PROBLEM SET 1
DUE: FEB.24

Problem 1

(1) Let G be the set of integers which can be written as the sum of two squares, that is, $G = \{x \in \boldsymbol{Z} | x = a^2 + b^2, a, b \in \boldsymbol{Z}\}$. Show that G is a monoid under the multiplication of integers.

(2) Let H be the set of integers which can be written as the sum of four squares, that is, $H = \{x \in \boldsymbol{Z} | x = a^2 + b^2 + c^2 + d^2, a, b, c, d \in \boldsymbol{Z}\}$. Show that H is also a monoid under the multiplication of integers.

*(3) Let N be the set of integers which can be written as the sum of eight squares, that is, $N = \{x \in \boldsymbol{Z} | x = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2, a_1, ..., a_8 \in \boldsymbol{Z}\}$. Is it true that N is also a monoid under the multiplication of integers?

Problem 2

Consider the lattice $\boldsymbol{Z}^2 \subset \boldsymbol{R}^2$, the points on the plane which has integral coordinates, and let C be the convex cone formed by two rays starting from the origin $(0, 0)$ (where we assume these two rays do not lie in the same straight line). Then show that $S = C \cap \boldsymbol{Z}^2$ is a monoid. Try to prove that the group generated by the monoid S is $\boldsymbol{Z}^2$.

Problem 3

Show that every group of order ≤ 5 is abelian.

Problem 4

Show that there are only two non-isomorphic groups of order 4, namely the cyclic one, and the product of two cyclic groups of order 2.

Problem 5

Let G be a group such that $\mathrm{Aut}(G)$ is cyclic. Prove that G is abelian.

Problem 6

Prove the following statement:

Let G be a group, S a set of generators for G, and $G^{'}$ another group. Let $f : S \to G^{'}$ be a map. If there exists a homomorphism $\overline{f}$ of G into $G^{'}$ whose restriction to S is f, then there is only one.

Problem 7

Let G be a group and let H ,$H^{'}$ be subgroups. By a **double coset** of H ,$H^{'}$ one means a subset of G of the form $HxH^{'}$.

(1) Show that G is a disjoint union of double cosets.

(2) Let $\{c\}$ be a family of representatives for the double cosets. For each $a \in G$ denote by $[a]H^{'}$ the conjugate $aH^{'}a^{-1}$ of $H^{'}$. For each c we have a decomposition into ordinary cosets

$$H = \bigcup_{c} x_c(H \cap [c]H^{'})$$

where x_c is a family of elements of H ,depending on C. Show that the elements $x_c c$ form a family of left coset representatives for $H^{'}$ in G; that is,

$$G = \bigcup_{c} \bigcup_{x_c} x_c c H^{'},$$

and the union is disjoint.

Problem 8

Let G be a group and H a subgroup of finite index. Prove that there is only a finite number of right cosets of H , and that the number of right cosets is equal to the number of the left cosets.