard*. Especially, if all the weights must be ones, then the T-spline is *standard*. Otherwise, the T-spline is *non-standard*, i.e., there exists a valid set of knot intervals such that $\sum_{i=1}^{n} \omega_i T_i(s,t) \neq 1$ for any weights ω_i. Sederberg et al. posed an open problem in [45]: the above definition is an algebraic statement of necessary and sufficient conditions for a T-spline space to be standard and semi-standard. What is the topological interpretation of those conditions? The above Corollary 1 can answer part of the open problem: an admissible AS++ T-spline is semi-standard and it is standard when it is an admissible AS T-spline.

5 Approximation

In this section, we develop the approximation theorems for arbitrary degree AS++ T-spline spaces. In order to do this, one needs firstly define a suitable projection operator to project a function in L^2 to the AS++ T-spline space. We are using the dual basis for B-splines in [41] to construct the projection operator. However, we can construct the projector only when the AS++ T-splines satisfy the DE condition (Definition 4). As proved in [30], the bi-cubic AS++ T-splines satisfy the DE condition which means they achieve the optimal approximation order. How to prove that the arbitrary degrees AS++ T-splines satisfy the DE condition will be left as future work.

Before giving the details, we introduce some additional notations. As shown in Fig. 9, let $\mathbf{s} = [s_0, s_1, \ldots, s_{c+d_1-1}, s_{c+d_1}]$ and $\mathbf{t} = [t_0, t_1, \ldots, t_{r+d_2-1}, t_{r+d_2}]$ be the global knot vectors, and $\mathfrak{D} = [s_{d_1}, s_c] \times [t_{d_2}, t_r]$, a rational AS++ T-spline geometric map $G : \mathfrak{D} \to \Omega$ is given by

$$G(s,t) = \frac{\sum_{i=1}^{n_A} \omega_i G_i T_i(s,t)}{\sum_{j=1}^{n_A} \omega_j T_j(s,t)} \doteq \sum_{i=1}^{n_A} \omega_i G_i R_i(s,t), \tag{1}$$



(a) Index T-mesh **(b)** Parametric T-mesh **(c)** Physical T-mesh

Fig. 9 A geometric map is defined by an AS++ T-spline. **a** is the index T-mesh, **b** shows the parametric domain of the T-spline and **c** is the geometric map

where

$$R_i(s,t) = \frac{\omega_i T_i(s,t)}{\sum_{j=1}^{n_A} \omega_j T_j(s,t)} \doteq \frac{\omega_i T_i(s,t)}{\omega(s,t)}.$$

We have three different spline spaces. The AS++ T-spline space $\mathbf{S}(\mathsf{T}, \mathbf{s}, \mathbf{t})$ which is the linear space spanned by the blending functions $T_i(s,t)$, the AS++ T-spline space with weight function $\mathbf{R}(\mathsf{T}, \mathbf{s}, \mathbf{t})$ which is the linear space spanned by $R_i(s,t)$ and the AS++ T-spline space $\mathbf{V}(\mathsf{T}, \mathbf{s}, \mathbf{t})$ on Ω, which is the push-forward of the space $\mathbf{R}(\mathsf{T}, \mathbf{s}, \mathbf{t})$ on the patch $\mathfrak{D}$, i.e.,

$$\mathbf{V}(\mathsf{T}, \mathbf{s}, \mathbf{t}) = span\{R_i(s,t) \circ G^{-1}\}.$$

Let Q be a generic (open) element in the element T-mesh T_{elem}, $Q^i = (s_{\sigma_i^0}, s_{\sigma_i^{d_1+1}}) \times (t_{\tau_i^0}, t_{\tau_i^{d_2+1}})$, $I(Q) = \{j | Q^j \bigcap Q \neq \emptyset\}$, and $\widetilde{Q}^i = \bigcup_{j \in I(Q)} \{Q^j\}$. Let $K = G(Q)$, $K^i = G(Q^i)$ and $\widetilde{K}^i = G(\widetilde{Q}^i)$. Furthermore, we denote $h_s(Q)$ and $h_t(Q)$ be the length in the s and t coordinate directions of the element Q, and $h_{s,i} = s_{\sigma_i^{d_1+1}} - s_{\sigma_i^0}$, $h_{t,i} = t_{\tau_i^{d_2+1}} - t_{\tau_i^0}$. Throughout this section, we will make use of the classical Lebesgue spaces $L^p(\Omega)$ with the norm $||\cdot||_{L^p}$ for $1 \leq p \leq \infty$. We also will need the Sobolev spaces $W^{k,p}(\Omega)$, for k a positive integer and $1 \leq p \leq \infty$, endowed with the usual norm $||\cdot||_{W^{k,p}}$ and semi-norm $|\cdot|_{W^{k,p}}$. For the Hilbert space $W^{k,2}$, we will switch to the notation $H^k(\Omega)$, and accordingly $||\cdot||_{H^k}$ and $|\cdot|_{H^k}$ stand for the norm and semi-norm, respectively. We also assume the T-mesh satisfies the mesh regularity assumption [15].

5.1 Dual basis for univariate B-splines

We briefly introduce univariate B-splines with the aim of recalling a few results needed in the next subsections.

Given the integers $n, d > 0$, let a knot vector $[s_0, s_1, s_1, \ldots, s_{n+d}, s_{n+d}]$ be given, with ordered knots $s_i \leq s_{i+1}$ and $s_i < s_{i+d+1}$. For $0 \leq i \leq n-1$, we denote by $B_i(x) = B[s_i, s_{i+1}, \ldots, s_{i+d+1}](x)$ the degree d B-spline function associated with the knots $\{s_i, s_{i+1}, \ldots, s_{i+d+1}\}$.

Following [41], we can define suitable functionals which are dual to the B-spline basis functions. Let

$$G_j(x) = g\left(\frac{2x - s_{j+d+1} - s_j}{s_{j+d+1} - s_j}\right),$$

where $g(x)$ is the transition function defined in [41],

$$g(x) = \begin{cases} 0, & x < -1; \\ \frac{d+1}{2}\int_{-1}^{x} B[y_0, y_1, \ldots, y_{d+1}](t)dt, & x \in [-1, 1); \\ 1, & x \geq 1. \end{cases} \tag{2}$$

with $y_i = \cos\frac{d+1-i}{d+1}\pi$ and $\phi_j(x) = \frac{(x-s_{j+1})\ldots(x-s_{j+d})}{d!}$.

Then the Schumaker's functional $\lambda[s_j, \ldots, s_{j+d+1}](\cdot)$ associated with knot vector $\{s_j, \ldots, s_{j+d+1}\}$ can be defined as follows,

$$\lambda[s_j, \ldots, s_{j+d+1}](f) = \int_{s_j}^{s_{j+d+1}} f(x)D^{(d+1)}(G_j(x)\phi_j(x))dx,$$

and it has the property that

$$\lambda[s_i, \ldots, s_{i+d+1}](B_j(x)) = \delta_{i,j}, \tag{3}$$

where δ_{ij} is the Kronecker delta function.

According to Theorem 4.41 in [41], we have the following lemma.

Lemma 2 *For any $f(x) \in L^2[s_j, s_{j+d+1}]$, there exists a constant C independent of the knot s_j, such that*

$$|\lambda[s_j, \ldots, s_{j+d+1}](f)| \leq Ch_j^{-\frac{1}{2}}||f||_{L^2[I_j]},$$

where $I_j = (s_j, s_{j+d+1})$ and $h_j = s_{j+d+1} - s_j$.

5.2 Dual basis for AS++ T-splines

For each vertex $\mathbf{A}_i$, suppose ${}_i = (\alpha_i^j)$, $j = 0, 1, \ldots, d_1 + 1$ and $\mathbf{fi}_i = (\beta_i^j)$, $j = 0, 1, \ldots, d_2 + 1$ be two local index vectors, where $\alpha_i^0 \geq \sigma_i^0, \alpha_i^{d_1+1} \leq \sigma_i^{d_1+1}, \beta_i^0 \geq \tau_i^0, \beta_i^{d_2+1} \leq \tau_i^{d_2+1}$, and $\alpha_i^p = \sigma_i^p, \beta_i^q = \tau_i^q$. And then, we associate a suitable functional $\lambda_i(\cdot) = \lambda(s_{_i}) \otimes \lambda(t_{\mathbf{fi}_i})(\cdot)$ for the i-th anchor, where $\lambda(s_{_i})$ and $\lambda(t_{\mathbf{fi}_i})$ are the dual bases defined in Sect. 5.1 in terms of the knot vectors $s_{_i}$ and $t_{\mathbf{fi}_i}$. The dual matrix D is defined to be a $n_A \times n_A$ matrix with $D_{i,j} = \lambda_i(T_j)$.

Definition 4 An AS++ T-spline satisfies the DE (dual-basis-existence) condition if there exist local knot index vectors ${}_i$ and $\mathbf{fi}_i$ such that the dual matrix is invertible.

Lemma 3 *If an AS++ T-spline satisfies the DE condition, then the blending functions for the AS++ T-spline are linearly independent.*

Proof It is sufficient to prove that if $\sum_{i=1}^{n_A} c_i T_i(s,t) = 0$ then $c_i = 0$. And we can apply all the functionals $\lambda_i(\cdot)$ on the above equation and we can get that $DC = 0$, where $C = \left[c_1, \ldots, c_{n_A}\right]^T$. As the matrix D is invertible, so we can conclude that $c_i = 0$, which completes the proof. □

An important consequence of satisfying the DE condition is that we can build a projection operator $\Pi : L^2(\mathfrak{D}) \to \mathbf{S}(\mathsf{T}, \mathbf{s}, \mathbf{t})$, defined by

$$\Pi(f)(s,t) = \sum_{i=1}^{n_A} \Lambda_i(f) T_i(s,t), \forall f \in L^2(\mathfrak{D}) \tag{4}$$

where $\left(\Lambda_1, \ldots, \Lambda_{n_A}\right)^T = M\left(\lambda_1, \ldots, \lambda_{n_A}\right)^T$, $M = D^{-1}$.

Similarly, we can build a projection operator for spline space $\mathbf{R}(\mathsf{T}, \mathbf{s}, \mathbf{t})$, $\Pi_r : L^2(\mathfrak{D}) \to \mathbf{R}(\mathsf{T}, \mathbf{s}, \mathbf{t})$, defined by

$$\Pi_r(f)(s,t) = \frac{\sum_{i=1}^{n_A} \Lambda_i(\omega f) T_i(s,t)}{\omega(s,t)}, \forall f \in L^2(\mathfrak{D}). \tag{5}$$

It is straightforward to check that Π and Π_r are both projection operators. The dual basis grants a very powerful tool to prove approximation properties for AS++ T-spline spaces, which is one of the fundamental conditions for a spline space to be used in the IGA problems. According to Lemma 2 and the construction of dual basis for AS++ T-splines, we have the following lemma.

Lemma 4 *For any $f \in L^2(\mathfrak{D})$, there exists a constant C independent of the knot vectors, such that*

$$|\lambda_i(f)| \leq C(h_{s,i} h_{t,i})^{-\frac{1}{2}} ||f||_{L^2(\Omega)}.$$

Proof This can be directly derived from Lemma 2. □

5.3 Approximation for arbitrary degree AS++ T-splines

In this subsection, we derive the approximation properties of the AS++ T-spline space. Let $d = \min\{d_1, d_2\}$.

Lemma 5 *Given an AS++ T-mesh T and two knot vectors* $\mathbf{s}, \mathbf{t}$, *suppose the AS++ T-spline satisfies the DE condition, then there exists a constant C independent of T,* $\mathbf{s}$ *and* $\mathbf{t}$ *such that for any generic element Q in the element T-mesh T_{elem},*

$$||\Pi(f)||_{L^2(Q)} \leq C||f||_{L^2(\widetilde{Q})}. \tag{6}$$

Proof Since we have proved that all the constant functions belong to the AS++ T-spline space $\mathbf{S}(\mathsf{T}, \mathbf{s}, \mathbf{t})$, there exist some weights $\omega_i \geq 0$ such that

$$\sum_{i=1}^{n_A} \omega_i T_i = 1. \tag{7}$$

On the other hand, when we apply $f = 1$ in the equation (4) and we can get $\omega_i = \Lambda_i(1)$. Since $\left(\Lambda_1, \ldots, \Lambda_{n_A}\right)^T = M \left(\lambda_1, \ldots, \lambda_{n_A}\right)^T$, so

$$\left(\Lambda_1(1), \ldots, \Lambda_{n_A}(1)\right)^T = M \left(\lambda_1(1), \ldots, \lambda_{n_A}(1)\right)^T = M \left(1, \ldots, 1\right)^T.$$

Thus,

$$\omega_i = \sum_{j=1}^{n_A} m_{i,j},$$

where $m_{i,j}$ is the i-th row j-th column element of matrix M.

Now for any given Q, and any point $(s, t) \in Q$,

$$\begin{aligned}
|\Pi(f)(s,t)|^2 &= |\sum_{i=1}^{n_A} \Lambda_i(f) T_i(s,t)|^2 = |\Lambda(f)T|^2 \\
&= \left| \sum_{i=1}^{n_A} \lambda_i(f) \left(\sum_{j=1}^{n_A} m_{i,j} T_j(s,t) \right) \right|^2 \\
&\leq C_1 \max |\lambda_i(f)|^2 \\
&\leq C \max_{i \in I(Q)} (h_{s,i} h_{t,i})^{-1} ||f||^2_{L^2(Q)} \\
&\leq C \max_{i \in I(Q)} (h_s(Q) h_t(Q))^{-1} ||f||^2_{L^2(Q)}.
\end{aligned}$$

The first inequality satisfies because $\sum_{j=1}^{n_A} m_{i,j} T_j(s,t) \leq \max\{\omega_i\} \doteq \omega_k$. On the other hand, since $\sum \omega_i T_i = 1$ and $\omega_i \geq 0$, so $\omega_k \leq \frac{1}{\max_{(s,t)\in\mathfrak{D}} T_k(s,t)}$. Denote $M = \max_{(s,t)\in\mathfrak{D}} T_k(s,t)$, and we can get

$$M \left(s_{\sigma_k^{d_1+1}} - s_{\sigma_k^0} \right) \left(t_{\tau_k^{d_2+1}} - t_{\tau_k^0} \right) \geq \int T_k(s,t) ds dt = \frac{\left(s_{\sigma_k^{d_1+1}} - s_{\sigma_k^0} \right) \left(t_{\tau_k^{d_2+1}} - t_{\tau_k^0} \right)}{(d_1+1)(d_2+1)},$$

which leads $M \geq \frac{1}{(d_1+1)(d_2+1)}$. So $\omega_k \leq (d_1+1)(d_2+1) \leq (d+1)^2$, i.e., we can choose C_1 to be $(d+1)^4$.

Note that the constant C appearing above is independent of any other variable or parameter. Since the bound above holds for any $(s, t) \in Q$, integrating on the element Q and applying the above equation yields

$$||\Pi(f)||_{L^2(Q)} \leq h_s(Q)h_t(Q)||\Pi(f)||_{L^\infty(Q)} \leq C||f||_{L^2(\widetilde{Q})}.$$

□

Theorem 2 *For any Q in an AS++ T-mesh which satisfies the DE condition, denote $\widehat{Q}$ be the smallest rectangle containing $\widetilde{Q}$ and h be the diameter of $\widehat{Q}$, then there exists a constant C independent of T, $\mathbf{s}$ and $\mathbf{t}$ such that for any $0 \leq k \leq r \leq d+1$,*

$$||f - \Pi_r(f)||_{H^k(Q)} \leq Ch^{r-k}||f||_{H^r(\widehat{Q})}, \forall f \in H^r(\widehat{Q}), \tag{8}$$

where $H^r(\Omega)$ indicates the Sobolev space of order r.

Proof First, we prove the theorem for the projection operator Π. Let p be any bi-degree d polynomials on $[0, 1]^2$. Since $p \in \mathbf{S}(\mathsf{T}, \mathbf{s}, \mathbf{t})$, it follows for any $0 \leq k \leq r \leq d+1$,

$$\begin{aligned} ||f - \Pi(f)||_{H^k(Q)} &= ||f - p + p - \Pi(f)||_{H^k(Q)} \\ &\leq ||f - p||_{H^k(Q)} + ||\Pi(f - p)||_{H^k(Q)} \\ &\leq (1 + C_1)||f - p||_{H^k(\widetilde{Q})} \\ &\leq (1 + C_1)||f - p||_{H^k(\widehat{Q})}. \\ &\leq Ch^{r-k}||f||_{H^r(\widehat{Q})}. \end{aligned} \tag{9}$$

As the result for the projection operator Π_r, it can be proved that

$$\begin{aligned} ||f - \Pi_r(f)||_{H^k(Q)} &= \left|\left|\frac{\omega f - \Pi(\omega f)}{\omega}\right|\right|_{H^k(Q)} \\ &\leq C\sum_{i=0}^{k}\left|\frac{1}{\omega}\right|_{W^{i,\infty}(Q)}\left|\left|\frac{\omega f - \Pi(\omega f)}{\omega}\right|\right|_{H^{k-i}(Q)} \\ &\leq Ch^{r-k}\sum_{i=0}^{k}\left|\frac{1}{\omega}\right|_{W^{i,\infty}(Q)}||\omega f||_{H^{r-i}(\widehat{Q})} \\ &\leq Ch^{r-k}\sum_{i=0}^{k}\left|\frac{1}{\omega}\right|_{W^{i,\infty}(Q)}\sum_{j=0}^{r-i}\sum_{Q'\cap\widehat{Q}\neq\emptyset}|\omega|_{W^{j,\infty}(Q')}||f||_{H^{r-(i+j)}(Q')}. \end{aligned}$$

Since $0 \leq i + j \leq r$ in the last summations, we complete the proof with the constant C depending on the weight function ω. □

The above theorem can directly derive the following optimal convergence rates for bi-cubic AS++ T-spline spaces in the parametric domain.

Proposition 2 *For any Q in a bi-cubic AS++ T-mesh, denote $\widehat{Q}$ be the smallest rectangle containing $\widetilde{Q}$ and h be the diameter of $\widehat{Q}$, then there exists a constant C independent of* T*,* **s** *and* **t** *such that for any $0 \le k \le r \le 4$,*

$$||f - \Pi_r(f)||_{H^k(Q)} \le Ch^{r-k}||f||_{H^r(\widehat{Q})}, \forall f \in H^r(\widehat{Q}). \tag{10}$$

Proof The result follows immediately from Theorem 4.10 in [30] and Theorem 2. □

5.4 Approximation with AS++ T-splines in the physical domain

Define the projector $\Pi_v : L^2(\Omega) \to \mathbf{V}(\mathsf{T}, \mathbf{s}, \mathbf{t})$ as

$$\Pi_v(f) = (\Pi_r(f \circ G)) \circ G^{-1}. \tag{11}$$

And we can give the estimates for the change of variable from the patch to the physical domain in the following Theorem 3.

Theorem 3 *For any Q in an AS++ T-mesh which satisfies the DE condition, denote $\widehat{Q}$ be the smallest rectangle containing $\widetilde{Q}$ and $K = G(Q)$, $\widehat{K} = G(\widehat{Q})$, h be the diameter of $\widehat{Q}$, then there exists a constant C independent of* T*,* **s** *and* **t** *such that for any $0 \le k \le r \le d+1$,*

$$||f - \Pi_v(f)||_{H^k(K)} \le Ch_K^{r-k} \sum_{i=0}^{r} ||\nabla G||^{i-r}_{L^\infty(\widehat{K})} ||f||_{H^i(\widehat{K})}, \forall f \in L^2(\Omega) \cap H^r(\widehat{K}),$$

where $h_K = ||\nabla G||_{L^\infty(Q)} h$ is the element size in the physical domain.

Proof According to the definition of $\Pi_v(f)$, we have

$$\begin{aligned} ||f - \Pi_v(f)||_{H^k(K)} &= ||f - (\Pi_r(f \circ G)) \circ G^{-1}||_{H^k(K)} \\ &\le C_1 ||det \nabla G||^{1/2}_{L^\infty(Q)} ||\nabla G||^{-k}_{L^\infty(Q)} \sum_{i=0}^{k} ||f \circ G - \Pi_r(f \circ G)||_{H^i(Q)}. \end{aligned}$$

Apply Theorem 2 for $f \circ G$ on each term of the above inequality, we can obtain

$$\begin{aligned} ||f \circ G - \Pi_r(f \circ G)||_{H^i(Q)} &\le C_2 h^{l-k} ||f \circ G||_{H^{l+i-k}(\widehat{Q})} \\ &\le C_2 h^{l-k} \sum_{j=0}^{l+i-k} ||f \circ G||_{H^j(\widehat{Q})}. \end{aligned}$$

Combine the above two estimates and we get

$$||f - \Pi_v(f)||_{H^k(K)} \leq C_1 ||det\nabla G||^{1/2}_{L^\infty(Q)} ||\nabla G||^{-k}_{L^\infty(Q)} h^{l-k} \sum_{j=0}^{l} ||f \circ G||_{H^j(\widehat{Q})}$$

$$\leq C_1 ||det\nabla G||^{1/2}_{L^\infty(Q)} ||\nabla G||^{-k}_{L^\infty(Q)} h^{l-k} \sum_{j=0}^{l} \sum_{Q' \cap \widehat{Q} \neq \emptyset} ||f \circ G||_{H^j(Q')}.$$

Again, for the term of $||f \circ G||_{H^j(Q')}$, we can also get

$$||f \circ G||_{H^j(Q')} \leq C ||det\nabla G^{-1}||^{1/2}_{L^\infty(K')} \sum_{i=0}^{j} ||\nabla G||^{i}_{L^\infty(Q')} ||f||_{H^i(K')},$$

where $K' = G(Q')$.

By coalescing the above summation onto a single sum, we have

$$||f - \Pi_v(f)||_{H^k(K)} \leq C ||\nabla G||^{-k}_{L^\infty(Q)} h^{l-k} \sum_{j=0}^{l} \sum_{K' \cap \widehat{K} \neq \emptyset} ||\nabla G||^{j}_{L^\infty(Q')} ||f||_{H^j(K')}$$

$$\leq C ||\nabla G||^{-k}_{L^\infty(\widehat{Q})} h^{l-k} \sum_{j=0}^{l} ||\nabla G||^{j}_{L^\infty(\widehat{Q})} ||f||_{H^j(\widehat{K})}$$

$$\leq C \frac{||\nabla G||^{-k}_{L^\infty(\widehat{Q})}}{||\nabla G||^{-k}_{L^\infty(Q)}} h_K^{l-k} \sum_{j=0}^{r} ||\nabla G||^{j-r}_{L^\infty(\widehat{Q})} ||f||_{H^j(\widehat{K})},$$

which completes the proof. □

Similarly the optimal convergence rates for bi-cubic AS++ T-spline spaces in the physical domain can be derived directly.

Proposition 3 *For any Q in a bi-cubic AS++ T-mesh, denote $\widehat{Q}$ be the smallest rectangle containing $\widetilde{Q}$ and $K = G(Q)$, $\widehat{K} = G(\widehat{Q})$, h be the diameter of $\widehat{Q}$, then there exists a constant C independent of* T, **s** *and* **t** *such that for any $0 \leq k \leq r \leq 4$,*

$$||f - \Pi_v(f)||_{H^k(K)} \leq C h_K^{r-k} \sum_{i=0}^{r} ||\nabla G||^{i-r}_{L^\infty(\widehat{K})} ||f||_{H^i(\widehat{K})}, \forall f \in L^2(\Omega) \cap H^r(\widehat{K}),$$

where $h_K = ||\nabla G||_{L^\infty(Q)} h$ is the element size in the physical domain.

Proof The result follows immediately from Theorem 4.10 in [30] and Theorem 3. □

6 Conclusion

We have established several important properties for AS++ T-spline spaces. We developed a characterization for a B-spline basis function belonging to an AS++ T-spline space and provided the condition for two AS++ T-spline spaces to be nested. The basic technique we are using is a simple recursive process to construct the linear combination of blending functions in an AS++ T-spline space. This provides the theoretical foundation for the AS++ local refinement algorithm in [51] and also the approximation properties. Using the nestedness of AS++ T-spline space and the assumption of the existence of dual basis, we then proved several basic approximation results for arbitrary degree AS++ T-spline spaces in the parametric domain and physical domain.

For the mathematical properties of arbitrary degree AS++ T-splines, the main left future work is to prove the sufficient condition for the existence of dual basis. We conjecture that the AS++ T-splines satisfy the sufficient condition for the existence of dual basis, which will be left for future work. On the other hand, since we have defined a less restricted T-splines which satisfy all the good properties, so how to construct the local refinement algorithm with as less as possible control points and how to apply the arbitrary degree AS++ T-splines for the real IGA applications are both very important and interesting future problems.

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