PROBLEM SET 10
DUE: May. 12

Problem 1

Let $E = \boldsymbol{Q}(\alpha)$, where α is a root of the equation

$$\alpha^3 + \alpha^2 + \alpha + 2 = 0.$$

Express $(\alpha^2 + \alpha + 1)(\alpha^2 + \alpha)$ and $(\alpha + 1)^{-1}$ in the form

$$a\alpha^2 + b\alpha + c$$

with $a, b, c \in \boldsymbol{Q}$.

Problem 2

Let $E = F(\alpha)$ where α is algebraic over F, of odd degree. Show that $E = F(\alpha^2)$.

Problem 3

Let α be the real positive fourth root of 2. Find all intermediate fields in the extension $\boldsymbol{Q}(\alpha)$ of $\boldsymbol{Q}$.

Problem 4

If α is a complex root of $x^6 + x^3 + 1 = 0$, find all homomorphisms $\sigma : \boldsymbol{Q}(\alpha) \to \boldsymbol{C}$.

Problem 5

Show that $\sqrt{2} + \sqrt{3}$ is algebraic over $\boldsymbol{Q}$ of degree 4.

Problem 6

Let E, F be two finite extensions of a field k, contained in a larger field K. (1). Show that

$$[EF : k] \leq [E : k][F : k].$$

(2). If $[E : k]$ and $[F : k]$ are relatively prime, show that one has an equality sign in the above relation.

(3). Give an example in which the equality sign fails in the above relation.

Problem 7

Find the minimal polynomials of the following elements over $\boldsymbol{Q}$. (1). $a+bi$, where $a, b \in \boldsymbol{Q}, b \neq 0$. (2). $e^{\frac{2\phi i}{p}}$ where p is an odd prime.

Problem 8

(1). Let E/F be a finite extension. A field homomorphism $\sigma : E \to E$ is called an F-homomorphism, if $\sigma|_F = id_F$. Prove that any F-homomorphism is an F-isomorphism.

(2). Prove the above statement for any algebraic extension E/F.

Problem *9(**transcendental numbers**)

(1). Show that there exist transcendental numbers(over $\boldsymbol{Q}$).

Following Hermite Charles,(1873), we are going to show that e is transcendental.

Otherwise, there exists a polynomial $p(x) = a_0 + a_1 x + ... + a_m x^m \in \boldsymbol{Z}[x]$ such that $p(e) = 0$.

(2). Let $f(x)$ be a polynomial of degree n, and denote $F(x) = f(x) + f'(x) + f''(x) + ... + f^{(n)}(x)$. Using integration by parts show that

$$F(0)e^b = F(b) + e^b \int_0^b f(x)e^{-x}dx.$$

and henceforth the identity

$$\sum_{i=0}^{m} F(i)a_i + \sum_{i=0}^{m} a_i e^i \int_0^i e^{-x} f(x)dx = F(0)p(e) = 0.$$

(3). Now take $f(x) = \frac{1}{(p-1)!}x^{p-1}(x-1)^p(x-2)^p....(x-m)^p$ where p is a prime greater than $max\{m, |a_0|\}$. Try to show $p \not| a_0 F(0)$, $p | a_j F(j)$ for $j \neq 0$. Hence $\sum_{i=0}^{m} a_i F(i)$ is a nonzero integer.

(4). Next show that

$$\left| \sum_{i=0}^{m} a_i e^i \int_0^i e^{-x} f(x)dx \right| < ae^m m^m \frac{(m^{m+1})^{p-1}}{(p-1)!}.$$

while the right hand side tends to 0 for p sufficiently large.

(5). Combining (2),(3) and (4) we deduce that for a sufficiently l arge prime p, an integer plus a number whose absolute value is less than 1 equals 0, which is impossible. So e is transcendental.

(6). Using the following theorem, show that π is also transcendental.

Lindemann-Hermite: If α is algebraic over $\boldsymbol{Q}$ and is not 0, then e^α is transcendental.

(You may read the appendix 1: the transcendence of e and π, or any other books on analytic number theory, if you are interested in this topic).