USTC, School of Mathematical Sciences — *Winter semester 2018/12/20*
Algebraic topology by Prof. Mao Sheng — *Exercise sheet 12*
MA04311 Tutor: Lihao Huang, Han Wu — *12 points*
Posted online by Dr. Muxi Li — *2 points for a problem*

Ex1. (a) Using the cup product structure, show there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m;\mathbb{Z}_2) \to H^1(\mathbb{R}P^n;\mathbb{Z}_2)$ if $n > m$. What is the corresponding result for maps $\mathbb{C}P^n \to \mathbb{C}P^m$?
(b) Prove the Borsuk-Ulam theorem by the following argument. Suppose on the contrary that $f : S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all x. Then define $g : S^n \to S^{n-1}$ by $g(x) = (f(x) - f(-x))/|f(x) - f(-x)|$, so $g(-x) = -g(x)$ and g induces a map $\mathbb{R}P^n \to \mathbb{R}P^{n-1}$. Show that part (a) applies to this map. [Hint: the quotient map $S^n \to \mathbb{R}P^n$ sends each path connecting a pair of antipodal points to a closed 1-simplex which represents the generator of $H_1(\mathbb{R}P^n, \mathbb{Z}_2)$.]

Ex2. Use cup products to show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

Ex3. Show that the cross product map $H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \to H^*(X \times Y;\mathbb{Z})$ is not an isomorphism if X and Y are infinite discrete sets. [This shows the necessity of the hypothesis of finite generation in Theorem 3.15.]

Ex4. Using cup products, show that every map $S^{k+\ell} \to S^k \times S^\ell$ induces the trivial homomorphism $H^{k+\ell}(S^{k+\ell}) \to H^{k+\ell}(S^k \times S^\ell)$, assuming $k > 0$ and $\ell > 0$.

Ex5. Show that the spaces $(S^1 \times \mathbb{C}P^\infty)/(S^1 \times \{x_0\})$ and $S^3 \times \mathbb{C}P^\infty$ have isomorphic cohomology rings with $\mathbb{Z}$ or any other coefficients. [An exercise for §4.L is to show these two spaces are not homotopy equivalent.]

Ex.6 Show the ring $H^*(\mathbb{R}P^\infty;\mathbb{Z}_{2k})$ is isomorphic to $\mathbb{Z}_{2k}[\alpha,\beta]/(2\alpha, 2\beta, \alpha^2 - k\beta)$ where $|\alpha| = 1$ and $|\beta| = 2$. [Use the coefficient map $\mathbb{Z}_{2k} \to \mathbb{Z}_2$ and the proof of Theorem 3.19.]

Please hand in this homework on 26th Dec. 2018.