PROBLEM SET 5
DUE: Mar.24

Problem 1

Let $\alpha_0 < \alpha_1 \leq \alpha_2$ be integers, and p be a prime. Consider the diagonal homomorphism:

$$\phi : \boldsymbol{Z}/p^{\alpha_0}\boldsymbol{Z} \to \boldsymbol{Z}/p^{\alpha_1}\boldsymbol{Z} \times \boldsymbol{Z}/p^{\alpha_2}\boldsymbol{Z}$$

$$x \mapsto (p^{\alpha_1 - \alpha_0}x, p^{\alpha_2 - \alpha_0}x)$$

We shall denote $\boldsymbol{Z}/p^{\alpha_1}\boldsymbol{Z} \times \boldsymbol{Z}/p^{\alpha_2}\boldsymbol{Z}$ by G and $H = im(\phi)$. Determine the quotient group G/H as a direct product.

Problem 2 **Permutation groups**

(1). Let $\sigma = [123....n]$ in S_n. Show that the conjugacy class of σ has $(n-1)!$ elements. Show that the centralizer of σ is the cyclic group generated by σ.

(2). Prove the following formula:

$$\gamma[i_1 \quad i_2 \quad ....i_k]\gamma^{-1} = [\gamma(i_1) \quad \gamma(i_2) \quad ....\gamma(i_k)]$$

where $\gamma \in S_n$ and $k \leq n$.

(3). Suppose that a permutation σ in S_n can be written as a product of r disjoint cycles, and let $d_1, d_2....d_r$ be the number of elements in each cycle, in increasing order. Let τ be another permutation which can be written as a product of disjoint cycle, whose cardinalities are $d_1^{'}, d_2^{'}, ..., d_s^{'}$ in increasing order. Prove that σ is conjugate to τ if and only if $r = s$ and $d_i = d_i^{'}$ for all $i = 1, 2, ..., r$.

(4). Show that S_n is generated by the transpositions $[12], [23], [34], ..., [n-1, n]$.

(5). Show that S_n is generated by the cycles $[12]$ and $[12....n]$.

(6). Assume that n is prime. Let $\sigma = [12...n]$ and let $\tau = [rs]$ be any transposition. Show that σ, τ generate S_n.

Problem 3

Consider the following game:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	*

Each time we can transpose the * block with a nearby block. Is it possible that we can get the following status after finite steps?

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	*

Problem 4

Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G which is contained in H and also of finite index.

Problem 5

Let G be a finite group operating on a finite set S with $\#(S) \geq 2$. Assume that there is only one orbit. Prove that there exists an element $x \in G$ which has no fixed point, i.e. $xs \neq s$ for all $s \in S$.

Problem 6

Let X, Y be finite sets and let C be a subset of $X \times Y$. For $x \in X$ let $\phi(x) =$ number of elements of $y \in Y$ such that $(x, y) \in C$. Verify that

$$\#(C) = \sum_{x \in X} \phi(x).$$

remark: A subset C as in the above exercise is often called a **correspondence**, and $\phi(x)$ is the number of elements in Y which correspond to a given element $x \in X$.