USTC, School of Mathematical Sciences — Winter semester 2018/19
Algebraic topology by Prof. Mao Sheng — Exercise sheet 8
MA04311 Tutor: Lihao Huang, Han Wu — 11 points
Posted online by Dr. Muxi Li

Ex 1. (3 pt) *Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is the wedge sum of n-spheres:*

(a) *If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.*
(b) *$H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.*
(c) *If X has k n-cells, then $H_n(X)$ is generated by at most k elements.*

Ex 2. (2 pt) *Show that the second barycentric subdivision of a $\triangle$-complex is a simplicial complex. Namely, show that the first barycentric subdivision produces a $\triangle$-complex with the property that each simplex has all its vertices distinct, then show that for a $\triangle$-complex with this property, barycentric subdivision produces a simplicial complex.*

Ex 3. (2 pt) *Show that a simplicial complex structure on the torus needs at least 14 triangles, 21 edges and 7 vertices. Write down this simplicial structure explicitly. [hint: using Euler formula for torus to get the relation of the number of triangles, edges and vertices.]*

Ex 4. (2 pt) *Let $f : (X, A) \to (Y, B)$ be a map such that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences.*

(a) *Show that $f_* : H_n(X, A) \to H_n(Y, B)$ are isomorphism for all n.*
(b) *For the case of the inclusion $f : (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs—there is no $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs.*

Ex 5. (2 pts) *Recall that for a good pair (X, A), the relative homology $H_n(X, A)$ can be expressed as reduced absolute homology $\tilde{H}_n(X/A)$. Now, for an arbitary pair (X, A), show that $H_n(X, A) \cong \tilde{H}_n(X \cup_A CA)$, where CA is the cone $(A \times I)/(A \times \{0\})$, and $\cup_A$ means identifying $A \subset X$ with the base of the cone $A \times \{1\}$.*

Note: Please hand in this homework on 28$^{\text{th}}$ Nov. 2018.