*USTC, School of Mathematical Sciences* — *Winter semester 2018/26*
*Algebraic topology by Prof. Mao Sheng* — *Hint to the exercise sheet 2*
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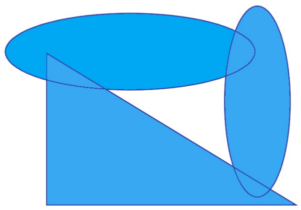
**1.** Let $X \subset \mathbb{R}^m$ be the union of convex open sets $X_1, ..., X_n$ such that $X_i \cap X_j \neq \varnothing$ for all $i, j$. Give an example to show that $X$ is not necessarily simply-connected.

**Hint.** The following figure shows a union of three convex open sets in $\mathbb{R}^2$ which is not simply-connected, but those convex open sets intersect pairwise.



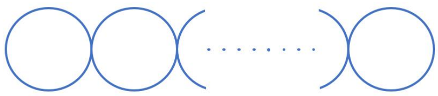
**2.** Let $X \subset \mathbb{R}^m$ be the union of simply-connected open sets $X_1, ..., X_n$ such that $X_i \cap X_j \cap X_k \neq \varnothing$ are path-connected for all $i, j, k$. Show that $X$ is simply-connected.

**Hint.** We prove $X_1 \cup ... \cup X_n$ is simply-connected by induction on $n$. The convex open set $X_n$ is simply-connected and so is $X_1 \cup ... \cup X_{n-1}$ from the induction, hence $X_1 \cup ... \cup X_n$ is simply-connected if Van Kampen's theorem works. So the key is to show $(X_1 \cup ... \cup X_{n-1}) \cap X_n$ path-connected.

Noticed $(X_1 \cup ... \cup X_{n-1}) \cap X_n = (X_1 \cap X_n) \cup ... \cup (X_{n-1} \cap X_n)$ and $X_k \cap X_n$ is path-connected, $\forall k$, then recall when the union of some path-connected sets is path-connected.

**3.** Let $X \subset \mathbb{R}^3$ be the union of $n$ lines through the origin. Compute $\pi(\mathbb{R}^3 - X)$.

**Hint.** Let $\mathbb{R}^3$ deformation retracts onto the unit sphere $S^2$, and we observe $\mathbb{R}^3 - X$ retracts onto $S^2 - Y$, where $Y \subset S^2$ consists of $2n$ points. Moreover, $S^2 - Y$ is homeomorphic to $\mathbb{R}^2 - Z$, where $Z$ consists of $2n - 1$ points. Let $\mathbb{R}^2 - Z$ deformation retract onto an union of $2n - 1$ circles like the following picture shows.



For the rigorousness, we give a proof of it that $\mathbb{R}^2 - N$ is homeomorphic to $\mathbb{R}^2 - K$, where both $N$ and $K$ are finite and have the same number, in addition, $K$ lies in a line. In fact we would like to find an auto-homeomorphism of $\mathbb{R}^2$ which sends $N$ to $K$. For every pair $a, b \in N$, there are two points in the unit circle $S^1$ perpendicular to $a - b$. We go through all of the pairs in $N$ and pick out finitely many points in $S^1$ as the above. Now choose a remaining point $c \in S^1$, and we observe $\langle a, c\rangle \neq \langle b, c\rangle$, for any two different points $a, b \in N$, that means the orthogonal projection $K$ of $N$ onto the line $\mathbb{R}c$ consists of the same number of points as $N$. By rotating, we consider $\mathbb{R}c$ as $\mathbb{R} \times \{0\}$, and choose some pairwise disjoint open intervals of $\mathbb{R} \times \{0\}$ s.t. every interval contains a point of $K$. Now it is sufficient to give a auto-homeomorphism of $\mathbb{R}^2$, which sends $(x_0, y_0)$ to $(x_0, 0)$ and keep every point in $(\mathbb{R} - I) \times \mathbb{R}$ still, where $I$ is an open interval contains $x_0$. Choose a continuous function $f$ s.t. its support (closure of the set of non-zero points) is contained in $I$ and $f(x_0) = 1$, then we construct

$$F(x, y) = \begin{cases} (x, y - f(x)y_0), & x \in I, \\ (x, y), & x \notin I, \end{cases}$$

that is the homeomorphism we want.

**4.** Let $X \in \mathbb{R}^2$ be a connected graph that is the union of a finite number of straight line segments. Show that $\pi_1(X)$ is free with a basis consisting of loops formed by the boundaries of the bounded complementary regions of $X$, joined to a basepoint by suitably chosen paths in $X$. [Assume the Jordan curve theorem for polygonal simple closed curves, which is equivalent to the case that $X$ is homeomorphic to $S^1$.]
**Hint.** Choose a maximal tree $T$ of $X$, and consider the quotient space $X/T$, which is a wedge sum of circles, where every circle comes from a edge not in $T$. The quotient mapping $X \to X/T$ is a homotopy equivalence because $T$ is contractible and the CW-pair $(X, T)$ has the homotopy extension property.

For the basis of $\pi_1(X)$, we induct on the of the bounded complementary regions. We obtain a subspace $X'$ by removing an edge of $X$ touching the unbounded complementary region, and not in $T$. The subspace $X'$ still has the maximal tree $T$ and loses a bounded complementary region whose boundary contains the removed edge. Let $\{a_1, ..., a_n\}$ denote the standard basis of $\pi_1(\vee^n S^1) \cong \pi_1(X/T)$, then the boundary of the region lost in $X'$ stands for $x * a_n * y$, where $x, y$ are elements of $\pi_1(\vee^{n-1} S^1) \cong \pi_1(X'/T)$. Now it is not hard to find $\{x_1, ..., x_{n-1}, x * a_n * y\}$ a minimal set of generators of $\pi_1(\vee^n S^1)$, where $\{x_1, ..., x_{n-1}\}$ is the basis of $\pi_1(\vee^{n-1} S^1)$ represented by the boundaries of bounded complementary regions of $X'$.

**5.** Suppose a space $Y$ is obtained from a path-connected subspace $X$ by attaching $n$-cells for a fixed $n \geqslant 3$. Show that the inclusion $X \hookrightarrow Y$ induces an isomorphism on $\pi_1$. Apply this to show that the complement of a discrete subspace of $\mathbb{R}^n$ is simply-connected if $n \geqslant 3$.

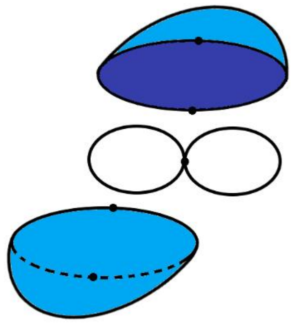
**Hint.** It is easy to solve the first part by using Van Kampen's theorem when the $n$-cells are finitely many. When the $n$-cells are infinitely many, we observe that a loop of $Y$ meets finitely many $n$-cells otherwise its image has a infinite subset intersecting every $n$-cell in one point at most. It is easy to check such a set closed , further, discrete since its every subset is closed for the same reason. That is a contradiction to the compactness of the image of the loop. So the inclusion induces a surjection $\pi_1(X) \cong \pi_1(X') \to \pi_1(Y)$, where $X'$ forms by attaching finite number of cells onto $X$. Similarly, if a loop is nullhomotopic in $Y$, the image of the homotopy lies in some $X''$, then the loop represents zero in $\pi_1(X'') \cong \pi_1(X)$.

The rest part of the original problem lacks an assumption that the discrete subspace is closed, which is supplied in the corrections (Section 1.2, page 53, Exercise 6.) of *Algebraic Topology* by Allen Hatcher, although the result still holds without this condition. A quite complicated proof of the original result which is said to be given by Professor Hatcher himself can be seen here. Now we only prove it with the additional condition and shall give another proof of the original problem at the end of this sheet.

The image of an arbitrary loop must be contained in an bounded open ball $B$, and there is only finite number of points of the closed discrete set in $B$. So it is sufficient to prove $B - N$ simply-connected, where $N$ is a finite subset. We replace $B - N$ by $B - N'$ for homotopy equivalence, where $N'$ is an union of finitely many disjoint open balls whose closures are contained in $B$. We obtain $B$ by attaching finite number of $n$-cells onto $B - N'$, but it does not change $\pi_1$ to attach $n$-cells, hence $\pi_1(B - N') = \pi_1(B)$ is trivial.

**6.** Let $X$ be the quotient space of $S^2$ obtained by identifying the north and south poles to a single point. Put a cell complex structure on $X$ and use this to compute $\pi_1(X)$.

**Hint.** The following picture shows a cell complex stucture of $X$. And use Prop 1.26 (in *Algebraic Topology* by Allen Hatcher) to compute $\pi_1(X)$. The result is $\pi_1(X) \approx \mathbb{Z}$.



**7.** Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.
**Hint.** It is easy to obtain that

$$\pi_1(X) \approx \mathbb{Z}^2 * \mathbb{Z}^2 / \langle (1,0) * (-1,0) \rangle$$

from Van Kampen's theorem. And the result can be simplified as $\mathbb{Z} \times (\mathbb{Z} * \mathbb{Z})$. We can also observe that $X$ is homeomorphisc to $S^1 \times (S^1 \vee S^1)$ and get the same result from the conclusion on the $\pi_1$ of the product spaces.

**8.** Show that $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ is uncountable.
**Hint.** It is sufficient to prove that loops $L_a$ and $L_b$ are not homotopic, $\forall a, b \in \mathbb{R} - \mathbb{Q},\ a > 0 > b$, where

$$L_x = \overline{(0,c)(x,c)}\ \overline{(x,c)(x,d)}\ \overline{(x,d)(0,d)}\ \overline{(0,d)(0,c)},$$

$\forall x \in \mathbb{R} - \mathbb{Q}$, where $c \neq d$ are fixed irrational numbers. Consider $L_a \cdot L_b^{-1}$, which is homotopic to the rectangular loop with vertex $(a,c),\ (a,d),\ (b,d)$ and $(b,c)$. So what we need to do is to show that such a rectangular loop is not nullhomotopic (in $\mathbb{R}^2 - \mathbb{Q}^2$). Recall the proof of it that the image of $f : D^2 \to \mathbb{R}^2$ covers $D^2$ if $f|_{S^1} = \mathrm{id}_{S^1}$.

**A proof of the second part in the original Ex.5**

This elegant proof was given by an Internet user named Martin M. W. at this page (Maybe there is someone credited with this proof earlier but we have not verified that).

In fact one can prove the complement of any countable set $X$ in $\mathbb{R}^n$ simply-connected, for $n \geqslant 3$. First, $C(D^2, \mathbb{R}^n)$, the space consisting of all the continuous mappings $D^2 \to \mathbb{R}^n$, can be equipped the uniform norm

$||F|| := \max_{x\in D^2} |F(x)|$ and forms a complete metric space. For an arbitrary loop $f$ lies in $\mathbb{R}^n - X$, we consider

$$\mathcal{H} := \{F \in C(D^2, \mathbb{R}^n) |\ F|_{S^1} = f\}.$$

It is easy to check that $\mathcal{H}$ is a closed subspace of $C(D^2, \mathbb{R}^n)$, hence is also complete. Let $x_1, x_2, ...$ denote all the points in $X$, and

$$\mathcal{U}_i = \{F \in \mathcal{H} |\ x_i \notin F(D^2)\},\ \forall i.$$

An element in $\bigcap_i \mathcal{U}_i$ is a mapping $D^2 \to \mathbb{R}^n - X$, s.t. $F|_{S^1} = f$, hence gives a homotopy from $f$ to a constant mapping. Therefore, what we want to prove is that $\bigcap_i \mathcal{U}_i \neq \varnothing$. We assert that $\mathcal{U}_i$ is open and dense (in $\mathcal{H}$), $\forall i$, then obtain that $\bigcap_i \mathcal{U}_i$ is dense, and of course nonempty, from the Baire category theorem.

Now we should prove the above assertion. It is easy to verify that $\mathcal{U}_i$ is open if $\mathcal{U}_i \neq \varnothing$. To prove the density of $\mathcal{U}_i$, of course which implies $\mathcal{U}_i$ nonempty, we should find an $F$ s.t. $||F - F_0|| < \varepsilon$, for any given $F_0 \in \mathcal{H}$ and $\varepsilon > 0$. We assume the loop $f$ does not go in the closed ball $\overline{B_{\varepsilon/2}(x_i)}$, then choose a disk $D' \subset D^2$ with diameter $r < 1$, and also centered at the origin, which contains $F_0^{-1}(\overline{B_{\varepsilon/2}(x_i)})$. According to the Stone-Weierstrass theorem, there exists an smooth mapping $F_1 : D^2 \to \mathbb{R}^n$ (not in $\mathcal{H}$) s.t. $||F_1 - F_0|| < \varepsilon/4$. Choose a continuous function $\varphi$ defined in $[0, 1)$ with compact support, $\varphi|_{[0,r]} = 1$, and $0 \leqslant \varphi \leqslant 1$, then we construct a mapping

$$F_2(x) = (1 - \varphi(|x|))F_0(x) + \varphi(|x|)F_1(x).$$

It is easy to find that $F_2 \in \mathcal{H}$, $\widetilde{F}|_{D'} = F_1|_{D'}$, and

$$||F_2 - F_0|| < \varepsilon/4,$$

where the inequality implies $F_2^{-1}(\overline{B}) \subset D'$, and $B$ denote $B_{\varepsilon/4}(x_i)$. Because $F_2$ is smooth in $D'$, the (Lebesgue) measure of $F_2(D')$ is 0. So $B$ is not contained in $F_2(D^2)$. Choose a point $y \in B \backslash F_2(D^2)$, and we construct

$$F(x) = \begin{cases} \text{the intersection of } \partial B \text{ and} & \\ \text{the half-line from } y \text{ to } F_2(x), & F_2(x) \in B, \\ & \\ F_2(x), & F_2(x) \notin B. \end{cases}$$

Then $F \in \mathcal{H}$,

$$||F - F_0|| \leqslant ||F - F_2|| + ||F_2 - F_0|| < \frac{3}{4}\varepsilon < \varepsilon,$$

and $x_i \notin F(D^2)$.