PROBLEM SET 7
DUE: Apr. 7

Problem 1

Let P be a p-group. Let A be a normal subgroup of order p. Prove that A is contained in the center of P.

Problem 2

Let G be a finite group and H a subgroup. Let P_H be a p-Sylow subgroup H. Prove that there exists a p-Sylow subgroup P of G such that $P_H = P \cap H$.

Problem 3

Let H be a normal subgroup of a finite group G and assume that $\#(H) = p$. Prove that H is contained in every p- Sylow subgroup of G.

Problem 4

Let $P, P^{'}$ be p-Sylow subgroups of a finite group G.

(a). If $P^{'} \subset N(P)$ (normalizer of P), then $P^{'} = P$.

(b). If $N(P^{'}) = N(P)$, then $P^{'} = P$.

(c). We have $N(N(P)) = N(P)$.

Problem 5

Let p be a prime number. Show that a group of order p^2 is abelian, and that there are only two such groups up to isomorphism.

Problem 6

Let G be a group of order p^3, where p is a prime, and G is not abelian. Let Z be its center. Let C be a cyclic group of order p.

(a). Show that $Z \cong C$ and $G/Z \cong C \times C$.

(b). Every subgroup of G of order p^2 contains Z and is normal.

(c). Suppose $x^p = 1$ for all $x \in G$. Show that G contains a normal subgroup $H \cong C \times C$.

Problem 7

(a). Let G be a group of order pq, where $p,\ q$ are primes and $p < q$. Assume that $q \not\equiv 1 \bmod p$. Prove that G is cyclic.

(b). Show that every group of order 15 is cyclic.

Problem 8

Let p, q be distinct primes. Prove that a group of order p^2q is solvable, and that one of its Sylow subgroups is normal.

Problem 9

Let p, q be odd primes. Prove that a group of order $2pq$ is solvable.

Problem 10

(a). Prove that one of the Sylow subgroups of a group of order 40 is normal.

(b). Prove that one of the Sylow subgroups of a group of order 12 is normal.

Problem 11

Determine all groups of order ≤ 10 up to isomorphism. In particular, show that a non-abelian group of order 6 is isomorphic to S_3.

Problem 12

Let S_n be the permutation group on n elements. Determine the p-Sylow subgroups of $S_3,\ S_4,\ S_5$ for $p = 2$ and $p = 3$.