USTC, School of Mathematical Sciences — Winter semester 2018/19
Algebraic topology by Prof. Mao Sheng — Hint to exercise sheet 5
MA04311 Tutor: Lihao Huang, Han Wu — Posted by Dr. Muxi Li

Ex 1. (2 pt) *Let X be a path-connected space with base point x_0. Let $\gamma : S^1 \to X$ be a loop and $f : (D^n, \partial D^n) \to (X, x_0)$ be a map. Define $\gamma f := (D^n, \partial D^n) \to (X, x_0)$ by*

$$\gamma f = \begin{cases} f(2x) & if \quad 0 \le |x| < 1/2 \\ \gamma(2|x|-1) & if \quad 1/2 \le |x| \le 1 \end{cases}$$

Show that this induces a well-defined action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Hint. *It is easy to see that If $f_1 \overset{F_t}{\simeq} f_2$ and $\gamma_1 \overset{G_t}{\simeq} \gamma_2$ then $\gamma_1 f_1 \overset{H}{\simeq} \gamma_2 f_2$, here*

$$H = \begin{cases} G_t(2x) & if \quad 0 \le |x| < 1/2 \\ F_t(2|x|-1) & if \quad 1/2 \le |x| \le 1 \end{cases}$$

So $[\gamma f] = [\gamma][f]$ is well defined. In order to prove this is a right action, we need to prove that $(\gamma_1 \cdot \gamma_2) f \overset{H_t'}{\simeq} \gamma_2(\gamma_1 f)$. By definition,

$$(\gamma_1 \cdot \gamma_2) f = \begin{cases} f(2x) & if \quad 0 \le |x| < 1/2 \\ \gamma_1(4|x|-2) & if \quad 1/2 \le |x| \le 3/4 \\ \gamma_2(4|x|-3) & if \quad 3/4 \le |x| \le 1 \end{cases}$$

and

$$\gamma_2(\gamma_1 f) = \begin{cases} f(4x) & if \quad 0 \le |x| < 1/4 \\ \gamma_1(4|x|-2) & if \quad 1/4 \le |x| \le 1/2 \\ \gamma_2(4|x|-3) & if \quad 1/2 \le |x| \le 1 \end{cases}$$

Construct H_t' by

$$H_t' = \begin{cases} f(2(1+t)x) & if \quad 0 \le |x| < 1/[2(1+s)] \\ \gamma_1(4|x| - 2/(1+t)) & if \quad 1/[2(1+s)] \le |x| \le 1/[2(1+s)] + 1/4 \\ \gamma_2(4(1+s)/(1+3s)|x| - (3+s)/(1+3s)) & if \quad 1/[2(1+s)] + 1/4 \le |x| \le 1 \end{cases}$$

Then it is the required homotopy.

Ex 2. (2 pt) *Let (X, A) be a CW pair and let (Y, B) be any pair with $B \neq \emptyset$. For each n such that $X - A$ has cells of dimension n, assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f : (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$.*

Hint. *Refer to the proof of* ***compression lemma*** *given in textbook P347.*

Ex 3. (2 pt) *Given the definition of complex and its exactness, then split a long exact sequence into short exact sequences.*

Hint. *Let $\mathscr{A}$ be an abelian category. Let $A_i \in Object(\mathscr{A})$ and $d_{i+1} \in$* $\mathrm{Hom}(A_{i+1}, A_i)$ *with $i \in \mathbb{Z}$. Then the sequence*

$$\cdots \overset{d_{i+2}}{\to} A_{i+1} \overset{d_{i+1}}{\to} A_i \overset{d_i}{\to} A_{i-1} \overset{d_{i-1}}{\to} \cdots$$

with $d_{i+1} \circ d_i = 0 \forall i \in \mathbb{Z}$ is called a complex. If in addition, $\ker(d_i) = \mathrm{im}(d_{i+1}) \forall i \in \mathbb{Z}$, then it is called an exact complex. If it is exact, we can split it into short exact sequence

$$0 \to \ker(d_i) \to A_i \to \mathrm{im}(d_i) \to 0$$

Ex 4. (3 pt) *Let X be connected space with base point x_0. For any positive integer n, show that the following conditions are equivalent.*

(1) *Every map $S^n \to X$ is homotopic to a constant map.*

(2) *Every map $S^n \to X$ extends to a map $D^{n+1} \to X$.*

(3) $\pi_n(X, x_0) = 0$.

Hint. (1) $\Rightarrow$ (2)

Let $f : S^n \to X$, then by (1) *there exists $F(s,t) : S^n \times I \to X$ with $F(s,0) : S^n \to x_0 \to X$ and $F(s,1) = f(s)$. Consider the quotient map $p : S^n \times I \to D^{n+1}$ given by $(s,t) \mapsto t \cdot s$. Then there exists an unique map such that the following diagram commutes*

$$\begin{array}{ccccc} S^n & \overset{i}{\hookrightarrow} & S^n \times I & \xrightarrow{F(s,t)} & X \\ & & \downarrow p & \nearrow \exists! G & \\ & & D^{n+1} & & \end{array}$$

Here $i(s) = (s,1)$. View S^n as a subcomplex by $p \circ i$, then the map G is the required extension. (2) $\Rightarrow$ (3)

Take $[f] \in \pi_n(X, x_0)$ then by (2) *there exists an extension $G : D^{n+1} \to X$. Define a homotopy F by the composition of* $S^n \times I \xrightarrow{p} D^{n+1} \xrightarrow{G} X$.

Then $[f] = 0$.

(3) $\Rightarrow$ (1) *trivial.*

Ex 5. (2 pt) *Given a CW pair (X, A) and a map $f : A \to Y$ with Y path-connected, then f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X - A$ has cells of dimension n.*

Hint. *Refer to the proof of* ***extension lemma*** *givne in textbook P348.*

Ex 6. (2 pts) *Let $f : X \to Y$ be a cellular map between CW complexes. Show that the mapping cylinder $M(f)$ of f has a natural CW-structure such that X, Y are CW-subcomplexes of $M(f)$.*

Hint. *Let* $X^n = X^{n-1} \sqcup_\alpha \; e_\alpha^n$ *with attaching maps* $\varphi_\alpha : S^{n-1} \to X^{n-1}$, $Y^n = Y^{n-1} \sqcup_\beta \; e_\beta^n$ *with attaching maps* $\varphi_\beta : S^{n-1} \to Y^{n-1}$ *and* $f(X^n) \subset Y^n \; \forall n$. *The mapping cylinder* $M(f) = X \times I \sqcup Y/ \sim = X \sqcup X \times e^1 \sqcup Y$. *So* $M(f)^{n+1} = M(f)^n \sqcup_\alpha \; e_\alpha^{n+1} \sqcup_\alpha \; e_\alpha^n \times e^1 \sqcup_\beta \; e_\beta^{n+1}$. *Attaching maps for* e_α^{n+1} *and* e_β^{n+1} *are induced by* φ_α, φ_β *naturally. For the boundary* $\partial(D_\alpha^{n+1} \times I) = D_\alpha^{n+1} \times 0 \sqcup S^n \times e^1 \sqcup D_\alpha^{n+1} \times 1$, *define the map* $\partial(D_\alpha^{n+1} \times I) \to M(f)^n$ *by the first two pieces induced by* φ_α *and the last piece given by the composition* $f \circ \varphi_\alpha$, *which is the attaching map of* $e_\alpha^n \times e^1$.

Ex 7. (2 pts) *Let* (X, A) *be a CW-pair and* $x_0 \in A$. *Show that the sequence* $\pi_1(A, x_0) \to \pi_1(X, x_0) \xrightarrow{\partial_3} \pi_1(X, A, x_0) \xrightarrow{\partial_2} \pi_0(A, x_0) \xrightarrow{\partial_1} \pi_0(X, x_0)$ *is exact.*

Hint. *Exactness at* $\pi_1(X, x_0)$ *refer to theorem 4.3 given in textbook P344. It is easy to see* $\partial_2 \circ \partial_3 = 0$ *and* $\partial_1 \circ \partial_2 = 0$. *Let* $f : (I, \partial I, 0) \to (X, A, x_0)$. *If* $f(1), x_0$ *are in the same path-connected component of* A, *then choose a path* $\gamma : I \to A$ *such that* $\gamma(0) = f(1)$, $\gamma(1) = x_0$. *Then* $[f \cdot \gamma] \in \pi_1(X, x_0)$ *and* $[f \cdot \gamma] = [f]$ *in* $\pi_1(X, A, x_0)$, *which means the exactness at* $\pi_1(X, A, x_0)$. *It is easy to see the exactness at* $\pi_0(A, x_0)$.