**PROBLEM SET 4**
**DUE: Mar.17**

Problem 1

Let $\phi : \boldsymbol{Z}/5\boldsymbol{Z} \to Aut(\boldsymbol{Z}/11\boldsymbol{Z})$ be a nontrivial homomorphism. Then we can construct a semidirect product

$$G = \boldsymbol{Z}/5\boldsymbol{Z} \rtimes_\phi \boldsymbol{Z}/11\boldsymbol{Z}.$$

Show that $G$ is not isomorphic to the direct product of $\boldsymbol{Z}/5\boldsymbol{Z}$ and $\boldsymbol{Z}/11\boldsymbol{Z}$.

Problem 2

Recall that the **dihedral group $\boldsymbol{D_{2n}}$** is the symmetric group of a regular n-gon, generated by the rotation and certain reflection. That is,

$$\boldsymbol{D_{2n}} =< a, b|a^n = b^2 = 1, bab = b^{-1} >,$$

where $a$ is the rotation and $b$ is the reflection.

(1). Give the order of each element of $\boldsymbol{D_8}$.

(2). Compute $Aut(\boldsymbol{D_8})$.

Problem 3

Let $G$ ba a group. Show that $G$ is abelian if and only if the map $\alpha : g \to g^{-1}$ is an automorphism.

Problem 4

(1). Show that $S_3$, $S_4$ are solvable, and write down all the composition series of them. (Recall that a tower

$$1 = G_0 \lhd G_1 \lhd .... \lhd G_n = G$$

is a composition series, if each factor $G_{i+1}/G_i$ is abelian.)

**Definition**: Let $G$ be a group, and $G^{(1)} = [G, G]$ denote its commutator subgroup. Then we define the **higher commutator subgroup** inductively by $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.

(2). Show that if

$$1 = G_0 \lhd G_1 \lhd .... \lhd G_n = G$$

is a solvable series, that is, $G_{i+1}/G_i$'s are abelian groups, then $G_i \geq G^{(i)}$.

(3). A subgroup $H$ of $G$ is said to be **fully invariant**, if $\phi(H) = H$ for every homomorphism $\phi \in End(G)$. Show that the higher commutator subgroups are fully invariant.

Problem 5

Show that the center of a group of order $p^n$ is nontrivial, where $p$ is a prime. (Hint: use the class formula.)

Problem *6 **nilpotent groups**

**Definition**: If $H, K \leq G$, then

$$[H, k] = < [h, k] | h \in H \quad and \quad k \in K >,$$

where $[h, k]$ is the commutator $hkh^{-1}k^{-1}$.

(1). If $K \lhd G$ and $K \leq H \leq G$, then $[H, G] \leq K$ if and only if $H/K \leq Z(G/K)$.

(2). If $H, K \leq G$ and $f : G \to L$ is a homomorphism, then $f([H, K]) = [f(H), f(K)]$.

**Definition**: The **lower central series** (or descending central series) of $G$ is the series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq ....$$

where $\gamma_{i+1}(G) = [\gamma_i(G), G]$. (This need not to be a normal series because it may not reach 1).

**Definition**: The **higher centers** $\zeta^i(G)$ are the subgroups defined by induction:

$$\zeta^0(G) = 1; \qquad \zeta^{i+1}(G)/\zeta^i(G) = Z(G/\zeta^i(G));$$

that is, if $\nu_i : G \to G/\zeta^i(G)$ is the natural map, then $\zeta^{i+1}(G)$ is the inverse image of the center.

**Definition**: The **upper central series** (or ascending central series) of $G$ is

$$1 = \zeta^0(G) \leq \zeta^1(G) \leq ....$$

(3). Prove the theorem: If $G$ is a group, then there is an integer $c$ with $\zeta^c(G) = G$ if and only if $\gamma_{c+1}(G) = 1$. Moreover, in this case,

$$\gamma_{i+1}(G) \leq \zeta^{c-i}(G)$$

for all $i$.

**Definition**: A group $G$ is **nilpotent** if there is an integer $c$ such that $\gamma_{c+1}(G) = 1$; the least such $c$ is call the **class** of the nilpotent group $G$.

(4). Show that every finite group of order $p^n$ is nilpotent, where $p$ is a prime. (Hint: use problem 6).

(5). Show that every nilpotent group is solvable.

(6). If $G \neq 1$ is nilpotent, then $Z(G) \neq 1$.

(7). $S_3$ is a solvable group but not nilpotent.

(8). Show that every subgroup $H$ of a nilpotent group $G$ is nilpotent. Moreover, if $G$ is nilpotent of class $c$, then $H$ is nilpotent of class $\leq c$.

(9). If $G$ is nilpotent of class $c$, and $H \lhd G$, then $G/H$ is nilpotent of class $\leq c$.