PROBLEM SET 15

DUE: June 16

Problem 1

What is the Galois groups of the following polynomials?

(1). $x^3 - x - 1$ over $\boldsymbol{Q}$.

(2). $x^3 - 10$ over $\boldsymbol{Q}$.

(3). $x^3 - 10$ over $\boldsymbol{Q}(\sqrt{2})$.

(4). $x^3 - 10$ over $\boldsymbol{Q}(\sqrt{-3})$.

(5). $(x^2-2)(x^2-3)(x^2-5)(x^2-7)$ over $\boldsymbol{Q}$.

(6). $x^n - t$ where t is transcendental over the complex numbers $\boldsymbol{C}$ and n is a positive integer, over $\boldsymbol{C}(t)$.

Problem 2

Let k be a field of characteristic $\neq 2$. Let $c \in k,\ c \notin k^2$. Let $F = k(\sqrt{c})$. Let $\alpha = a + b\sqrt{c}$ with $a, b \in k$ and not both $a, b = 0$. Let $E = F(\sqrt{\alpha})$. Prove that the following conditions are equivalent:

(1). E is Galois over k.

(2). $E = F(\sqrt{\alpha'})$, where $\alpha' = a - b\sqrt{c}$.

(3). Either $\alpha\alpha' \in k^2$ or $c\alpha\alpha' \in k^2$.

Show that when these conditions are satisfied then E is cyclic over k of degree 4 if and only if $c\alpha\alpha' \in k^2$.

Problem 3

Show that the regular pentagon is ruler and compass constructible.

Problem 4(**cyclotomic polynomials**)

(1). Let $\Phi_d(x) = \prod_{period \zeta = d}(x - \zeta)$, where the product is taken over all $n-th$ roots of unity of period d. Show that

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{d|n, d<n} \Phi_d(x)}.$$

And compute $\Phi_n(x)$ for $1 \leq n \leq 12$.

(2). $\Phi_n(x)$ is called the **n-th cyclotomic polynomial**. Show that if

p is a prime number, then

$$\Phi_p(x) = x^{p-1} + x^{p-2} + ... + 1.$$

and for an integer $r \geq 1$,

$$\Phi_{p^r}(x) = \Phi_p(x^{p^{r-1}}).$$

(3). Let $n = p_1^{r_1}....p_s^{r_s}$ be a positive integer with its prime factorization. Then

$$\Phi_n(x) = \Phi_{p_1....p_s}(x^{p_1^{r_1-1}....p_s^{r_s-1}}).$$

(4). If n is odd > 1, then $\Phi_{2n}(x) = \Phi_n(-x)$.

(5). If p is a prime number not dividing n, then

$$\Phi_{pn}(x) = \frac{\Phi_n(x^p)}{\Phi_n(x)}.$$

On the other hand, if $p|n$, then $\Phi_{pn} = \Phi_n(x^p)$.

(6). We have

$$\Phi_n(x) = \prod_{d|n}(x^{\frac{n}{d}} - 1)^{\mu(d)}.$$

As usual, μ is the Mobius function.

Problem 5

Show that $\sum_{x \in F_q} x^m = 0$ for q not dividing m, where F_q is a finite field and the summation is taken over all elements $x \in F_q$. While $\sum_{x \in F_q} x^m = -1$ for q dividing m.

Problem 6

(1). Let $K_1, ..., K_n$ be Galois extensions of k with Galois groups $G_1, ..., G_n$. Assume that $K_{i+1} \cap (K_1....K_i) = k$ for each $i = 1, 2, ..., n-1$. Then the Galois group of $K_1....K_n$ is isomorphic to the product $G_1 \times .... \times G_n$ in a natural way.

(2). Let $p_1 < p_2 < .... < p_n < ....$ be a sequence of prime numbers, and $K_1 = \boldsymbol{Q}(\sqrt{p_1})$, $K_{i+1} = K_i(\sqrt{p_{i+1}})$. Compute the Galois group of $K_1....K_n$ over $\boldsymbol{Q}$.