Hint 1. *Let the exact sequence be* $A \xrightarrow{d_1} B \xrightarrow{d_2} C \xrightarrow{d_3} D \xrightarrow{d_4} E$.
$\Rightarrow$ *By exactness,* $\ker d_2 = \operatorname{im} d_1 = B$, $\ker d_4 = \operatorname{im} d_3 = 0$ *so* d_1 *is surjective and* d_4 *is injective.*
$\Leftarrow$ *By exactness and surjection of* d_1*, have* $\operatorname{im} d_1 = \ker d_2 = B$*, which implies* $d_2 = 0$*. By exactness and injection of* d_4*, have* $\operatorname{im} d_3 = \ker d_4 = 0$*, which implies* $d_3 = 0$*. So* $0 = \operatorname{im} d_2 = \ker d_3 = C$.
The long exact sequence $\cdots \to H_{n+1}(X,A) \to H_n(A) \xrightarrow{d_n} H_n(X) \to H_n(X,A) \to \cdots$ *implies that all* d_n *are isomorphisms (injection and surjection) iff* $H_n(X,A) = 0, \ \forall n$.

Hint 2. *(a)By exactness of* $H_0(A) \xrightarrow{i} H_0(X) \to H_0(X,A) \to 0$*, the* $H_0(X,A) = 0$ *iff* i *is surjective.* $H_0(-)$ *is free* $\mathbb{Z}$*-module generated by path-component of the given space. Thus, the surjection of* i *means that* A *meets each path-component of* X.
(b) By exactness of $H_1(A) \to H_1(X) \to H_1(X,A) \xrightarrow{\partial} H_0(A) \xrightarrow{i} H_0(X)$ *and exercise 1 above,* $H_1(X,A) = 0$ *iff* $H_1(A) \to H_1(X)$ *is surjective and* i *is injective. The same analyses as (a),the injective of* i *means that each path-component of* X *contains at most one path-component of* A.

Hint 3. *(a) Let the number of points of* A *be* m*, then*

$$H_n(A) = \begin{cases} \mathbb{Z}^{\oplus m} & if \quad n = 0 \\ 0 & if \quad n > 0. \end{cases}$$

It is easy to calculate that

$$H_n(S^2) = \begin{cases} \mathbb{Z} & if \quad n = 0 \ or \ n = 2 \\ 0 & otherwise \end{cases} \qquad H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & if \quad n = 0 \ or \ n = 2 \\ \mathbb{Z}^{\oplus 2} & if \quad n = 1 \\ 0 & otherwise \end{cases}$$

By the exact sequence $H_2(A)(=0) \to H_2(X) \xrightarrow{\approx} H_2(X,A) \to H_1(A)(=0) \to H_1(X) \to H_1(X,A) \to H_0(A)(= \mathbb{Z}^{\oplus m}) \twoheadrightarrow H_0(X)(= \mathbb{Z}) \to H_0(X,A)(= 0) \to 0$*, we have*

$$H_n(S^2, A) = \begin{cases} \mathbb{Z}^{\oplus m-1} & if \quad n = 1 \\ \mathbb{Z} & if \quad n = 2 \\ 0 & otherwise \end{cases} \quad H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z}^{\oplus m+1} & if \quad n = 1 \\ \mathbb{Z} & if \quad n = 2 \\ 0 & otherwise \end{cases}$$

(b) ***Method 1,*** *using the* $\triangle$ *complex structure of* X *pictured in textbook P102 second picture, having one vertex, four edges, six 2-simplices, we can*

calculate the simplicial homology group of X

$$H_n(X) = H_n^{\triangle}(X) = \begin{cases} \mathbb{Z} & if \quad n=0 \ or \ n=2 \\ \mathbb{Z}^{\oplus 4} & if \quad n=1 \\ 0 & otherwise \end{cases}$$

And from example 2,2 in textbook P106,

$$H_n(S^1) = H_n^{\triangle}(S^1) = \begin{cases} \mathbb{Z} & if \quad n=0 \ or \ n=1 \\ 0 & otherwise \end{cases}$$

We have a long exact sequence $\cdots \to H_2(S^1)(=0) \to H_2(X)(=\mathbb{Z}) \hookrightarrow H_2(X,S^1) \xrightarrow{\partial} H_1(S^1)(=\mathbb{Z}) \xrightarrow{d} H_1(X)(=\mathbb{Z}^{\oplus 4}) \to H_1(X,S^1) \xrightarrow{0} H_0(S^1)(=\mathbb{Z}) \xrightarrow{\approx} H_0(X)(=\mathbb{Z}) \to H_0(X,S^1)(=0) \to 0$.
For $S^1 = A$ *as in the picture, the image of the generator of* $H_1(A)$ *in* $H_1(X)$ *is* 0 *because this is the boundary of three adjunctive 2-simplices. So we have*

$$H_n(X,A) = \begin{cases} \mathbb{Z}^{\oplus 4} & if \quad n=1 \\ \mathbb{Z}^{\oplus 2} & if \quad n=2 \\ 0 & otherwise \end{cases}$$

For B *as in the picture, the image of the generator of* $H_1(A)$ *in* $H_1(X)$ *is one generator of* $H_1(X)$*, So* d *is an injection and* $\partial = 0$.

$$H_n(X,B) = \begin{cases} \mathbb{Z}^{\oplus 3} & if \quad n=1 \\ \mathbb{Z} & if \quad n=2 \\ 0 & otherwise \end{cases}$$

Method 2, *using the fact that* $H_n(X,S^1) = \tilde{H}_n(X/S^1)$*. For* $S^1 = A$ *as in the picture,* $X/A = T^1 \vee T^1$*. By Mayer-Vietoris sequences or Corollary 2.25 in textbook P126, we have*

$$H_n(X,A) = \tilde{H}_n(X/A) = \begin{cases} \mathbb{Z}^{\oplus 4} & if \quad n=1 \\ \mathbb{Z}^{\oplus 2} & if \quad n=2 \\ 0 & otherwise \end{cases}$$

For B *as in the picture,* X/B *is homotopy to* $T^1 \vee S^1$*. Similarly, we have*

$$H_n(X,B) = \tilde{H}_n(X/B) = \begin{cases} \mathbb{Z}^{\oplus 3} & if \quad n=1 \\ \mathbb{Z} & if \quad n=2 \\ 0 & otherwise \end{cases}$$

Hint 4. *By the exact sequence* $H_1(\mathbb{R})(=0) \to H_1(\mathbb{R},\mathbb{Q}) \xrightarrow{\partial} H_0(\mathbb{Q})(=\bigoplus_{\mathbb{Q}}\mathbb{Z}) \xrightarrow{i} H_0(\mathbb{R})(=\mathbb{Z})$*, we have* $H_1(\mathbb{R},\mathbb{Q}) = \bigoplus_{\mathbb{Q}^*}\mathbb{Z}$.

Hint 5. *Let this space be* X*, and let* $A = I \times [0, 2/3] \cap X,\ B = I \times [1/3, 1] \cap X$*. Using Mayer-Vietoris sequences, we have* $\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to \cdots$*. For* A*,* B *are contractible, we have* $\tilde{H}_n(A) = \tilde{H}_n(B) = 0$*. And* $A \cap B$ *is homotopy to* $\mathbb{Q} \cap I$*. So we have*

$$H_n(X) = \begin{cases} \mathbb{Z} & if \quad n = 0 \\ \bigoplus_{\mathbb{Q}^* \cap I} \mathbb{Z} & if \quad n = 1 \\ 0 & otherwise \end{cases}$$

Hint 6. *For the first part, consider* $X \hookrightarrow CX \twoheadrightarrow CX/X \cong SX$*. There is a long exact sequence* $\cdots \to \tilde{H}_{n+1}(SX) \overset{\partial_{n+1}}{\to} \tilde{H}_n(X) \to \tilde{H}_n(CX) \to \tilde{H}_n(SX) \overset{\partial_n}{\to} \cdots$*. For* $\tilde{H}_n(CX) = 0 \quad \forall n$*, we have* $\tilde{H}_n(X) = \tilde{H}_{n+1}(SX) \quad \forall n$*. For the second part, let* $S^r X$ *be the union of* $r + 1$ *cones* CX *with their basis identified. For* $r \geq 2$*, consider* $S^{r-1}X \overset{i}{\hookrightarrow} S^r X \twoheadrightarrow S^r X / S^{r-1} X \cong SX$*. There is a long exact sequence* $\cdots \to \tilde{H}_{n+1}(SX) \overset{\partial_{n+1}}{\to} \tilde{H}_n(S^{r-1}X) \to \tilde{H}_n(S^r X) \to \tilde{H}_n(SX) \overset{\partial_n}{\to} \cdots$*. There exists a section of* i*, which means every* $\partial_n = 0$ *and* $\tilde{H}_n(S^r X) = \tilde{H}_n(S^{r-1}X) \oplus \tilde{H}_n(SX)$*. Inductively, we have*

$$\tilde{H}_n(S^r X) = \begin{cases} \tilde{H}_n(SX)^{\oplus r} = \tilde{H}_{n-1}(X)^{\oplus r} & if r > 0 \\ 0 & if \quad r = 0 \end{cases}$$

Hint 7. *Let* $\sigma : \Delta^n \to X$*. It induces* $\tilde{s} : C\Delta^n \to CX$*. Let linear map* $\Delta^{n+1} \cong C\Delta^n$ *given by mapping the last vertex of* Δ^{n+1} *to the vertex of the cone. Define* $s(\sigma) : \Delta^{n+1} \cong C\Delta^n \overset{\tilde{s}}{\to} CX \to CX/X = SX$*. It is easy to check that* $\partial \circ s = s \circ \partial$*. This chain map induces an isomorphisms* $\tilde{H}_n(X) = \tilde{H}_{n+1}(SX)$*, which is the same as that in exercise 6.*