The 1st homework

of *Algebraic Topology*, Autumn, 2018

Please submit the homework on next Wednesday (19th Sep) morning. The symbol '$*$' means that the marked problems are a little difficult.

1.(1 point)(a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.
(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.
(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

2.(1pt) Show that a homotopy equivalence $f : X \to Y$ induces a bijection between the set of path-components of X and the set of path-components of Y, and that f restricts to a homotopy equivalence from each path-component of X to the corresponding path-component of Y. Prove also the corresponding statements with components instead of path-components. Deduce that if the components of a space X coincide with its path-components, then the same holds for any space Y homotopy equivalent to X.

3.(1pt) Given positive integers v, e, and f satisfying $v - e + f = 2$, construct a cell structure on S^2 having v 0-cells, e 1-cells, and f 2-cells.

4.(1pt) Show that composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.

5.(1pt) A subspace $X \subset \mathbb{R}^n$ is said to be star-shaped if there is a point $x_0 \in X$ such that, for each $x \in X$, the line segment from x_0 to x lies in X. Show that if a subspace $X \subset \mathbb{R}^n$ is locally star-shaped, in the sense that every point of X has a star-shaped neighborhood in X, then every path in X is homotopic in X to a piecewise linear path, that is, a path consisting of a finite number of straight line segments traversed at constant speed. Show this applies in particular when X is open or when X is a union of finitely many closed convex sets.

6.(1pt) Show that for a space X, the following three conditions are equivalent:
(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.
(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.
(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.
Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

7.(1pt) Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism φ_* of a map $\varphi : S^1 \to S^1$.

8.(1pt) Show that the isomorphism $\pi_1(X \times Y) \approx \pi_1(X) \times \pi_1(Y)$ in Proposition 1.12 is given by $[f] \mapsto (p_{1*}([f]), p_{2*}([f]))$ where p_1 and p_2 are the projections of $X \times Y$ onto its two factors.

9.*(2pts) Show that S^∞ is contractible.