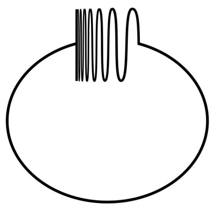
USTC, School of Mathematical Sciences *Winter semester 2018/10/11*
Algebraic topology by Prof. Mao Sheng *Exercise sheet 4*
MA04311 Tutor: Lihao Huang, Han Wu *10 points*
Posted online by Dr. Muxi Li *2 points for every problem*

Ex1. Construct a simply-connected covering space of the space $X \in \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

Ex2. Let Y be the quasi-circle shown in the figure, a closed subspace of $\mathbb{R}^2$ consisting of a portion of the graph of $y = \sin(1/x)$, the segment $[1, 1]$ in the y axis, and an arc connecting these two pieces. Collapsing the segment of Y in the y axis to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$. Thus local path-connectedness of Y is a necessary hypothesis in the lifting criterion.



Ex3. Let $\widetilde{X}$ and $\widetilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ then $\widetilde{X} \simeq \widetilde{Y}$.

Ex4. Given a group G and a normal subgroup N, show that there exists a normal covering space $\widetilde{X} \to X$ with $\pi_1(X) \approx G$, $\pi_1(\widetilde{X}) \approx N$, and deck transformation group $G(\widetilde{X}) \approx G/N$.

Ex5. For a path-connected, locally path-connected, and semilocally simply-connected space X, call a path-connected covering space $\widetilde{X} \to X$ abelian if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X, and that such a universal abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $X = S^1 \vee S^1 \vee S^1$.