USTC, School of Mathematical Sciences — *Winter semester 2018/12/06*
Algebraic topology by Prof. Mao Sheng — *Exercise sheet 10*
MA04311 Tutor: Lihao Huang, Han Wu — *10 points*
Posted online by Dr. Muxi Li — *2 points for a problem*

Ex1. Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.

Ex2. A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an even map. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show there exist even maps of any given even degree. [Hints: If f is even, it factors as a composition $S^n \to \mathbb{R}P^n \to S^n$. Using the calculation of $H_n(\mathbb{R}P^n)$ in the text, show that the induced map $H_n(S^n) \to H_n(\mathbb{R}P^n)$ sends a generator to twice a generator when n is odd. It may be helpful to show that the quotient map $\mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1}$ induces an isomorphism on H_n when n is odd.]

Ex3. Show the isomorphism between cellular and singular homology is natural in the following sense: A map $f : X \to Y$ that is *cellular* $-$ satisfying $f(X^n) \subset Y^n$ for all n $-$ induces a chain map f_* between the cellular chain complexes of X and Y , and the map $f_* : H_n^{CW}(X) \to H_n^{CW}(Y)$ induced by this chain map corresponds to $f_* : H_n(X) \to H_n(Y)$ under the isomorphism $H_n^{CW} \approx H_n$.

Ex4. What happens if one defines homology groups $h_n(X; G)$ as the homology groups of the chain complex

$$\cdots \to \mathrm{Hom}(G, C_n(X)) \to \mathrm{Hom}(G, C_{n-1}(X)) \to \cdots ?$$

More specifically, what are the groups $h_n(X; G)$ when $G = \mathbb{Z}$, $\mathbb{Z}_m$, and $\mathbb{Q}$?

Ex5. Let X be a Moore space $M(\mathbb{Z}_m, n)$ obtained from S^n by attaching a cell e_{n+1} by a map of degree m.
(a) Show that the quotient map $X \to X/S_n = S_{n+1}$ induces the trivial map on $\widetilde{H}_i(-;\mathbb{Z})$ for all i, but not on $H^{n+1}(-;\mathbb{Z})$. Deduce that the splitting in the universal coefficient theorem for cohomology cannot be natural.
(b) Show that the inclusion $S_n \hookrightarrow X$ induces the trivial map on $\widetilde{H}^i(;\mathbb{Z})$ for all i, but not on $H_n(-;\mathbb{Z})$.

Please hand in this homework on 12th Dec. 2018.