PROBLEM SET 12
DUE: May. 26

Problem 1

Let α be a real number such that $\alpha^4 = 5$.

(1). Show that $\boldsymbol{Q}(i\alpha^2)$ is normal over $\boldsymbol{Q}$.

(2). Show that $\boldsymbol{Q}(\alpha + i\alpha)$ is normal over $\boldsymbol{Q}(i\alpha^2)$.

(3). Show that $\boldsymbol{Q}(\alpha + i\alpha)$ is not normal over $\boldsymbol{Q}$.

Problem 2

Let $f(x) = \sum_{i=0}^{i=n} a_i x^i \in k[x]$. Define its derivative formally to be $f'(x) = \sum_{i=0}^{i=n} i a_i x^{i-1}$. Show that $f(x)$ has no multiple roots if and only if $g.c.d.(f(x), f'(x)) = 1$.

Problem 3

Let *char* $K = p$. Let L be a finite extension of K, and suppose $[L : K]$ prime to p. Show that L is separable over K.

Problem 4

If the roots of a monic polynomial $f(x) \in k[x]$ in some splitting field are distinct, and form a field, then *char* $k = p$ and $f(x) = x^{p^n} - x$ for some $n \geq 1$.

Problem 5

Suppose *char* $K = p$. Let $a \in K$. If a has no p-th root in K, show that $x^{p^n} - a$ is irreducible in $K[x]$ for all positive integers n.

Problem 6

Let *char* $K = p$. Let α be algebraic over K. Show that α is separable if and only if $K(\alpha) = K(\alpha^{p^n})$ for all positive integers n.

Problem 7

Prove that the following two properties are equivalent:

(1). Every algebraic extension of K is separable.

(2). Either *char* $K = 0$, or *char* $K = p$ and every element of K has a p-th root in K.

Problem 8

Let E be an algebraic extension of F. Show that every subring of E which contains F is actually a field. Is this necessarily true if E is not algebraic over F? Prove or give a counterexample.

Problem 9

Let $k = \boldsymbol{F}_p(t)$, and $f(x) = x^p - t \in k[x]$. Then $f(x)$ is irreducible but not separable.