USTC, School of Mathematical Sciences — Winter semester 2018/19
Algebraic topology by Prof. Mao Sheng — Hint to exercise sheet 4
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Hint 1. *The ideal to construct universal covering space of a given space is to break some parts of the space where the fundamental group arise. In the first part, if we break the diameter in the meddle, then this picture is simply connected, so we can glue the infinite copy of those broken space like a tree. In the second part, if we break the circle inner and out part of the sphere, then it is simply connected, so we can glue the infinite copy of those broken space like a tree.*

Hint 2. *Removing the segment $[-1,1]$in the y axis of Y, we get an open subset of Y denoted by X. Then $f : X \to S^1 \backslash f(0,0)$ is an isomorphism and let g be the inverse of f. For $\mathbb{R} \to S^1$ is the universal covering map, if there exists a lift $\tilde{f}$, then $\lim\limits_{s\to f(0,0)^+} \tilde{f}(g(s)) \neq \lim\limits_{s\to f(0,0)^-} \tilde{f}(g(s))$, which implies $\tilde{f}$ is not continuous.*

Hint 3. *Let $f : X \to Y,\ g : Y \to X$ be morphisms such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. Take y' and view X, Y as pointed space (X, x_0), (Y, y_0) with $x_0 = g(y'), y_0 = f(x_0)$. Let $(\tilde{X}, \tilde{x_0})$, $(\tilde{Y}, \tilde{y_0})$ be the universal covering spaces given. Let $F : X \times I \to X$ (resp. $H : Y \times I \to Y$) be a homotopy of id_X and $g \circ f$ (resp. id_Y and$f \circ g$). Then we have commutative diagrams*

$$\begin{array}{ccc} (I\times \tilde{X},(0,\tilde{x}_0)) & \xrightarrow{\tilde{F}} & (\tilde{X},\tilde{x}_0) \\ \downarrow & & \downarrow \\ (I\times X,(0,x_0)) & \xrightarrow{F} & (X,x_0) \end{array} \quad , \quad \begin{array}{ccc} (I\times \tilde{Y},(0,\tilde{y'})) & \xrightarrow{\tilde{H}} & (\tilde{Y},\tilde{y'}) \\ \downarrow & & \downarrow \\ (I\times Y,(0,y')) & \xrightarrow{H} & (Y,y') \end{array} .$$

For $F|_{(1,X)} = g \circ f$ (resp. $H|_{(1,X)} = f \circ g$), so $\tilde{F}|_{(1,\tilde{X})}$ (resp. $\tilde{H}|_{(1,\tilde{Y})}$) is a lift of $g \circ f$ (resp. $f \circ g$). We have $\tilde{F}(1, \tilde{x_0})$ is a lift of $g \circ f(x_0)$ and $\tilde{H}(1, \tilde{y'})$ is a lift of y_0. By HLP, we have the following diagrams

$$\begin{array}{ccccc} (\tilde{X},\tilde{x}_0) & \xrightarrow{\tilde{f}} & (\tilde{Y},\tilde{H}(1,\tilde{y'})) & \xrightarrow{\tilde{g}} & (\tilde{X},\tilde{F}(1,\tilde{x}_0)) \\ \downarrow & & \downarrow & & \downarrow \\ (X,x_0) & \xrightarrow{f} & (Y,y_0) & \xrightarrow{g} & (X,g\circ f(x_0)) \end{array} .$$

Then $\tilde{F}$ (resp. $\tilde{H}$) is a homotopy of $id_{\tilde{X}}$ and $\tilde{g} \circ \tilde{f}$ (resp. $id_{\tilde{Y}}$ and $\tilde{f} \circ \tilde{g}$).

Hint 4. *By corollary 1.28 in textbook, for the given G, there exists a 2-dimensional connected cell complex X with $\pi_1(X, x_0) = G$. By the 1-1 correspondence between normal subgroups in $\pi_1(X, x_0) = G$ and the isomorphism class of covering $(\tilde{X}, \tilde{x_0})$ with $\pi_1(\tilde{X}, \tilde{x_0}) = N$. Then the deck transformation group is $G(\tilde{X}/X) = G/N$.*

Hint 5. *Let* $G := \pi_1(X)$*, then the normal subgroup* $[G, G]$*, the commutative subgroup of* G*, correspondence to a unique (up to isomorphism) covering space* $(\tilde{X}, \tilde{x_0})$ *with* $\pi_1(\tilde{X}, \tilde{x_0}) = [G, G]$*, which is the required abelian covering space. The universal abelian covering space of* $S^1 \vee S^1$ *is* $\mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}$*. The universal abelian covering space of* $S^1 \vee S^1 \vee S^1$ *is* $\mathbb{R} \times \mathbb{Z} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}$.