USTC, School of Mathematical Sciences — Winter semester 2018/19
Algebraic topology by Prof. Mao Sheng — Hint to exercise sheet 2
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Ex 1. (1 pt) *Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to identity on the two bundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles. [Consider what f does to the path $s \mapsto (\theta_0, s)$ for fixed $\theta_0 \in S^1$.]*

Hint. *For the first part, consider a map $F_t : S^1 \times I \to S^1 \times I$ by $F_t(\theta, s) = (\theta + 2\pi st, s)$. For the second part, consider projection $p : S^1 \times I \to S^1$ and the path $s \mapsto (\theta_0, s)$.*

Ex 2. (1 pt) *Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?*

Hint. *Consider the projection $p_1 : S^1 \times S^1 \to S^1$ by $p_1(s_1, s_2) = s_1$ and the natural imbedding $i : S^1 \hookrightarrow \mathbb{R}^2$. Let $f = i \circ p_1$, then Borsuk-Ulam theorem doesn't hold in this case.*

Ex 3. (1 pt) *Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Use the Borsuk-Ulam theorem to show that there is one plane $P \subset \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.*

Hint. ***Method 1**, take $s \in S^2 \subset \mathbb{R}^3$, then $\exists\,!$ one plane P_1^s in $\mathbb{R}^3$ with normal vector $\overrightarrow{0s}$ such that P_1^s divides A_1 into two pieces of equal measure. Take $p_s \in P_1^s$, then define $P_s = \{v \in \mathbb{R}^3 | \overrightarrow{vp_s} \cdot \overrightarrow{0s} \geq 0\}$ (note: this is independent of the choice of p_s). Let $f_1(s)$ (resp. $f_2(s)$) be the measure of $P_s \cap A_2$ (resp. $P_s \cap A_3$). In this way, we get a map $f : S^2 \to \mathbb{R}^2$ by $f(s) = (f_1(s), f_2(s))$. By the Borsuk-Ulam theorem, we get a $s_0 \in S^2$ such that $f(s_0) = f(-s_0)$, then $P_1^{s_0}$ is just the plane we want.*
***Method 2**, using the Borsuk-Ulam theorem for maps $S^3 \to \mathbb{R}^3$.*
Take $s = (s_1, s_2, s_3, s_4) \in S^3 \subset \mathbb{R}^4$, then consider $P_s = \{(x, y, z) \in \mathbb{R}^3 | xs_1 + ys_2 + zs_3 + s_4 \geq 0\}$. Let $f_1(s)$ (resp. $f_2(s), f_3(s)$) be the measure of $P_s \cap A_1$ (resp. $P_s \cap A_2, P_s \cap A_3$). In this way, we get a map $f : S^3 \to \mathbb{R}^3$ by $f(s) = (f_1(s), f_2(s), f_3(s))$. By the Borsuk-Ulam theorem, we get a $v \in S^3$ such that $f(v) = f(-v)$. For $(v_1, v_2, v_3) \in \mathbb{R}^3 \backslash \{0\}$, the plane $xv_1 + yv_2 + zv_3 + v_4 = 0$ is just what we want.

Ex 4. (1 pt) *From the isomorphism $\pi_1(X \times Y, (x_0, y_0)) \approx \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0\} \times Y$ represent commuting elements of $\pi_1(X \times Y, (x_0, y_0))$. Construct an explicit homotopy demonstrating this.*

Hint. *Let $[f], [g]$ be elements in $\pi_1(X, x_0), \pi_1(Y, y_0)$ respectively. We need to construct an explicit homotopy in $X \times Y$ from $f \cdot g$ to $g \cdot f$ with base point (x_0, y_0). By definition*

$$f \cdot g(s) = \begin{cases} (f(2s), y_0) & \text{for } 0 \le s \le 1/2 \\ (x_0, g(2s-1)) & \text{for } 1/2 < s \le 1 \end{cases}$$

$$g \cdot f(s) = \begin{cases} (x_0, g(2s)) & \text{for } 0 \le s \le 1/2 \\ (f(2s-1), y_0) & \text{for } 1/2 < s \le 1 \end{cases}$$

Let

$$F_{1t}(s) = \begin{cases} x_0 & \text{for } 0 \le s \le t/2 \\ f(2s-t) & \text{for } t/2 < s \le (t+1)/2 \\ x_0 & \text{for } (t+1)/2 < s \le 2 \end{cases}$$

$$F_{2t}(s) = \begin{cases} y_0 & \text{for } 0 \le s \le (1-t)/2 \\ g(2s+t-1) & \text{for } (1-t)/2 < s \le (2-t)/2 \; . \\ y_0 & \text{for } (2-t)/2 < s \le 1 \end{cases}$$

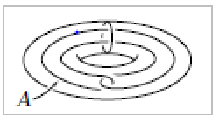
Then $F(t, s) : I \times I \to X \times Y$ given by $F(t, s) = (F_{1t}(s), F_{2t}(s))$ is a homotopy from $f \cdot g$ to $g \cdot f$ with base point (x_0, y_0).
***Alternatively**, we can construct an explicit homotopy in $X \times Y$ from $f \cdot g$ to (f, g) with base point (x_0, y_0). Let*

$$F_{1t}(s) = \begin{cases} f(2s/(1+t)) & \text{for } 0 \le s \le (1+t)/2 \\ x_0 & \text{for } (1+t)/2 < s \le 1 \end{cases}$$

$$F_{2t}(s) = \begin{cases} y_0 & \text{for } 0 \le s \le (1-t)/2 \\ g(2(s-1)/(1+t)+1) & \text{for } (1-t)/2 < s \le 1 \end{cases}$$

Then $F(t, s) : I \times I \to X \times Y$ given by $F(t, s) = (F_{1t}(s), F_{2t}(s))$ is a homotopy from $f \cdot g$ to (f, g) with base point (x_0, y_0).
Similarly, we can construct an explicit homotopy in $X \times Y$ from $g \cdot f$ to (f, g) with base point (x_0, y_0).

Ex 5. (3 pts) *Show that there are no retractions $r : X \to A$ in the following cases:*



(a) *$X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.*
(b) *$X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.*
(c) *$X = S^1 \times D^2$ and A the circle shown in the figure.*
(d) *$X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.*
(e) *X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.*
(f) *X the Möbius band and A its boundary circle.*

Hint. *If there exists retraction* $r : X \to A$*, then the inclusion* $i : A \to X$ *induces an isomorphism* $i_* : \pi_1(A) \to \pi_1(X)$.

(a) $\pi_1(A) = \mathbb{Z}, \pi_1(X) = 0.$
(b) $\pi_1(A) = \mathbb{Z} \times \mathbb{Z}, \pi_1(X) = \mathbb{Z}.$
(c) $\pi_1(A) = \mathbb{Z}, \pi_1(X) = \mathbb{Z}, i_* = 0$
(d) $\pi_1(A) = \mathbb{Z} * \mathbb{Z}, \pi_1(X) = 0.$
(e) $\pi_1(A) = \mathbb{Z} * \mathbb{Z}, \pi_1(X) = \mathbb{Z}.$
(f) $\pi_1(A) = \mathbb{Z}, \pi_1(X) = \mathbb{Z}, i_* = 2\times.$

Ex 6. (2 pts) *Using the technique in the proof of Proposition 1.14, show that if a space* X *is obtained from a path-connected subspace* A *by attaching a cell* e^n *with* $n \geq 2$*, then the inclusion* $A \hookrightarrow X$ *induces a surjection on* π_1*. Apply this to show:*

(a) *The wedge sum* $S^1 \vee S^2$ *has fundamental group* $\mathbb{Z}$.
(b) *For a path-connected CW complex* X *the inclusion map* $X^1 \hookrightarrow X$ *of its 1-skeleton induces a surjection* $\pi_1(X^1) \to \pi_1(X)$*. [For the case that* X *has infinitely many cells, see Proposition A.1 in Appendix.]*

Hint. *The same as in the proof of Proposition 1.14, every path in* X*, is homotopy to a path in* A.

(a) *Consider the natural inclusion* $S^1 \hookrightarrow S^1 \vee S^2 \hookrightarrow S^1 \vee D^3$*, the second injection using* $\partial(D^3) = S^2$.
(b) *If dim* $(X) < \infty$*, inductively use the result we get.*
If dim $(X) = \infty$*, consider a path* f *in* X*. By compactness of* $f(I)$*, this* $f(I) \subset X^n$ *for some* n.