中国科学技术大学数学科学学院
2018～2019学年第一学期考试试卷(A卷)

课程名称：代数拓扑　课程编号：MA04311

开课院系：数学科学学院　考试形式：开卷

姓名：__________　学号：____________　专业：____________

题 号	一	二	三	四	五	总 分
得 分						

第一题 (20 分)

1. Let $D^* = \{z \mid 0 < |z| \leqslant 1\}$ and $\Gamma = \{z \mid |z| = 1/2\}$ as the subspaces of $\mathbb{C}$.

(a) Show that D^* deformation retracts to Γ.

(b) Let X denote the quotient space of D^* obtained by identifying every point and its antipodal point in the unit circle S^1, with the quotient map q. Prove X cannot retract to $q(\Gamma)$.

(a) $F(z,t) = (1-t)z + tz/(2|z|)$ gives a deformation retraction.

(b) Let $H([z],t) = [(1-t)z + tz/|z|]$. It is not hard to verify H a well defined deformation retraction from X to $q(S^1) \cong S^1$, hence $\pi_1(X) \cong \mathbb{Z}$ and $X \xrightarrow{H_1} q(S^1)$ is homotopy equivalence.

The homeomorphisms $S^1 \xrightarrow{f} q(S^1)$, $e^{i\theta} \mapsto [e^{i\theta/2}]$ and $S^1 \xrightarrow{g} q(\Gamma)$, $e^{i\theta} \mapsto [e^{i\theta}/2]$ are the generator of $\pi_1(q(S^1))$ and $\pi_1(q(\Gamma))$ respectively. By comparing $H \circ g(e^{i\theta}) = [e^{i\theta}]$ and f, the homomorphism $\mathbb{Z} \cong \pi_1(q(\Gamma)) \to \pi_1(q(S^1)) \cong \mathbb{Z}$ induced by H_1 is to time 2. And the homomorphism induced by inclusion $q(\Gamma) \xrightarrow{i} X$ is also to time 2 according to the following commutative diagram

$$\begin{array}{ccc} q(\Gamma) & \xrightarrow{H_1} & q(S^1) \\ {\scriptstyle i}\downarrow & & \parallel \\ X & \xrightarrow{H_1} & q(S^1). \end{array}$$

So the homomorphism of the fundamental groups induced by i has no left inverse. That is contradictory to the existence of retraction $X \to q(\Gamma)$.

第二题 (20分)

2. Let K be m distinct points in $\mathbb{R}^n$ away from zero. Calculate $\pi_1(\mathbb{R}^n - K, \mathbf{0})$ and $\pi_1(\mathbb{R}^n/K, [\mathbf{0}])$.

Without loss of generality, we assume that the points $x_1, ..., x_k$ of K lie in a line in order.

When $n = 1$, $\mathbb{R}^n - K$ is the union of $m+1$ open intervals, whose fundamental group is always trivial whenever the base point is. And $\mathbb{R}^n/K$ is homeomorphic to the wedge sum of $m-1$ circles and a line, hence $\pi_1(\mathbb{R}^n/K)$ is the free groups with $m-1$ generator.

When $n > 1$, $\mathbb{R}^n - K$ deformation retracts to $\bigcup_i s_i$, where s_i is an $n-1$-dim sphere with the center x_i and radius r_i, $\forall i$, s.t. $r_i + r_{i+1} = |x_i - x_{i+1}|$, $\forall i < m$. Therefore $\pi_1(\mathbb{R}^n - K)$ is the free group with m generators for $n = 2$ and trivial for $n > 2$.

We choose a deformation retraction of $\mathbb{R}^n$ to the segment L connecting x_1 and x_m, which induces a deformation retraction of $\mathbb{R}^n/K$ to L/K. And L/K is homeomorphic to the wedge sum of $m-1$ circles, so $\pi_1(\mathbb{R}^n/K)$ is still the free group with $m-1$ generators for $n > 1$.

第三题 (20分)

3. Let X and Y are connected spaces with base points x_0 and y_0 respectively. The smash product of X and Y is denfined as $X \wedge Y := X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y)$.

(a) Prove the quotient map $X \times Y \to X \wedge Y$ inducing the trivial homomorphism of fundamental groups.

(b) Prove $\pi_1(X \wedge Y)$ trivial when X and Y are finite CW-complexes and x_0 and y_0 are their respective 0-cells.

(a) We show that $[(p_1\gamma, y_0)][(x_0, p_2\gamma)] = [\gamma]$ for any loop γ in $X \times Y$ with $\gamma(0) = (x_0, y_0)$, where p_1 and p_2 are projections onto X and Y respectively. Because $[\gamma] \mapsto (p_{1*}[\gamma], p_{2*}[\gamma])$ is an isomorphism, it is sufficient to check $p_{i*}[\gamma] = p_{i*}([(p_1\gamma, y_0)][(x_0, p_2\gamma)])$, $i = 1, 2$, which is easy. As a result, every $[\gamma] \in \pi_1(X \times Y)$ equals $[(f, y_0)][(x_0, g)]$ for some loop f in (X, x_0) and g in (Y, y_0).

According the above discussion, it is sufficient to show that $X \times Y \to X \wedge Y$ sends the loops with the form $[(f, y_0)]$ and $[(x_0, g)]$ to trivial loops in $X \wedge Y$. In fact they are obviously constant loops.

(b) $X \vee Y := X \times \{y_0\} \cup \{x_0\} \times Y$ is a CW-subcomplex of $X \times Y$, so the CW-pair $(X \times Y, X \vee Y)$ has homotopy extension property.

Let (Z, A) be a pair with the HEP, CA the cone over A, and $Z \cup CA$ the quotient space from identifying $(a, 0) \in CA$ and $a \in Z$, $\forall a \in A$. It is not difficult to check that $(Z \cup CA, CA)$ also has HEP. Hence $Z \cup CA$ is homotopy equivalent to $Z \cup CA/CA = Z/A$. Consider the $U = Z \cup CA - \{\text{pt}\}$ and $V = CA - A \times \{0\}$ as the subspaces of $Z \cup CA$, where 'pt' denotes the top point of cone CA, then V is contractible, $U \cap V$ is homeomorphic to $A \times I$ hence deformation retracts to A, and U retracts to Z similarly. So Van Kampen's theorem implies $\pi_1(Z \cup CA)$ is the cokernel of $\pi_1(A) \to \pi_1(Z)$, where the homomorphism is induced by inclusion $A \to Z$. Applying this to $(X \times Y, X \vee Y)$, we obtain $\pi_1(X \wedge Y)$ is trivial since the homomorphism $\pi_1(X \vee Y) \to \pi_1(X \times Y)$ is surjective.

第四题 (20 分)

4. Recall that the covering spaces of S^1 up to isomorphisms of covering spaces are given by

$$\{\varphi_\infty : \mathbb{R} \to S^1,\ t \mapsto e^{\mathrm{i}t}\} \cup \{\varphi_n : S^1 \to S^1,\ z \mapsto z^n |\, n \in \mathbb{N}_+\},$$

where S^1 is considered as $\{z \in \mathbb{C} |\, |z| = 1\}$. Now classify the covering spaces of $S^1 \times S^1$ up to isomorphisms of covering spaces in a similar form.

The covering spaces of T^2 1-1 correspond to the conjugacy classes of subgroups of $\pi_1(T^2, x_0) \cong \mathbb{Z}^2$. But $\pi_1(T^2)$ is abelian, the covering spaces 1-1 correspond to the subgroups of $\pi_1(T^2, x_0)$ in fact. We treat T^2 as $\mathbb{R}^2/\mathbb{Z}^2$, then for a subgroup G of $\mathbb{Z}^2$, the covering space corresponding to it is $\mathbb{R}^2/G$, and the covering projection is quotient homomorphism $\mathbb{R}^2/G \to (\mathbb{R}^2/G)/(\mathbb{Z}^2/G) \cong T^2$, i.e. the map s.t. every point has the same representation as its image.

To describe the projection more visually, we consider the covering spaces as standard cylinder $S^1 \times \mathbb{R}$ or standard torus $S^1 \times S^1$ via a automorphism of $\mathbb{R}^2$ and homeomorphism $\mathbb{R}/\mathbb{Z} \to S^1,\ \theta \mapsto e^{2\pi \mathrm{i}\theta}$.

For a subgroup with rank 2 generated by $\{x, y\}$, a unique automorphism $(\theta, \varphi)^T \mapsto x\theta + y\varphi$ is determined, so the covering space corresponding to this subgroup is

$$p_{x,y} = T^2 \to T^2, \quad (e^{2\pi \mathrm{i}\theta}, e^{2\pi \mathrm{i}\varphi}) \mapsto (e^{2\pi \mathrm{i}(x_1\theta + y_1\varphi)}, e^{2\pi \mathrm{i}(x_2\theta + y_2\varphi)}).$$

But $\{x, y\}$ and $\{x', y'\}$ generate the same subgroup if $(x, y)A = (x', y')$ for some invertible 2×2 matrix A over $\mathbb{Z}$, therefore the equivalence classes of covering spaces corresponding to rank-2 subgroups are $\{p_{[x,y]} |\ [x, y] \in M_2/GL(2, \mathbb{Z})\}$, where M_2 denotes all the 2×2 matrices over $\mathbb{Z}$ with rank 2, and $p_{[x,y]}$ denotes the class in which $p_{x,y}$ is.

For the subgroup generated by $x \neq 0$, we need to choose a automorphism of $\mathbb{R}^2$ s.t. $(1, 0)^T \mapsto x$. We fix the automorphism $(\theta, r)^T \mapsto x\theta + (0, 1)^T r$ for $x_1 \neq 0$, and $(\theta, r)^T \mapsto x\theta + (1, 0)^T r$ for $x_1 = 0$, then the covering space corresponding to the subgroup $\mathbb{Z}x$ is

$$p_x : S^1 \times \mathbb{R} \to T^2, \quad (e^{2\pi \mathrm{i}\theta}, r) \mapsto (e^{2\pi \mathrm{i}x_1\theta}, e^{2\pi \mathrm{i}(x_2\theta + r)})$$

when $x_1 \neq 0$ and

$$(e^{2\pi \mathrm{i}\theta}, r) \mapsto (e^{2\pi \mathrm{i}(x_1\theta + r)}, e^{2\pi \mathrm{i}x_2\theta})$$

when $x_1 = 0$. Therefore the equivalence classes of covering spaces corresponding to rank-1 subgroups are $\{p_{[x]} |\ [x] \in M_1/\{\pm 1\}\}$, where M_1 denotes $\mathbb{Z}^2 - \{0\}$.

The two remaining equivalence classes of covering spaces are represented by a homeomorphism $T^2 \to T^2$ and the universal covering $\mathbb{R}^2 \to T^2$.

第五题 (20 分)

5. Let X be a connected, locally path-connected and semilocally simply-connected space, and $p : \widetilde{X} \to X$ be a connected covering space.

(a) Prove that $\widetilde{X}$ is noncompact when $|p^{-1}(x)|$ is infinite for a point $x \in X$.

(b) Prove that $|p^{-1}(x)|$ is constant with respect to x, which is denoted by $\deg\ p$.

(c) Prove that $|G| \leqslant \deg\ p$, where G denotes the group of deck transformations of the covering space $p : \widetilde{X} \to X$.

(d) Assume that $\deg p < \infty$. Prove that $|G| = \deg p$ if and only if $p : \widetilde{X} \to X$ is a normal covering space.

(a) We assume that $\widetilde{X}$ is compact and $p^{-1}(x)$ is infinite. Let $y \in \widetilde{X}$ is a point s.t. its every neighbourhood contains infinitely many points of $p^{-1}(x)$, U is a neighbourhood of y s.t. $p : U \to p(U)$ is homeomorphism. There must be two distinct point $x_1, x_2 \in p^{-1}(x) \cap U$. Hence there is a contradiction that p is injective in U but $p(x_1) = p(x_2) = x$.

Because the cardinality of the set $p^{-1}(x)$ is locally constant over X and X is connected, this cardinality is constant as x ranges over all of X.

(b) It is sufficient to show that the action of G on $p^{-1}(x)$ is free, i.e. $f = g$ if $fx_0 = gx_0$ for a given $x_0 \in p^{-1}(x)$, or equivalently, $g = 1_{\widetilde{X}}$ if $gx_0 = x_0$. A deck transformation g can be seen as a lifting of the map p to the covering space $\widetilde{X}$, and the lifting is unique if it keeps the base points, so $g = 1_{\widetilde{X}}$ is the only lifting s.t. $gx_0 = x_0$.

(c) Let $\pi := \pi_1(X, x)$, $H =: p_*(\pi_1(\widetilde{X}, x_0))$ and $N(H)$ denotes the normalizer of H in π. The order $|N(H)| = |G||H|$ since $N(H)/H \cong G$. In addition, $|\pi| = \deg p|H|$ since $\deg p = [\pi : H]$. So $|N(H)| = |\pi|$ iff $|G| = \deg p$. And the former is equivalent to $N(H) = \pi$, i.e. H is a normal subgroup of π.