PROBLEM SET 11
DUE: May. 19

Problem 1

Show that the followings are equivalent:

(1). $f(x) \in k[x]$ is irreducible.

(2). $(f(x))$ is a maximal ideal of $k[x]$.

(3). $(f(x))$ is a prime ideal of $k[x]$.

(We shall see in Problem 7 that the same conclusion fails for polynomial ring in n variables, $n > 1$.)

Problem 2

Let F/K be an algebraic extension, and $K \subset D \subset F$ where D is an integral domain. Show that D is a field.

Problem 3

(1). Let $\phi : E \to F$ be a homomorphism of fields. Show that ϕ is either an embedding or the zero mapping.

(2). Construct a field K and a homomorphism $\psi : K \to K$ such that ψ is not surjective. (Hint: Compare with Lemma 2.1).

Problem 4

Let F be a field of characteristic p where p is a prime.

(1). (The Freshman's dream)Show that for $a, b \in F$, we have $(a+b)^{p^n} = a^{p^n} + b^{p^n}$.

(2). Show that $x^p - c$ is irreducible in $F[x]$, if and only if $x^p - c = 0$ is not solvable in F.

(*3). Show that $x^p - x - c$ is irreducible in $F[x]$ if and only if $x^p - x - c = 0$ is not solvable in F.

Problem 5

Let $f(x) \in k[x]$ be a polynomial of degree n. Show that there exists a fields K in which $f(x)$ splits into linear factors, and $[K : k]$ divides $n!$.

Problem 6

Describe the splitting fields of the following polynomials over $\boldsymbol{Q}$, and find the degree of each such splitting field.

(1). $x^2 - 2$

(2). $x^2 - 1$

(3). $x^3 - 2$

(4). $(x^3 - 2)(x^2 - 2)$

(5). $x^2 + x + 1$

(6). $x^6 + x^3 + 1$

(7). $x^5 - 7$

Problem *7

(1). Recall that an element α is integral over the ring R if it is a zero of a monic polynomial ring $f(x) \in R[x]$. And we say a ring S integral over R if every element $s \in S$ is integral over R. Show that if T is integral over S, and S is integral over R, then T is integral over R.

(2). Recall that an R-module M is an abelian group over a ring R. Prove that if $R \subset S$ are two commutative rings with identity and S is integral over R, then R is a field if and only if S is a field.

(3). A finitely generated k-algebra is a finitely generated ring which is also a k-vector space. Give an example of a finitely generated k-algebra.

Noether's Normalization theorem: Let R be a finitely generated k-algebra. Then there exist an integer n and n algebraic independent elements $y_1, y_2, ..., y_n \in R$, such that R is integral over the ring $k[y_1, y_2, ..., y_n]$.

(4). Using Noether's Normalization theorem prove the **Hilbert's Nullstellensatz**: Let k be an algebraically closed field, then every maximal ideal of the polynomial ring in n variables over k has the form $(x - a_1, ..., x - a_n)$ where $a_i \in k$.

(The great importance of this result is that it gives us a way to translate affine space k^n into pure algebra. We have a bijection between k^n , on the one hand, and the set of maximal ideals in $k[x_1, ..., x_n]$ on the other hand. This is the origin of the connection between algebra and geometry that gives rise to the whole subject.)