USTC, School of Mathematical Sciences *Winter semester 2018/12/28*
Algebraic topology by Prof. Mao Sheng *Exercise sheet 13*
MA04311 Tutor: Lihao Huang, Han Wu *10 points*
Posted online by Dr. Muxi Li

Ex1. (2 pts) Show that $M \times N$ is orientable iff M and N are both orientable.

Ex2. (1 pt) For a map $f : M \to N$ between connected closed orientable n-manifolds with fundamental classes $[M]$ and $[N]$, the degree of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \to S^n$.

Ex3. (1 pt) For a map $f : M \to N$ between connected closed orientable n-manifolds, suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is the disjoint union of balls B_i each mapped homeomorphically by f onto B. Show the degree of f is $\sum_i \varepsilon_i$ where ε_i is $+1$ or -1 according to whether $f : B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

Ex4. (2 pts) Show that for a degree 1 map $f : M \to N$ of connected closed orientable manifolds, the induced map $f_* : \pi_1(M) \to \pi_1(N)$ is surjective, hence also $f_* : H_1(M) \to H_1(N)$. [Lift f to the covering space $\widetilde{N} \to N$ corresponding to the subgroup $\mathrm{Im} f_* \subset \pi_1(N)$, then consider the two cases that this covering is finite-sheeted or infinite-sheeted.]

Ex5. (1 pt) If M_g denotes the closed orientable surface of genus g, show that degree 1 maps $M_g \to M_h$ exist iff $g \geqslant h$.

Ex6. (2 pts) Let M be a closed connected 3 manifold, and write $H_1(M;\mathbb{Z})$ as $\mathbb{Z}^r \oplus F$, the direct sum of a free abelian group of rank r and a finite group F. Show that $H_2(M;\mathbb{Z})$ is $\mathbb{Z}^r$ if M is orientable and $\mathbb{Z}^{r-1} \oplus \mathbb{Z}_2$ if M is nonorientable. [Hint: Corollary 3.37 and its proof may be helpful.]

Ex7. (1 pt) Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M;\mathbb{Z})$ torsionfree, then $H_k(M;\mathbb{Z})$ is also torsionfree.

Please hand in this homework before the final examination.