Hint1. For the good pair $(\mathbb{R}P^2, \mathbb{R}P^1 = S^1)$, we have the sequence

$$0 = H_2(\mathbb{R}P^2) \to H_2(S^2) \xrightarrow{\times 2} H_1(S^1) \to H_1(\mathbb{R}P^2) \approx \mathbb{Z}_2,$$

$(\mathbb{R}P^2/S^1 \cong S^2,)$ then obtain

$$0 \to h^1(S^1) \approx \mathbb{Z} \xrightarrow{\times 2} h^2(S^2) \approx \mathbb{Z} \to 0$$

by the using functor $\mathrm{Hom}(-, \mathbb{Z})$ on the above sequence. This sequence of h^n is not exact obviously.

Hint2. $C_n(X, A)$ is isomorphic to the free abelian group generated by the singular n-simplices whose image is not contained in A, hence the short exact sequence

$$0 \to \widetilde{C}_*(A) \to \widetilde{C}_*(X) \to C_*(X, A) \to 0$$

splits. Then it is not difficult to verify that

$$0 \to C^*(X, A; G) \to \widetilde{C}^*(X; G) \to \widetilde{C}^*(A; G) \to 0$$

is exact. Similarly,

$$0 \to \mathrm{Hom}(\widetilde{C}_*(A) + \widetilde{C}_*(B), G) \to \widetilde{C}^*(A; G) \oplus \widetilde{C}^*(B; G) \to \widetilde{C}^*(A \cap B; G) \to 0$$

is also exact.

The inclusion $\iota_* : C^{\mathcal{U}}_*(X) \to C_*(X)$ is a chain homotopy equivalence, which means that $\iota_n p_n - \mathrm{Id}_{C_n(X)} = S_{n-1}\partial + \partial S_n$ and $p_n \iota_n - \mathrm{Id}_{C^{\mathcal{U}}_n(X)} = T_{n-1}\partial + \partial T_n$. Let the functor $\mathrm{Hom}(-, G)$ act on these equalities, we obtian $\iota^n p^n - \mathrm{Id}_{\mathrm{Hom}(C^{\mathcal{U}}_n(X), G)} = \delta T^{n-1} + T^n \delta$ and $p^n \iota^n - \mathrm{Id}_{C^n(X;G)} = \delta S^{n-1} + S^n \delta$, i.e. $\iota^* : C^*(X; G) \to \mathrm{Hom}(C^{\mathcal{U}}_n(X), G)$ is also a chain homotopy equivalence. So we have $\widetilde{H}^n(X; G) \approx H_n(\mathrm{Hom}(\widetilde{C}_*(A) + \widetilde{C}_*(B), G))$ and the excision theorem of cohomology, and as result of the letter, $H^n(X, A; G) \approx \widetilde{H}^n(X/A; G)$.

Eventually, we still have the long exact sequence

$$\to \widetilde{H}^{n+1}(A; G) \to \widetilde{H}^n(X/A; G) \to \widetilde{H}^n(X; G) \to \widetilde{H}^n(A; G) \to$$

and

$$\to \widetilde{H}^{n+1}(A \cap B; G) \to \widetilde{H}^n(X; G) \to \widetilde{H}^n(A; G) \oplus \widetilde{H}^n(B; G) \to \widetilde{H}^n(A \cap B; G) \to$$

according to all of the above discussion, which can be used to calculate $H^k(S^n; G)$. Ditto we also have the non-reduced versions of the above exact

sequences.
(c) $ri = \mathrm{Id}_A$ hence $i^n r^n = \mathrm{Id}_{H^n(A;G)}$, $\forall n$. Especially i^n is surjective, $\forall n$, therefore the exact sequence

$$\xrightarrow{i^{n+1}} H^{n+1}(A;G) \to H^n(X/A;G) \to H^n(X;G) \xrightarrow{i^n} H^n(A;G) \to$$

is a short exact sequence

$$0 \to H^n(X/A;G) \to H^n(X;G) \xrightarrow{i^n} H^n(A;G) \to 0$$

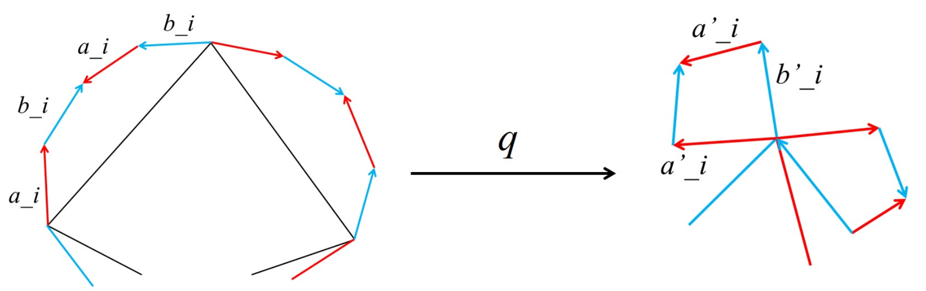
in fact, and splits for $i^n r^n = \mathrm{Id}_{H^n(A;G)}$.

Hint3. According to the universal coefficient theorem and its natruality, we have the commutative diagram

$$\begin{array}{ccccccc} 0 = \mathrm{Ext}(H_{n-1}(S^n), G) & \longrightarrow & H^n(S^n;G) & \longrightarrow & \mathrm{Hom}(H_n(S^n), G) & \longrightarrow & 0 \\ & & \downarrow f^* & & \downarrow & & \\ 0 = \mathrm{Ext}(H_{n-1}(S^n), G) & \longrightarrow & H^n(S^n;G) & \longrightarrow & \mathrm{Hom}(H_n(S^n), G) & \longrightarrow & 0, \end{array}$$

where the the both rows are exact, and the arrow in the right side is induced by $f_* : H_n(S^n) \to H_n(S^n)$.

Hint5. Let q denotes the quotient map $M_g \to \bigvee^g T^2$. We first calculate $q_* : H_*(M_g) \to H_*(\bigvee^g T^2)$ in the cellular homology. We use the CW structure of S^n containing one 0-cell v, $2g$ 1-cells a_i, b_i, $i = 1, ..., g$ and one 2-cell c, and the CW structure of $\bigvee^g T^2$ containing 0-cell v', $2g$ 1-cells a_i, b_i, $i = 1, ..., g$ and g 2-cells $c_1, ..., c_g$.



It is easy to calculate the cellular chain map: $q_*(v) = v'$, $q_*(a_i) = a'_i$, $q_*(b_i) = b'_i$ and $q_*(c) = c_1 + ... + c_g$, hence $q_*([v]) = [v']$, $q_*([a_i]) = [a'_i]$, $q_*([b_i]) = [b'_i]$ and $q_*([c]) = [c_1] + ... + [c_g]$ in cellular homology groups, and we know the all the homology groups of M_g and $\bigvee^g T^2$ are free abelian groups generated by these cells respectively. Now we return to the singular homology (because we do not want to involve the 'cellular cohomology'.) According to the above discussion ,the isomorphism between cellular homology and singular homology, and its naturality,

$$H_0(M_g) = \langle \nu \rangle, \quad H_1(M_g) = \bigoplus_{i=1}^{g} (\langle \alpha_i \rangle \oplus \langle \beta_i \rangle), \quad H_2(M_g) = \langle \gamma \rangle,$$

$$H_0(\bigvee^{g} T^2) = \langle \nu' \rangle, \quad H_1(\bigvee^{g} T^2) = \bigoplus_{i=1}^{g} (\langle \alpha'_i \rangle \oplus \langle \beta'_i \rangle), \quad H_2(\bigvee^{g} T^2) = \bigoplus_{i=1}^{g} \langle \gamma_i \rangle,$$

and $q_*(\nu) = \nu'$, $q_*(\alpha_i) = \alpha'_i$, $f_*(\beta_i) = \beta'_i$, $q_*(\gamma) = \gamma_1 + ... + \gamma_g$.

The universal coefficient theorem and its natruality shows that

$$H^0(M_g) = \langle \nu^* \rangle, \quad H_1(M_g) = \bigoplus_{i=1}^{g} (\langle \alpha^*_i \rangle \oplus \langle \beta^*_i \rangle), \quad H_2(M_g) = \langle \gamma^* \rangle,$$

$$H_0(\bigvee^{g} T^2) = \langle \nu'^* \rangle, \quad H_1(\bigvee^{g} T^2) = \bigoplus_{i=1}^{g} (\langle {\alpha'}^*_i \rangle \oplus \langle {\beta'}^*_i \rangle), \quad H_2(\bigvee^{g} T^2) = \bigoplus_{i=1}^{g} \langle \gamma^*_i \rangle,$$

and

$$q^*(\nu'^*) = \nu^*, \quad q^*(\alpha'^*_i) = \alpha^*_i, \quad q^*(\beta'^*_i) = \beta^*_i, \quad q^*(\gamma^*_i) = \gamma^*,$$

where $\{\gamma^*_i\}$ is the dual basis of $\{\gamma_i\}$, etc. Using the fact that $H^*(\bigvee^g T^2) \approx \prod^g H^*(T^2)$ as rings and recalling the cup product in $H^*(T^2)$, we know

$$\nu'^* \smile \gamma^*_i = \gamma^*_i, \quad \alpha'^*_i \smile {\beta'_j}^* = \delta_{ij}\gamma_i, \quad \alpha'^*_i \smile {\alpha'_j}^* = \beta'^*_i \smile {\beta'_j}^* = 0,$$

therefore

$$\begin{aligned} \nu^* \smile \gamma^* &= q^*(\nu'^* \smile \gamma^*_1) = \gamma^*, \\ \alpha^*_i \smile \beta^*_j &= q^*(\alpha'^*_i \smile {\beta'_j}^*) = \delta_{ij}\gamma, \\ \alpha^*_i \smile \beta^*_j &= q^*(\alpha'^*_i \smile {\alpha'_j}^*) = 0, \\ \beta^*_i \smile \beta^*_j &= q^*(\beta'^*_i \smile {\beta'_j}^*) = 0. \end{aligned}$$