USTC, School of Mathematical Sciences — Winter semester 2018/19
Algebraic topology by Prof. Mao Sheng — Exercise sheet 5
MA04311 Tutor: Lihao Huang, Han Wu — 15 points
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Ex 1. (2 pt) *Let X be a path-connected space with base point x_0. Let $\gamma : S^1 \to X$ be a loop and $f : (D^n, \partial D^n) \to (X, x_0)$ be a map. Define $\gamma f := (D^n, \partial D^n) \to (X, x_0)$ by*

$$\gamma f = \begin{cases} f(2x) & if \quad 0 \leq |x| < 1/2 \\ \gamma(2|x| - 1) & if \quad 1/2 \leq |x| \leq 1 \end{cases}$$

Show that this induces a well-defined action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Ex 2. (2 pt) *Let (X, A) be a CW pair and let (Y, B) be any pair with $B \neq \emptyset$. For each n such that $X - A$ has cells of dimension n, assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f : (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$.*

Ex 3. (2 pt) *Given the definition of complex and its exactness, then split a long exact sequence into short exact sequences.*

Ex 4. (3 pt) *Let X be connected space with base point x_0. For any positive integer n, show that the following conditions are equivalent.*

(1) *Every map $S^n \to X$ is homotopic to a constant map.*
(2) *Every map $S^n \to X$ extends to a map $D^{n+1} \to X$.*
(3) $\pi_n(X, x_0) = 0$.

Ex 5. (2 pt) *Given a CW pair (X, A) and a map $f : A \to Y$ with Y path-connected, then f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X - A$ has cells of dimension n.*

Ex 6. (2 pts) *Let $f : X \to Y$ be a cellular map between CW complexes. Show that the mapping cylinder $M(f)$ of f has a natural CW-structure such that X, Y are CW-subcomplexes of $M(f)$.*

Ex 7. (2 pts) *Let (X, A) be a CW-pair and $x_0 \in A$. Show that the sequence $\pi_1(A, x_0) \to \pi_1(X, x_0) \to \pi_1(A, X, x_0) \to \pi_0(A, x_0) \to \pi_0(X, x_0)$ is exact.*

Note: Please hand in this homework on 7^{th} Nov. 2018.