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2016年12月19日丘成桐教授在三亚清华国际数学论坛纪念Emmy Noether的报告。该报告将在数理人文出版社发表。该文件仅供自己学习使用，请勿传播导致侵犯版权。谢谢合作！

Geometry and Physics

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Master Lectures–the Legacy of Emmy Noether
and Göttingen mathematics

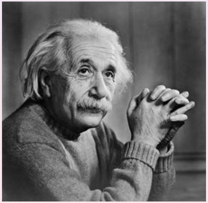
December 19, 2016

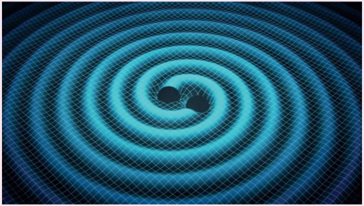
Mathematics, rightly viewed, possesses not only truth, but supreme beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Bertrand Russell

In February 2016, there was an announcement by NSF, Caltech and MIT that they have found gravitational wave as predicted by Einstein 100 years ago.

This is a triumph of the theory of Einstein who proposed that gravity should be looked at as effects of curvature of space-time. My title of Geometry and Physics is very much related to this proposal of Einstein.





Before this theory of Einstein, scientists followed the view point of Newton: the space is static and gravity allows action at a distance simultaneously.

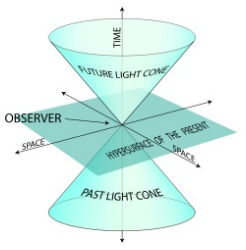
When special relativity was discovered at 1905 by Einstein, with the helps by several people including Lorentz and Poincaré, it was found that physical information should not travel faster than light, and the principle of action at a distance that was used by Newtonian gravity is not compatible to the newly found special relativity.

Lorentz

Poincaré

In a vague sense, Einstein and other physicists knew that space and time cannot be distinguished under the rules of relativity.

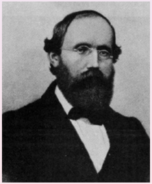
It was not until 1908 that Minkowski, teacher of Einstein, proposed the concept of Minkowski four dimensional space-time where a metric of Riemann type is used to describe all phenomena that appear in special relativity. The group of motion of this space-time is given by the Lorentzian group.



After this important discovery of Minkowski, Einstein realized that the description of gravity should be given by a four dimensional object. By thought experiments, he realized that the quantity to describe gravity has to depend on directions at each point. When an observer is moving in some direction, the distance he or she measures will change depending on the direction he is measuring. (When it is measured in the direction perpendicular to his movement, nothing changes but is different when it is parallel to his movement.)

After he consulted his college friend Grossmann, Einstein understood that gravity should be measured by a tensor: a quantity that was invented by geometers (Christoffel, inspired by Riemann's works) in late nineteenth century. In fact, the tensor he needed was first invented by Riemann in 1854, with different signature. This was a great breakthrough as Newtonian gravity was measured by a scalar function, not by a tensor.

Grossmann



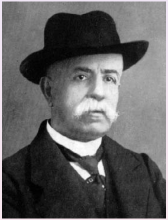
Riemann

Christoffel

When Einstein wanted to generalize the equation of Newton on gravity, he needed some quantity that was obtained by differentiating the above metric tensor two times. (In Newtonian gravity, it was determined by the Laplacian of the scalar gravitational potential.) But he wanted to make sure everything obeys the equivalence principle: every law of physics is the same independent of frame of observer. Hence the result of this second derivative of the metric tensor must be a tensor again.

Well, it is known that all tensors that are obtained by differentiating the metric tensor twice must be a combination of the curvature tensor of the metric and the curvature tensor is the only tensor that depends linearly on the second derivatives of the metric. If there are other physical fields coming from different matter (other than gravity) there is a matter stress tensor. Then a simple generalization of Newton's equation is try to equate the above curvature tensor with this matter tensor.

There is only a couple of curvature tensors that can do the job. One is called the Ricci tensor which was found in the library by Grossmann for Einstein. It was invented by Ricci in the end of nineteenth century.



Ricci

Einstein and Grossmann wrote two papers in 1912 and 1913, where they wrote down the equation for gravity in tensorial form. But Einstein was not able to use the equations to explain perihelion of Mercury. He was tempted to give up the equivalence principle by choosing a suitable coordinate system. The Einstein-Grossmann equation did not satisfy conservation law and has to be modified!

Einstein struggled on this question until he met Hilbert in 1915. By November, Hilbert and Einstein arrived at the derivation of the Einstein equation around the same time. Hilbert also found the Hilbert action whose variation will give the Einstein equation. It should be noted that Hilbert gave a lot of credit to Emmy Noether on helping him to achieve the works. The great accomplishment of Einstein also relied on his understanding of the physical meaning and the application of the equation to explain astronomical events.

Noether

Hilbert

The geometers Euler, Gauss, Riemann, Ricci, Christoffel, Bianchi, Minkowski, Hilbert, Levi-Civita and others had great impact on the creation of the subject of general relativity. But the creation of general relativity has tremendous input on the development of Riemannian geometry in the twentieth century up to present days.

We shall discuss about such development now.

Euler

Gauss

Levi-Civita

The idea of using symmetry to dictate geometry and physical phenomena:

Some physicists claimed that Einstein was the first one to use symmetry to derive an equation of interest. Actually his work was inspired by his teacher Minkowski who used Lorentzian group as the group of symmetry to derive the Minkowski space-time.

In fact, the idea of using symmetry to understand geometry went back to the nineteenth century where the works of Sophus Lie and Klein helped us to create invariants of geometry by continuous symmetries. Klein's famous Erlangen Program in 1872 was to classify geometry according to their group of symmetry.

Klein

Lie

The classification of the structure of the Lie groups is one of the most glorious chapter in mathematics. It starts from Sophus Lie, Killing, Klein and continued into the twentieth century by E. Cartan, and H. Weyl. The power of representation theory of finite and compact groups has repeatedly appeared in geometry and physics.

Cartan

Killing

Weyl was the key person to pioneer the theory for compact Lie groups. Hermann Weyl, Eugene Wigner and others applied such theory to quantum mechanics and brought fruitful results. It has been one of the most important method in studying particle physics, where groups $U(1)$, $SU(2)$, $SU(3)$ and $SU(5)$ are related to electric magnetic field, weak interaction, strong interaction and grand unified fields.

Weyl

Wigner

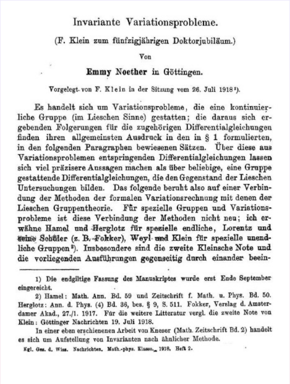
The work of Emmy Noether on the action principle had direct influence on modern physics and geometry. In fact, she was in Göttingen in 1915 when Hilbert was working on the action principle of general relativity. Hilbert acknowledged the influence of her ideas on his work.

The Noether theorem, which was published in 1918, inspired the modern treatment of mechanics and modern symplectic geometry and the concept of moment map.

The Noether theorem:

If a system has a continuous symmetry property, then there corresponds a quantity that is conserved in time.

Noether E. *Invariante Variationsprobleme*. Nachr. König. Gesellsch. Wiss. Zu Göttingen, Math-phys. Klasse. (1918) 235–257.



Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

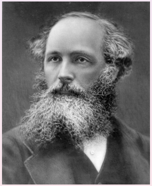
2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

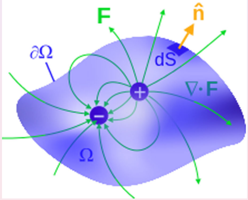
Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2.

Development of gauge theory:

Immediately after the success of Einstein on general relativity, there was great desire to unify all the known forces by using ideas similar to general relativity. At that time, the most important field is electricity and magnetism as is dictated by Maxwell equations.



Maxwell



There were two approaches: one is the gauge theory of Hermann Weyl and the other one is the Kaluza-Klein model of General relativity in five dimensions.

Both of these two developments lay the foundations of modern geometry and modern physics.

Kaluza

Klein

In fact, in the theory of electromagnetism, Riemann-Silberstein vector which combines electric and magnetic fields, is also attributed to Riemann. This is a complex vector of the form $F = E + \mathrm{c}\sqrt{-1}B$ where c is the speed of light.

This vector is an origin of what physicists later generally called 'dualities' in the framework of string theory and quantum field theory. For example, the interpretation of Geometric Langlands program in the work of Kapustin and Witten originates from a generalization of this 'electro-magnetic' duality to nonabelian contexts.

At the level of particles, this is the duality between electron and magnetic monopole, which gave birth to the Seiberg-Witten theory in physics.

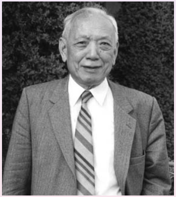
Hermann Weyl was the first one who introduced the concept of Gauge theory. (he was the one who coined this terminology.) While gravity can be considered as a gauge theory with gauge group given by the group of diffeomorphisms, Weyl succeeded to show that Maxwell equations is also a gauge theory with gauge group given by $U(1)$. The development went through a nontrivial process.

The group that Weyl proposed at the beginning was non compact and cannot preserve length. This was criticized by Einstein. But after a few years, Weyl learned from the works of London et al in quantum mechanics that the group should be $U(1)$. Once the group is chosen right, length is preserved under parallel transportation and Maxwell equation becomes a gauge theory.

While Weyl accomplished the remarkable interpretation of Maxwell equations in terms of gauge theory around 1928, the theory of connection was developed by several geometers. In 1917 Levi-Civita studied parallel transport of vectors in Riemannian geometry. In 1918 Weyl in his book "Raum, Zeit, Materie" introduced affine connections. Cartan in 1926 studied holonomy group for general connections.

Levi Civita and E. Cartan were interested in another approach to extend Einstein theory of general relativity by looking into connections with nontrivial torsion.(Einstein was using Levi-Civita connections which has no torsion.) The connection still preserves metric. This is in fact a form of gauge theory on the tangent bundle. But Weyl's point of view was different and did not restrict himself to tangent bundles.

In 1944, Chern studied Hermitian connections on complex bundles and, using the curvature of the Hermitian connections, introduced the Chern classes of the bundles. They give rise to the de Rham classes of the space which turns out to be integral classes.



Chern

de Rham

Upon seeing the definitions, Weil interpreted Chern's theory in terms of invariant theory. This is called the Chern-Weil theory. It is remarkable that Weil said that at that time Chern classes may be used to quantize physical theory.

The modern formulation of connections on arbitrary bundles was introduced by C. Ehresmann in 1950.

Weil

Ehresmann

Modern development of high energy physics and theory of condensed matter shows that the prediction of Weil is accurate. In fact, not only Chern classes play an important role in modern quantum field theory, the Chern-Simons invariant, which is derived from curvature representation of Chern classes, also play an important role in condensed matter physics and string theory, which in turn influence the study of knot theory in geometry, as was shown by Witten that it can be used to explain the Jones polynomial of the knots.

From this point of view, an important insight was gained to calculate the volume of a complete hyperbolic metric on the knot complement. This is called Volume conjecture due to Kashaev, Murakami and Murakami.

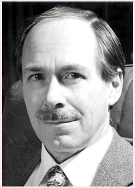
The volume conjecture states that in a certain limit when the number of colorings N approach infinity in the N-colored Jones polynomial for a knot, the value of the colored knot Jones polynomial evaluated at the N-th root of unity is the exponential of N-times the simplicial volume of the knot complement divided by 2π. The knot complement can be uniquely decomposed into hyperbolic pieces and Seifert fibered pieces. The simplicial volume is then the sum of the hyperbolic volumes of the hyperbolic pieces of the decomposition.

The volume conjecture dictates the convergence of the quantum Chern-Simons path integral with non-compact gauge group and has nontrivial connections with three dimensional quantum gravity. This connection was anticipated by Witten and later studied also by Gukov and Vafa.

In 1954, there were two independent developments related to gauge theory. One was the work of Yang and Mills who looked at an action on the space of connections on higher rank bundles over a manifold. The group of parallel transportation will preserve a higher dimensional Lie group whose dimension is, in general, greater than the circle group as was used by Weyl. This group is in general not commutative. Hence this generalization of gauge theory of Weyl is called nonabelian gauge theory. The action that Yang and Mills used was the L^2 norm of the curvature of the connection.

The theory of Yang-Mills was finally quantized by 't Hooft in early seventies, based on preliminary works of Faddeev-Popov. This was a difficult work as there is a problem of choice of gauge. The works of 't Hooft was continued by Veltman, Gross et al. It laid the foundation for the theory of standard model of modern particle physics.

Yang



't Hooft

The equation of motion defined by the Yang-Mills action gives rise to a nice elliptic system (in a suitable choice of gauge) which was not studied by geometers. At late sixties and early seventies, during the process of quantization of gauge theory, 't Hooft and Polyakov became interested in the concept of monopoles and instantons defined on four dimensional Euclidean space. They give important special solutions of Yang-Mills equation, by minimizing Yang-Mills energy in terms of topological data. Because of the last property, they played important roles in topological quantum field theory.

There were extensive efforts by physicists and mathematicians in the seventies to find these instantons. The equations of instantons were rewritten in 1977 by C.N. Yang in terms of Cauchy-Riemann equations. In fact, what Yang did was that the instanton bundle should be extended to be a holomorphic bundle over complex projective space and, after extension, the instanton connection satisfies certain equation on this holomorphic bundle which we later called Hermitian-Yang-Mills connection.

The concept of Hermitian-Yang-Mills connection can be generalized to higher dimensional Kähler manifolds. They are "super symmetric" and played important roles in the development of string theory and algebraic geometry.

The most important work in this direction was due to Donaldson for algebraic surfaces and due to Uhlenbeck-Yau for arbitrary Kähler manifolds.

In the meanwhile, Donaldson observed that the moduli space of instantons can be used to define topological invariants for four dimensional manifolds. Hence he made the first major achievement in the theory of topology of smooth four manifolds. While Donaldson invariants have been fundamental, it is not so easy to compute. Ten years later, Seiberg and Witten found a simpler invariant for four manifolds which enjoy similar properties as Donaldson invariants. Taubes made fundamental contributions to the subject of symplectic geometry by constructing pseudoholomorphic curves based on nonvanishing of Seiberg-Witten invariants. Many major results were solved by the work of Taubes.

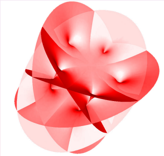
The other important development was due to E. Calabi. He was also interested in the Yang-Mills action on the space of metrics. Within the space of Kähler metrics with the same Kähler class, Calabi showed that the critical point of the Yang-Mills functional gives rise to the Kähler-Einstein metric if the Kähler class is proportional to the first Chern class.

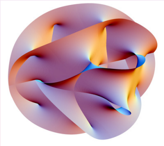
The existence of such critical points was not known at the time. It is called the Calabi conjecture when the first Chern class is zero.

The third major development, after the great discovery in 1915, was the work of Kaluza, followed by Oscar Klein. They proposed a remarkable approach to create the Maxwell equations from vacuum Einstein equations. They considered the vacuum Einstein equation on a four dimensional manifold product with a circle and demanded all the fields to be invariant under the rotation of the circle group.

In this way, they found a (Lorentzian) metric tensor, a vector field and a scalar on the four manifold. The vector field satisfies the Maxwell equations which couple with the metric tensor. This is a beautiful theory, except that the extra scalar field cannot be found in nature. Nonetheless , this is a beautiful theory and Einstein likes it.

This theory is the forerunner of the compactification theory in modern string theory. The circle is replaced by a six dimensional manifold satisfying certain constraints. Those constraints give rise to the Calabi-Yau manifolds which are Kähler manifolds with a non-vanishing holomorphic volume form.





The existence of a Kähler metric with zero Ricci curvature was proved by me in 1976. Its use in string theory was proposed by Candelas-Horowitz-Strominger-Witten in 1984. The proposal was that, with the right choice of such Calabi-Yau manifolds, we can calculate the basic physical quantities in nature, including number of generations of fermions, grand unification scale and Yukawa couplings. Many important algebraic and enumerative properties was proved for Calabi-Yau manifolds based on intuition coming from string theory.

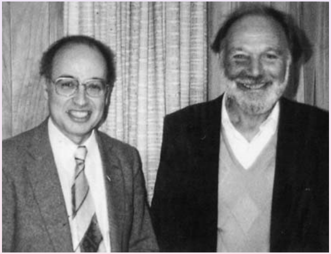
A very major property that arises in physics is duality between Calabi-Yau manifolds which gives a very effective tool to calculate interesting geometry objects that are of great interest to geometers. Different branches of mathematics were brought in to study such dualities.

It is rather exciting to watch these branches merge in a natural manner within string theory.

The basic tools to study above theories came from some classical theory in mathematics and in physics. Hilbert, in his systematical way to organize the theory of integral equations, introduced the concept of Hilbert space. The study of self-adjoint and non-self-adjoint operators on Hilbert spaces play fundamental roles in quantum mechanics.

It was a coincidence that abstractly defined spectrum of an operator coincide with the spectrum found in nature. Weyl also found the fundamental Weyl law for the asymptotic behavior of the spectrum of linear elliptic operators based on the question on black body radiation, which was a question raised by Lorentz.

Study of index of elliptic operators relating index of the operator to the topology of the manifold has made much contribution to modern geometry and particle physics. This gave rise to the famous Atiyah-Singer index formula.



Atiyah and Singer

One of many physics applications of the index formula is to the study of anomaly in Quantum Field Theory. Fujikawa among others derived the chiral anomaly using index theorem. Alvarez-Gaume and Witten applied similar methods to the study of anomaly in quantization of gravity. Green and Schwarz used it to derive the correct grand unification group for the Heterotic string, setting off the 1st string theory revolution.

The very important tool towards such development was pioneered by Hodge in 1941. This actually went back to the works of nineteenth century where periods of integral were studied extensively by Riemann, Abel, Lagrange, Jacobi and others.

Hodge

Abel

A significant part of the later interpretation of Riemann's work owes to Hermann Weyl. Weyl's book "Die Idee der Riemannschen Flache" formulated Riemann's results in modern terms concerning the existence of polarized Hodge structures.

Weyl's book was published in 1913 as the fifth volume in the series of Göttingen Lectures on mathematics, the previous four volumes were by Klein, Minkowski, Voigt, and Poincaré.

Weyl developed the theory later called Hodge theory in his book based on a motivation from fluid dynamics . His philosophy was influenced by the book of F. Klein, "On Riemann's Theory of Algebraic Functions and their Integrals".

Period is the integral of a closed form over a cycle. According to the theory of de Rham, this gives a pairing between topological cycles and space of closed forms modulo those which are exact. Hodge proposed forms that are closed and coclosed to be harmonic forms. He proved that periods can be realized by harmonic forms.

Historically, the question of period was studied for two dimensional surfaces at the beginning. Some part of this theory was developed by the theory of two dimensional fluid dynamics in nineteenth century. Klein wrote about it in his book. In the 1913 book on Riemann surface, Weyl established the theory of harmonic forms on Riemann surfaces based on the Dirichlet principle of Riemann.

In 1930s, Hodge generalized the theory to higher dimensional manifold based on the theory of parametrix of Hadamard. Hodge's work appeared in 1942.

Hodge stated that the main purpose of his book "The theory and applications of Harmonic integrals", published in 1941, was to prove, using differential forms and the then recent de Rham theorem, Lefschetz's results on the topology of algebraic varieties.

He derived several theorems of de Rham using his main existence theorem for harmonic integrals. This led to the Hodge decomposition theorem.

It is interesting to note that Weyl published a paper called "Method of Orthogonal Projection in Potential Theory", which he demonstrated how to apply the theory to study Hodge theory in Riemann surfaces. This method was later used extensively to study Hodge theory.

Right after Hodge published his work, a gap was found in the proof of existence of harmonic forms with prescribed period which was fixed by Weyl and Kodaria via different approaches.

Weyl's proof appeared in the paper "On Hodge's theory of Harmonic integrals" in 1943. Kodaira gave another proof independent of Weyl's for the prescribed period problem in 1942. Kodaira later generalized it to the existence of harmonic forms with prescribed singularities (and periods).

In early fifties, Milgram and Rosenblum introduced the heat equation method to give a proof of the Hodge theory. The method of heat flow has tremendous influence in later development in geometry.

In the above discussion on Weyl estimate on asymptotic behavior of eigenvalues, we should mention that estimate of eigenvalues of Laplacian went back to early works of applied mathematicians and physicists such as lord Kelvin, lord Rayleigh, Rellich, Hilbert in nineteenth century and also Polya, Szegö, Courant, Carleman in the early twentieth century.

Some approaches are based on studying the heat kernel and the Tauberian theorems (in the last fifty years, wave equation method was brought in by Hörmander, developed by Ivrii, Victor Guillemen, Sternberg, Duistermaat, Boutet de Monvel, Sjöstrand, Taylor, Zelditich, Melrose, Ralston and others), which was the one used by Weyl to deduce his asymptotic estimates. Another approach was based on variational principle, or min-max principle characterization of eigenvalues.

The later method brought in close relation of spectrum of Laplacian with geometry, especially the isoperimetric inequality of various kind. Polya and Szegö gave many important discussions on how those quantities give rise to effective estimates of eigenvalues of Laplacian. The idea appeared later in the paper of Cheeger where he discussed the estimate of first eigenvalue on a compact manifold. This is now called Cheeger inequality and is being studied extensively in the theory of graph.

While the idea of using wave kernel has been able to link eigenvalues of Laplacian with the length of closed geodesics of the manifold, the method of heat kernel gives rise to many delicate estimates in geometry, including the local index formula of Atiyah-Singer. This later work has played important roles in modern quantum field theory.

Several important events occurred in the nineteen seventies that changed geometers' view towards physics. Besides the important discussion of instanton solutions to Yang-Mills equations which led to the revolution of topology of four manifolds, we have the discussion of positive mass theorem in general relativity.

While this conjecture was settled by Schoen any myself in 1978, it has far reaching consequence in understanding geometry of manifolds with positive scalar curvature. The later work of Witten in proving the positive mass conjecture using Dirac spinors gives a different and powerful venue to understand classical general relativity.

In the past thirty years, both the methods of Schoen-Yau and Witten have developed to be important powerful tools in classical general relativity and the theory of manifolds with positive scalar curvature.

Perhaps one should mention that the method of harmonic spinor dated back to the important work of Lichnerowicz on his famous vanishing theorem, which, coupled with Atiyah-Singer index theorem, gave the first instinct of topological obstruction for metrics with positive scalar curvature.

A very major turning point for implication of ideas on geometry was the famous paper of Witten on analytic treatment of Morse theory appeared in 1984 in Journal of Differential Geometry. This paper has deep influence in later development of supersymmetric quantum field theory and differential geometry. Immediately afterwards, Floer extended the ideas to build the Floer theory in symplectic geometry where he was able to prove the Arnold conjecture in case the manifold has trivial higher homotopy groups.

The idea of Witten was motivated by quantum field theory and also became the foundation of later development of topological field theory.

The idea of keeping track of the change of a theory when some parameter is moving, is a very important one. In the case of the heat equation proof of the Hodge theory or the index formula, when the temperature is very high, we see the harmonic forms or solutions of some linear elliptic system. But when the temperature is low, the classical effect comes from the metric or from the coefficients of the operators. Hence if some object (such as the index of the elliptic operator) is invariant when the temperature varies, we can relate it on the one hand to quantum mechanical property and on the other hand to classical properties.

In Witten's interpretation of the Morse theory, space of harmonic forms is related to critical point of a function defined on the manifold. In 1994, Seiberg and Witten, based on similar philosophy, also connected the Donaldson invariants on four manifolds to some topological invariant which are easier to compute.

A very important contribution of the Seiberg-Witten type invariants is the fundamental result of Cliff Taubes who was able to make use of the non-vanishing of the (topological) Seiberg-Witten invariant to construct pseudo-holomorphic curves for four dimensional symplectic manifold with an almost complex structure.

Many important open problems in four dimensional symplectic geometry were solved by this theorem of Taubes. In particular, Taubes solved an old problem that the symplectic structure on the complex projective plane is unique.

The subject of symplectic geometry probably already started after Newtonian mechanics was invented. But the modern development mainly started from the work of Emmy Noether where she published the important foundational work in 1918. Many modern ideas such as moment map can be traced to her works.

The theory of geometric quantization and moment map played an important role in geometry and physics, influencing the theory of representation and differential geometry.

Atiyah and Bott initiated the idea of interpreting Yang-Mills action of bundles over Riemann surfaces. This point of view has deep influence on Donaldson and other mathematicians in Oxford, who started to look at similar situations on complex algebraic surfaces.

However, the theory becomes much more complicated as theory of nonlinear elliptic equation associated to the subject in this number of dimensions is not solid enough. In most cases, it was used as a motivation rather than a proof.

On the other hand, symplectic geometry has become much more developed in the last thirty years after it was realized that the theory of mirror symmetry, motivated by physical consideration, called for a symmetric treatment of symplectic geometry with complex geometry. Roughly speaking , the symplectic geometry of one Calabi-Yau manifold is supposed to be isomorphic to the complex geometry of the mirror Calabi-Yau manifold. The actual situation is much more complicated as we need to find the so called quantum corrections to the symplectic theory. But the quantum corrections contain many interesting geometric objects that we like to learn, for example pseudo-holomorphic curves.

Pseudo-holomorphic curves are also called worldsheet instantons by physicists. In fact, when Candelas et. al. computed the instanton correction for an important Calabi-Yau manifold, he found a closed formula for the counting of rational curves within the manifold, revealing its deep geometric properties.

It solved a long standing question in enumerative geometry. Nobody in algebraic geometry expected that it can be done in such an elegant way. Up to now , there is no other way to find the formula of Candelas et. al. based on algebraic geometry alone although the formula was proved rigorously to be true by Givental and Lian-Liu-Yau independently, a few years after Candelas and collaborators announced their result.

This new chapter of enumerative geometry is called the theory of Gromov-Witten invariants. Major contributions on the subject can be viewed as joint efforts of mathematicians and physicists.

In the past ten years, we have witnessed that topological quantum field theory is starting to play important roles in also condensed matter theory, for example, as seen in the works of Charles Kane, S. C. Zhang, on topological insulators and Kitaev and X. G. Wen on topological phases. Sophisticated Chern-Simons theory calculations and higher order tensor category theories were used in some of these studies, especially regarding questions on classification of topological orders.

Recently we proposed, in collaboration with J. Wang and X. G. Wen, the quantum statistics of anyon excitations in condensed matter systems satisfy consistency relations arising from surgery on three and four manifolds. This is deeply related to the work of Witten on TQFTs.

A very important goal in fundamental physics and geometry in the twenty first century is to build a solid foundation for a theory that is capable to incorporate quantum theory in the small scale of the spacetime. Insights from physics and geometry have to play a fundamental role.

String theory, theory of quantum entanglement and concept of noncommutative geometry were brought in. The understanding of quantum entanglement, may offer a deeper look into the nature of spacetime, and important geometric concepts related to gravitation such as quasi-local mass studied by me and collaborators.

There is no doubt that great activities in the interactions between geometry and physics will go on today and in the future. And both subjects will greatly benefit from it. We expect to see much further interactions between geometry and physics in the next fifty years.

Thank you!