USTC, School of Mathematical Sciences — *Winter semester 2019/1/5*
Algebraic topology by Prof. Mao Sheng — *Hints to exercise sheet 13*
MA04311 Tutor: Lihao Huang, Han Wu — *Posted online by Dr. Muxi Li*

Hint1. We use the following notations: $M|x := (M, M - \{x\})$, $N|y := (N, N - \{y\})$, then

$$\begin{aligned}
(M|x) \times (N|y) &= (M, M - \{x\}) \times (N, N - \{y\}) \\
&= (M \times N, (M \times (N - \{y\})) \cup ((M - \{x\}) \times N)) \\
&= (M \times N, M \times N - \{x\} \times N \cap M \times \{y\}) \\
&= (M \times N, M \times N - \{(x, y)\}) =: M \times N|(x, y).
\end{aligned}$$

Using the Künneth homomorphism of relative homology

$$\mu : H_m(M|x) \otimes H_n(N|y) \to H_{m+n}(M \times N|(x, y)),$$

we define $o(x, y) = \mu(o_M(x) \otimes o_N(y))$, where o_M and o_N are the orientation of M and N respectively. Since $H_i(N|y)$ are all free, $\forall i$, the Künneth formula tells us μ is an isomorphism. Hence $o(x, y)$ is a generater of $H_{m+n}(M \times N|(x, y))$, $\forall (x, y)$.

We need to show that o is continuous in the sense that there exists a neighbourhood W s.t. $o(x', y') = j_*^{x'y'}(\gamma)$,$\forall (x', y') \in W$, for some $\gamma \in H_{m+n}(M \times N|M \times N - W)$, where $j^{x'y'}$ is the inclusion $(M \times N|(x', y')) \to (M \times N, M \times N - W)$. We can assume that $o_M(x') = i_*^{x'}(\alpha)$ and $o_N(y') = \iota_*^{y'}(\beta)$ for some $\alpha \in H_{(}M, M - U)$ and $H_n(N, N - V)$ for the continuity of o_M and o_N. Künneth formula and its naturality implies the following commutative diagram:

$$\begin{array}{ccc}
H_m(M, M - U) \otimes H_n(N, N - V) & \xrightarrow{\nu} & H_{m+n}(M \times N, M \times N - U \times V) \\
\Big\downarrow {\scriptstyle i_*^{x'} \otimes \iota_*^{y'}} & & \Big\downarrow {\scriptstyle j_*^{x'y'}} \\
H_m(M|x) \otimes H_n(N|y) & \xrightarrow{\mu} & H_{m+n}(M \times N|(x', y')),
\end{array}$$

which shows that $o(x', y') = j_*^{x'y'}(\nu(\alpha \otimes \beta))$.

Hint2. Let $D \subset M$ is an open set whose closet $\overline{D}$ is homeomorphism to a closed n-ball. We assert that the quotient map $q : M \to M/(M - D) = \overline{D}/(\overline{D} - D) \cong S^n$ is a degree 1 map. For a point $x \in D$, f is a map between the pairs $(M, M - \{x\}) \to (S^n, S^n - \{q(s)\})$, hence we have the following commutative diagram

$$\begin{array}{ccc}
H_n(M) & \xrightarrow{f_*} & H_n(S^n) \\
\Big\downarrow {\scriptstyle j_*^x} & & \Big\downarrow {\scriptstyle j_*^{q(x)}} \\
H_n(M, M - \{x\}) & \xrightarrow{f_*} & H_n(S^n, S^n - \{q(s)\})
\end{array} \quad ,$$

where j^x and $j^{q(x)}$ are both inclusion of pairs. We can reset the orientation of S^n s.t. f_* sends $o(x)$, the local orientation at x to $o(q(x))$, the one at $q(x)$, since f is a local homeomorphism. According to the properties of the fundamental classes, $j_*^x([M]) = o(x)$, $j_*^{q(x)}([S^n]) = o(q(x))$, and j_*^x and $j_*^{q(x)}$ are isomorphisms. Therefore $f_*([M]) = [S^n]$ following from the above diagram.

Hint3. This problem can be solved by just emulating the proof of **Proposition 2.30** in the textbook.

Hint4. We follow the hint. Let $p : \widetilde{N} \to N$ be a covering space s.t. $p_*(\pi_1(\widetilde{N})) = f_*(\pi_1(M))$. Then there is a lifting of f, i.e. a map $F : M \to \widetilde{N}$ s.t. $f = pF$. The number of p's sheets m

is larger than 1 unless f_* is surjective, because it equals the index $[f_*\pi_1(M) : \pi_1(N)]$. When $1 < m < \infty$, $\deg p = m$ hence we obtain an absurdity $1 = \deg f = \deg f \deg F = m \deg F$. When $m = \infty$, $H_n(\tilde{N}) = 0$ since $\widetilde{N}$ is noncompact, and we also obtain a absudity $1 = \deg f = \deg f \deg F = 0$.

Hint5. We can immediately see that there is no map $M_g \to M_h$ with degree 1 when $g < h$, according to **Ex4**, because $H_1(M_g) \approx \mathbb{Z}^g$ and there is no epimorphism $\mathbb{Z}^g \to \mathbb{Z}^h$ for $g < h$.

Hint6. When M is orientable, $H_2(M) \approx H^1(M) \approx \mathrm{Hom}(\mathbb{Z}^r \oplus F, \mathbb{Z}) \approx \mathbb{Z}^r$, where the first isomorphism is from the Poincaré duality, the second one is from the universal coefficient theorem, and $\mathrm{Ext}(H_0(M), \mathbb{Z}) = 0$ since $H_0(M)$ is free.

When M is nonorientable, the betti numbers of M that we have known are $b_0 = 1$ (for M connected), $b_1 = r$, $b_3 = 0$ (for M nonorientable). Hence $b_2 = r - 1$ because the Euler number $\chi(M) := b_0 - b_1 + b_2 - b_3 = 0$. Therefore $H_2(M) \approx \mathbb{Z}^{r-1} \oplus G$, where G is a finite group, i.e. the torsion subgroup of $H_2(M)$, which is $\mathbb{Z}_2$ according to **Corollary 3.28** in the textbook.

Hint7. $H_k(M) \approx H^k(M) \approx \mathrm{Hom}(H_k(M), \mathbb{Z}) \oplus \mathrm{Ext}(H_{k-1}(M), \mathbb{Z}) = \mathrm{Hom}(H_k(M), \mathbb{Z})$ is torsionfree.