USTC, School of Mathematical Sciences — *Winter semester 2018/12/29*
Algebraic topology by Prof. Mao Sheng — *Hints to exercise sheet 10*
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Hint1. If $f(x) \neq x$, $\forall x$, then $H(x,t) = ((1-t)f(x)+tx)/|(1-t)f(x)-tx|$ gives a homotopy connecting f and A, where $A(x) := -x$. Similarly, $f \simeq \mathrm{Id}$ if $f(x) \neq -x$, $\forall x$. Neither $f \simeq A$ nor $f \simeq \mathrm{Id}$ holds for $\deg f = 0$ here. Hence there exist points x, y with $f(x) = x$ and $f(y) = -y$.

Let $G := F/|F| : D^n \to S^{n-1}$. The composition $f : S^{n-1} \hookrightarrow D^n \xrightarrow{G} S^{n-1}$ is nullhomotopy so $\deg f = 0$. There exist points $x, y \in S^{n-1}$ with $f(x) = x$ and $f(y) = -y$, i.e. $F(x) = |F(x)|x$ and $F(y) = -|F(y)|y$, according to the above discussion.

Hint2. Each even map f can be decomposed to $f : S^n \xrightarrow{q} \mathbb{R}P^n \to S^n$, where q is the quotient map. $H_n(\mathbb{R}P^n) = 0$, hence $\deg f = 0$, when n is even.

We knew that the degree of the composition

$$S^n \xrightarrow{q} \mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n$$

is $1 + (-1)^n$ when we calculated the cellular homology groups of projective spaces. Therefore $H_n(\mathbb{R}P^n) \approx \mathbb{Z}$ and $q_* : H_n(S^n) \to H_n(\mathbb{R}P^n)$ is multiplication by 2 when n is odd, since the homomorphism $H_n(\mathbb{R}P^n) \to H_n(\mathbb{R}P^n/\mathbb{R}P^{n-1})$ induced by quotient map is an isomorphism when n is odd, which can be prove by using the long exact sequences of homology groups of good pairs. Then it is distinct that the degree of $f : S^n \xrightarrow{q} \mathbb{R}P^n \to S^n$ is even.

To give a even map with degree $2k$, we construct $g : S^n \to S^n$ with degree k, then the composition $S^n \xrightarrow{q} \mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n \xrightarrow{g} S^n$ has the degree $2k$.

Hint3. We first emphasize that the cellular chain map f_n^{CW} is defined as $f_* : H_n(X^n, X^{n-1}) \to H_n(Y^n, Y^{n-1})$, although we leave out the verifying that it is a chain map indeed. We have the following commutative graph

$$\begin{array}{ccccc}
H_n(X^n) & \longrightarrow & H_n(X^{n+1}) & \longrightarrow & H_n(X) \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
H_n(X^n) & \longrightarrow & H_n(X^{n+1}) & \longrightarrow & H_n(X),
\end{array}$$

where all the vertical arrows are induced by inclusion, the arrows in left side are surjective, and arrows in right side are isomorphic in both rows. The isomorphism $I : H_n(X) \to H_n^{CW}(X)$ is given by the following process: for a given element α in $H_n(X)$, choose a preimage α_n in $H_n(X^n)$ along the first row in the above diagram, then $I(\alpha) := [j_n^X(\alpha_n)]$, where $j_n : H_n(X^n) \to H_n(X^n, X^{n-1}) = C_n^{CW}(X)$ is the homomorphism induced by the quotient homomorphism of chain complex, and $[-]$ means the homology class in the cellular homology. Notice that we have known that $j_n^X(\alpha_n)$ is closed and uniquely determined by α up to homology, i.e. I is well defined when we proved $H_n \approx H_n^{CW}$. We can choose $f_*(\alpha_n)$ as one of the preimages of $f_*(\alpha)$ for the above commutative diagram. Hence $I(f_*(\alpha)) = [j_n^Y \circ f_*(\alpha_n)] = [f_* \circ j_n^X(\alpha_n)]$ for the following obvious commutative diagram

$$\begin{array}{ccc}
H_n(X^n) & \xrightarrow{j_n^X} & H_n(X^n, X^{n-1}) \\
\downarrow f_* & & \downarrow f_* \\
H_n(Y^n) & \xrightarrow{j_n^Y} & H_n(Y^n, Y^{n-1}),
\end{array}$$

while $f_*^{CW}(I(\alpha)) = [f_*(j_n^X(\alpha_n))] = I(f_*(\alpha))$. So we have proved $f_*^{CW} \circ I = I \circ f_*$.

Moreover, we want to illustrate the cellular chain map f_*^{CW} when we treat $C_n^{CW}(X) = H_n(X^n, X^{n-1})$ as the free abelian group generated by n-cells of X. Precisely, we want to prove that the coefficient a_{ij}^n in the expansion $f_*^{CW}(e_i^n) = \sum_j a_{ij}^n c_j^n$ is the degree of the composition

$$D^n/S^{n-1} \xrightarrow{\bar{\varphi}_i} X^n/X^{n-1} \xrightarrow{\bar{f}} Y^n/(Y^n - c_j^n) \xrightarrow{\bar{\psi}_j^{-1}} D^n/S^{n-1},$$

where $\{e_i^n\}_{i\in I_n}$ and $\{c_j^n\}_{j\in J_n}$ are the n-cells of X and Y, φ_i and ψ_j are the characteristic maps of e_i^n and c_j^n respectively, and $\bar{f}$ means the map induced by f in the quotient spaces, etc. Recall that $H_n(X^n, X^{n-1})$ is a free abelian group generated by $\{\varphi_{i*}(o^n)\}_{i\in I_n}$, where o^n is a generator of $H_n(D^n, S^{n-1})$. Hence e_i^n just means $\{\varphi_{i*}(o^n)\}_{i\in I_n}$ when we treat $H_n(X^n, X^{n-1})$ as the free abelian group generated by n-cells of X. Now what we want results from the following diagram:

$$\begin{array}{ccccccc}
H_n(D^n, S^{n-1}) & \xrightarrow{\varphi_i} & H_n(X^n, X^{n-1}) & \xrightarrow{f} & H_n(Y^n, Y^n - c_j^n) & \underset{\approx}{\xleftarrow{\psi_j}} & H_n(D^n, S^{n-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\widetilde{H}_n(D^n/S^{n-1}) & \xrightarrow{\bar{\varphi}_i} & \widetilde{H}_n(X^n/X^{n-1}) & \xrightarrow{\bar{f}} & \widetilde{H}_n(Y^n/(Y^n - c_j^n)) & \underset{\approx}{\xleftarrow{\bar{\psi}_j}} & \widetilde{H}_n(D^n/S^{n-1}),
\end{array}$$

where all the vertical arrows are the isomorphism for the good pairs.

Hint4. Let $i_n(g)(k) = kg$, $\forall g \in C_n(X)$ and $d_n(\alpha) = \partial_n \circ \alpha$, $\forall \alpha \in \mathrm{Hom}(\mathbb{Z}, C_n(X))$, then it can be verified that $i_n : C_n(X) \to \mathrm{Hom}(\mathbb{Z}, C_n(X))$ is an isomorphism and the diagram

$$\begin{array}{ccc}
C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) \\
\downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_{n-1}} \\
\mathrm{Hom}(\mathbb{Z}, C_n(X)) & \xrightarrow{d_n} & \mathrm{Hom}(\mathbb{Z}, C_{n-1}(X))
\end{array}$$

is commutative, $\forall n$. Hence the chain complexes $(C_*(X), \partial_*)$ is isomorphic to $(\mathrm{Hom}(\mathbb{Z}, C_*(X)), d_*)$. So $h_n(X;\mathbb{Z}) \approx H_n(X)$.

We show that $\mathrm{Hom}(\mathbb{Z}_m, H) = \mathrm{Hom}(\mathbb{Q}, H) = 0$ when H is a free alelian group, hence $h_n(X;\mathbb{Z}_m) = h_n(X;\mathbb{Q}) = 0$. For an arbitrary $f \in \mathrm{Hom}(\mathbb{Z}_m, H)$, $mf(k) = f(mk) = f(0) = 0$, so $f(k) = 0$ since H is free, $\forall k \in \mathbb{Z}_m$. For an arbitrary $f \in \mathrm{Hom}(\mathbb{Q}, H)$, there is one non-zero coefficient at least when we expand $f(1)$ by a basis if H, i.e. $f(1) = ae_{i_0} + \sum_{i\neq i_0} a_i e_i$, $a \neq 0$ and $\{e_i\}$ is a basis of H. Let $f(1/(2a)) = be_{i_0} + \sum_{i\neq i_0} b_i e_i$, then $f(1) = 2af(1/(2a)) = 2abe_{i_0} + \sum_{i\neq i_0} 2ab_i e_i$. But the coefficients are uniquely determined by $f(1)$, so we obtain a absurdity that $a = 2ab \neq 0$. Hence $f(1) = 0$, then $f = 0$.

Hint5. It is easy to calculate the cellular homology groups: $H_n(X) \approx \mathbb{Z}_m$, $\widetilde{H}_i(X) = 0$, $\forall i \neq n$. And $H^{n+1}(X) \approx \mathbb{Z}_m$, $\widetilde{H}^i(X) = 0$, $\forall i \neq n$ follow from the universal coefficient theorem.
(a) It is obvious that the quotient map $q : X \to X/S^n = S^{n+1}$ induces the trivial map on $\widetilde{H}_i(-;\mathbb{Z})$. We have the following exact sequence

$$H^{n+1}(X/S^n) \to H^{n+1}(X) \to H^{n+1}(S^n) = 0$$

for good pair (X, S^n), hence $q^* : H^{n+1}(X/S^n) \to H^{n+1}(X) \approx \mathbb{Z}_m$ is surjective, and nontrivial.

If the splitting in the universal coefficient theorem for cohomology were natural, the diagram

$$\begin{array}{ccc}
H^{n+1}(X/S^n) & \xrightarrow{\approx} & \mathrm{Hom}(H_{n+1}(X/S^n), \mathbb{Z}) \oplus \mathrm{Ext}(H_n(X/S^n), \mathbb{Z}) \\
\downarrow{\scriptstyle q^*} & & \downarrow \\
H^{n+1}(X) & \xrightarrow{\approx} & \mathrm{Hom}(H_{n+1}(X), \mathbb{Z}) \oplus \mathrm{Ext}(H_n(X), \mathbb{Z})
\end{array}$$

would be commutative. The vertical arrow in the right side is the direct of $\mathrm{Hom}(H_{n+1}(Y), \mathbb{Z}) \to \mathrm{Hom}(H_{n+1}(X), \mathbb{Z})$ and $\mathrm{Ext}(H_n(Y), \mathbb{Z}) \to \mathrm{Ext}(H_n(X), \mathbb{Z})$, hence trivial, (for $\mathrm{Hom}(H_{n+1}(X), \mathbb{Z}) = 0$ and $\mathrm{Ext}(H_n(X/S^n), \mathbb{Z}) = 0$.) But q^* is nontrivial so the diagram is not commutative.
(b) Similarly.