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A note on the characteristic p nonabelian Hodge theory in the geometric case

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We provide a construction of associating a de Rham subbundle to a Higgs subbundle in characteristic p in the geometric case. As applications, we obtain a Higgs semistability result and a W_2-unliftable result.

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0. Introduction

This paper grows out of our efforts to understand the spectacular work of Ogus–Vologodsky [13] (see [14] for the logarithmic analogue) on the nonabelian Hodge theory in characteristic p. Let k be an algebraically closed field of positive characteristic p, and X a smooth variety over k which admits a $W_2(k)$-lifting. The authors loc. cit. establish a correspondence between a category of vector bundles with integrable connections and a category of Higgs bundles over X, the objects of which are subject to certain nilpotent conditions (see [13, Theorem 2.8]). The whole theory is analogous to the one over complex numbers (see [17]). Their construction relies either on the theory of Azumaya algebra or on a certain universal algebra $\mathcal{A}$ associated to a W_2-lifting of X on which both an integrable connection and a Higgs field act (see [13, Sec. 2]). The correspondence is generally complicated. However, there are two cases where the correspondence is known to be classical: the zero Higgs

field and the geometric case. In the former case this correspondence is reduced to a classical result of Cartier (see [13, Remark 2.2; 8, Theorem 5.1]), while in the latter case the Higgs bundle corresponding to the Gauß-Manin system of a geometric family is obtained by taking gradings of the de Rham bundle with respect to either the Hodge filtration or the conjugate filtration by the Katz's p-curvature formula (see [13, Remark 3.19; 9, Theorem 3.2]). Unlike the zero Higgs field case, an explicit construction of the converse direction in the geometric case is still unknown. In the complex case this amounts to solving the Hermitian–Yang–Mills equation (see [16] for the compact case), which is transcendental in nature.

The main finding of this paper is that one can profit from using the relative Frobenius of a geometric family, which behaves like the Hodge metric of the associated variation of Hodge structures over $\mathbb{C}$ in a certain sense. Indeed, we show that one can use them to construct a de Rham subbundle from a Higgs subbundle in the geometric case. Throughout this note p is an odd prime and n is an integer which is greater than or equal to $p-2$. Our main results read as follows.

Theorem 0.1. *Let X be a smooth scheme over $W = W(k)$ and $f : Y \to X$ a proper smooth morphism as given in Example* 1.1. *Let (H, ∇) (respectively, (E, θ)) be the associated de Rham bundle (respectively, Higgs bundle) of degree n to f. Then to any Higgs subbundle $(G, \theta) \subset (E, \theta)_0$ in characteristic p one can associate naturally a de Rham subbundle $(H_{(G,\theta)}, \nabla)$ of $(H, \nabla)_0$. For a subsystem of Hodge bundles in $(E, \theta)_0$ the Cartier–Katz descent of the associated de Rham subbundle $(H_{(G,\theta)}, \nabla)$ to (G, θ) is (G, θ) itself.*

In the above theorem, H is the nth hypercohomology of the relative de Rham complex associated to f, equipped with the Gauß-Manin connection ∇ and the Hodge filtration $\mathrm{Fil}^{\cdot}$; The associated Higgs bundle (E, θ) to a de Rham bundle is obtained by taking grading of (H, ∇) with respect to $\mathrm{Fil}^{\cdot}$, which is, more precisely,

$$E = \bigoplus_i E^{i,n-i}, \quad \theta = \bigoplus_i \theta^{i,n-i},$$

with

$$E^{i,n-i} = \frac{\mathrm{Fil}^i H}{\mathrm{Fil}^{i+1} H}, \quad \theta^{i,n-i} = \bar{\nabla} : E^{i,n-i} \to E^{i-1,n-i+1} \otimes \Omega_{X/W}.$$

Our construction is independent of that of Ogus–Vologodsky. It is interesting to compare it with the inverse Cartier transform of Ogus–Vologodsky loc. cit. in the situation of the above theorem. The construction works as well when the base scheme is equipped with a certain logarithmic structure or defined over $W_{n+1} = W_{n+1}(k)$. For a Higgs subsheaf of $(E, \theta)_0$, by which we mean a θ-stable coherent subsheaf in E_0, the above construction yields a de Rham subsheaf of $(H, \nabla)_0$. In the most general form, the construction works for a Higgs subsheaf of a Higgs bundle coming from the modulo p reduction of $\mathrm{Gr}_{\mathrm{Fil}}(H, \nabla)$, where $(H, \mathrm{Fil}^{\cdot}, \nabla, \Phi)$ is an object of the Faltings category $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ (see Sec. 1). We obtain two applications of the construction.

Proposition 0.2. *Assume that X is projective over W_{n+1}.*

(1) *For a subsystem of Hodge bundles or a Higgs subbundle with zero Higgs field $(G,\theta) \subset (E,\theta)_0$, one has the slope inequality $\mu(G) \leq 0$, where $\mu(G)$ is the μ-slope of G with respect to the restriction of an ample divisor of X to X_0.*
(2) *The following statements are true*:
   (i) *Let $g_0 : C_0 \to X_0$ be a morphism of a smooth projective curve C_0 to X_0 over k which is liftable to a morphism $g : C \to X$ over W_{n+1}. Then for any Higgs subbundle $(G,\theta) \subset (E,\theta)_0$, one has $\deg(g_0^* G) \leq 0$.*
   (ii) *Let C be a smooth projective curve in X over W_{n+1}. Then the Higgs bundle $(E,\theta)_0$ is Higgs semistable with respect to the μ_{C_0}-slope.*

Let F be a real quadratic field such that p is inert in F. Let $m \geq 3$ be an integer coprime to p. Let M be the smooth scheme over W which represents the fine moduli functor which associates a W-algebra R a principally polarized abelian surface over R with a real multiplication $\mathcal{O}_F$ and a full symplectic level m-structure (see e.g. [6]). Let $S_h \subset M_0$ be the Hasse locus which is known to be a $\mathbb{P}^1$-configuration in characteristic p (see [1, Theorem 5.1]).

Proposition 0.3. *Let D be an irreducible component in S_h. Let $C = \sum C_i$ (may be empty) be a simple normal crossing divisor in M_0 such that $D + C$ is again of simple normal crossing. If the intersection number $D \cdot C$ is less than or equal to $p-1$, then the curve $D + C$ is W_2-unliftable inside the ambient space M_1.*

As a convention, we denote the reduction modulo $p^{i+1}, i \geq 0$ of an object by attaching the subscript i. However for a connection or a Higgs field this rule will not be strictly followed for simplicity of notation.

1. The Category $\mathcal{MF}^{\nabla}_{[0,n]}(X)$

In his study of p-adic comparison over a geometric base, Faltings has introduced the category $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ in various settings. Its objects are the strong divisible filtered Frobenius crystals over X, which could be considered as the p-adic analogue of a variation of $\mathbb{Z}$-Hodge structures over a complex algebraic manifold. One shall be also aware of the fact that Ogus has developed systematically the category of $F-T$-crystals in the book [12], which is closely related to the category $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ (see particularly [12, Sec. 5.3]).

1.1. *Smooth case*

Let X be a smooth W-scheme. A small affine subset U of X is an open affine subscheme $U \subset X$ over W which is étale over $\mathbb{A}^d_W$.[a] As X is smooth over W there

[a]A small affine open subset in the sense of Faltings in the p-adic comparison in the p-adic Hodge theory is required to be étale over $\mathbb{G}^d_m$. Since nowhere in this paper the p-adic comparison is used, it suffices to take the above notion for a small affine subset.

exists an open covering consisting of small affine subsets of X. Let $U \subset X$ be a small affine subset. For it one could choose a Frobenius lifting $F_{\hat{U}}$ on $\hat{U}$, the p-adic completion of U. An object in $\mathcal{MF}^{\nabla}_{[0,n]}(\hat{U})$ (see [2, Chap. II], [4, Sec. 3]) is a quadruple $(H, \mathrm{Fil}^{\cdot}, \nabla, \Phi_{F_{\hat{U}}})$, where:

(i) $(H, \mathrm{Fil}^{\cdot})$ is a filtered free $\mathcal{O}_{\hat{U}}$-module with a basis e_i of $\mathrm{Fil}^i, 0 \leq i \leq n$.
(ii) ∇ is an integrable connection on H satisfying the Griffiths transversality:

$$\nabla(\mathrm{Fil}^i) \subset \mathrm{Fil}^{i-1} \otimes \Omega^1_{\hat{U}}.$$

(iii) The relative Frobenius is an $\mathcal{O}_{\hat{U}}$-linear morphism $\Phi_{F_{\hat{U}}} : F^*_{\hat{U}} H \to H$ with the strong p-divisible property: $\Phi_{F_{\hat{U}}}(F^*_{\hat{U}} \mathrm{Fil}^i) \subset p^i H$ and

$$\sum_{i=0}^{n} \frac{\Phi_{F_{\hat{U}}}(F^*_{\hat{U}} \mathrm{Fil}^i)}{p^i} = H.$$

(iv) The relative Frobenius $\Phi_{F_{\hat{U}}}$ is horizontal with respect to the connection $F^*_{\hat{U}}\nabla$ on $F^*_{\hat{U}} H$ and ∇ on H.

The filtered-freeness in (i) means that the filtration $\mathrm{Fil}^{\cdot}$ on H has a splitting such that each Fil^i is a direct sum of several copies of $\mathcal{O}_{\hat{U}}$. The pull-back connection $F^*_{\hat{U}}\nabla$ on $F^*_{\hat{U}} H$ is the composite

$$\begin{aligned} F^*_{\hat{U}} H &= F^{-1}_{\hat{U}} H \otimes_{F^{-1}_{\hat{U}} \mathcal{O}_{\hat{U}}} \mathcal{O}_{\hat{U}} \xrightarrow{F^{-1}_{\hat{U}} \nabla \otimes \mathrm{id}} (F^{-1}_{\hat{U}} H \otimes F^{-1}_{\hat{U}} \Omega^1_{\hat{U}}) \otimes_{F^{-1}_{\hat{U}} \mathcal{O}_{\hat{U}}} \mathcal{O}_{\hat{U}} \\ &= F^*_{\hat{U}} H \otimes F^*_{\hat{U}} \Omega^1_{\hat{U}} \xrightarrow{\mathrm{id} \otimes dF_{\hat{U}}} F^*_{\hat{U}} H \otimes \Omega^1_{\hat{U}}. \end{aligned}$$

The horizontal condition (iv) is expressed by the commutativity of the diagram

$$\begin{array}{ccc} F^*_{\hat{U}} H & \xrightarrow{\Phi_{F_{\hat{U}}}} & H \\ {\scriptstyle F^*_{\hat{U}}\nabla}\downarrow & & \downarrow{\scriptstyle \nabla} \\ F^*_{\hat{U}} H \otimes \Omega^1_{\hat{U}} & \xrightarrow{\Phi_{F_{\hat{U}}} \otimes \mathrm{id}} & H \otimes \Omega^1_{\hat{U}}. \end{array}$$

As there is no canonical Frobenius liftings on $\hat{U}$, one must know how the relative Frobenius changes under another Frobenius lifting. This is expressed by a Taylor formula. Let $\hat{U} = \mathrm{Spf}\, R$ and $F : R \to R$ a Frobenius lifting. Choose a system of étale local coordinates $\{t_1, \ldots, t_d\}$ of U (namely fix an étale map $U \to \mathrm{Spec}(W[t_1, \ldots, t_d])$). Let R' be any p-adically complete, p-torsion free W-algebra, equipped with a Frobenius lifting $F' : R' \to R'$ and a morphism of W-algebras $\iota : R \to R'$. Then the relative Frobenius $\Phi_{F'} : F'^*(\iota^* H) \to \iota^* H$ is the composite

$$F'^* \iota^* H \overset{\alpha}{\cong} \iota^* F^* H \xrightarrow{\iota^* \Phi_F} \iota^* H,$$

where the isomorphism α is given by the formula:

$$\alpha(e \otimes 1) = \sum_{\underline{i}} \nabla^{\underline{i}}_{\underline{\partial}}(e) \otimes \frac{z^{\underline{i}}}{\underline{i}!}.$$

Here $\underline{i} = (i_1, \ldots, i_d)$ is a multi-index, and $z^{\underline{i}} = z_1^{i_1} \cdots z_d^{i_d}$ with $z_i = F' \circ \iota(t_i) - \iota \circ F(t_i), 1 \leq i \leq d$, and $\nabla^{\underline{j}}_{\underline{\partial}} = \nabla^{i_1}_{\partial_{t_1}} \cdots \nabla^{i_d}_{\partial_{t_d}}$.

One defines then the category $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ (see [2, Theorem 2.3]). Its object will be denoted again by a quadruple $(H, \mathrm{Fil}^{\cdot}, \nabla, \Phi)$. Here $(H, \mathrm{Fil}^{\cdot}, \nabla)$ is a locally filtered free $\mathcal{O}_X$-module with an integrable connection satisfying the Griffiths transversality. For each small affine $U \subset X$ and each choice $F_{\hat{U}}$ of Frobenius liftings on $\hat{U}$, Φ defines $\Phi_{F_{\hat{U}}} : F^*_{\hat{U}} H|_{\hat{U}} \to H|_{\hat{U}}$ such that it, together with the restriction of $(H, \mathrm{Fil}^{\cdot}, \nabla)$ to $\hat{U}$, defines an object in $\mathcal{MF}^{\nabla}_{[0,n]}(\hat{U})$.

Example 1.1. Let $f : Y \to X$ be a proper smooth morphism of relative dimension $n \leq p-2$ between smooth W-schemes. Assume that the relative Hodge cohomologies $R^i f_* \Omega^j_Y, i+j = n$ has no torsion. By [2, Theorem 6.2],[b] the crystalline direct image $R^n f_*(\mathcal{O}_Y, d)$ is an object in $\mathcal{MF}^{\nabla}_{[0,n]}(X)$.

1.2. *Logarithmic case*

The category $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ has a logarithmic variant (see [2, IV(c); 3, Sec. 4(f); 4, Sec. 3]). A generalization of the Faltings category to a syntomic fine logarithmic scheme over W can be found in [18, Sec. 2]. We shall focus only on two special cases: the case of "having a divisor at infinity" and the semistable case. In the first case, X is assumed to be a smooth scheme over W, and $D \subset X$ a divisor with simple normal crossings relative to W, i.e. $D = \bigcup_i D_i$ is the union of smooth W-schemes D_i meeting transversally. In the second case, X is assumed to be a regular scheme over W such that *Zariski* locally there is an étale morphism to the affine space $\mathbb{A}^d_W$ or $\mathrm{Spec}(W[t_1, \ldots, t_{d+1}]/(t_1 \cdots t_{d+1} - p))$ over W. We call an open affine subset $U \subset X$ small if there is an étale morphism $U \to \mathbb{A}^d_W$ mapping $U \cap D$ to a union of coordinate hyperplanes (may be empty) in the first case, and if U satisfies one of the two conditions in the second case. In each case one associates a natural fine logarithmic structure to the scheme X such that the structural morphism $X \to \mathrm{Spec}\, W$ is log smooth (in the former case one equips $\mathrm{Spec}\, W$ with the trivial log structure and in the latter case with the log structure determined by the closed point of $\mathrm{Spec}\, W$). See [7, (1.5)(1) and Examples (3.7)(2)]. Note also that the assumptions made above use the Zariski topology on X instead of the étale topology as in [7]. The logarithmic crystalline site is then defined for $X \to \mathrm{Spec}\, W$ with the above logarithmic structure (see [7, Sec. 5]).

Let $X \to \mathrm{Spec}\, W$ be as above. Compared with the definition of $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ for the smooth case, we shall take the following modifications for the logarithmic

[b]Faltings loc. cit considered only the p-torsion objects. One obtains this by passing to the p-adic limit or applying another result of Faltings (see [4, Remark, p. 124]).

analogue. In the first case, for a small affine open subset $U \subset X$ a Frobenius lifting on $\hat{U}$ shall respect the divisor $\widehat{U \cap D} \subset \hat{U}$ (called a logarithmic Frobenius lifting by Faltings), and ∇ is an logarithmic integrable connection

$$\nabla(\mathrm{Fil}^i) \subset \mathrm{Fil}^{i-1} \otimes \Omega^1_{\hat{U}}(\log \widehat{U \cap D}).$$

In the second case, for an affine open subset $U \subset X$ which meets the singularities of X_0 it is *necessary* to consider a closed W-embedding $i : U \hookrightarrow Z$ in the category of logarithmic schemes together with a logarithmic Frobenius lifting on Z, by which we mean a Frobenius lifting respecting the logarithmic structure. In the current special case Z can be chosen to be smooth over W. Write J for the PD-ideal of i and $D^{\log}_U(Z)$ the logarithmic PD-envelope of U in Z (see [7, Proposition 5.3]). Denote by $\widehat{D^{\log}_U(Z)}$ the p-adic completion of $D^{\log}_U(Z)$. Then H is a free $\mathcal{O}_{\widehat{D^{\log}_U(Z)}}$-module and the decreasing filtration $\mathrm{Fil}^{\cdot}$ on H is compatible with the PD-filtration $J^{[\cdot]}$ on $\mathcal{O}_{\widehat{D^{\log}_U(Z)}}$ and is filtered free (see [4, P. 119]). For the formal logarithmic scheme $\widehat{D^{\log}_U(Z)}$ let $\Omega^1_{\widehat{D^{\log}_U(Z)}}$ be the sheaf of the formal relative logarithmic differentials on $\widehat{D^{\log}_U(Z)}$ (see [7, (1.7)]). For a choice of a logarithmic Frobenius lifting F_Z on Z let $F_{\widehat{D^{\log}_U(Z)}}$ be the induced morphism on $\widehat{D^{\log}_U(Z)}$. Then by replacing $\hat{U}$ in the definition of $\mathcal{MF}^{\nabla}_{[0,n]}(\hat{U})$ in Sec. 1 with $\widehat{D^{\log}_U(Z)}$ we get the description of the local category $\mathcal{MF}^{\nabla}_{[0,n]}(\widehat{D^{\log}_U(Z)})$. Taking a small affine covering $\mathcal{U} = \{U\}$ of X and a family of closed embeddings $i : U \to Z$ in the second case, one defines the global category $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ (see [3, Sec. 4(f); 18, Sec. 2]).

One basic example of objects in the category $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ is provided by the result of Faltings ([2, Theorem 6.2; 4, Remark, Sec. 3, p. 124]): For a W-morphism $f : Y \to X$ which is proper, log-smooth and generically smooth at infinity, if the relative Hodge cohomology $R^i f_* \Omega^j_{Y,\log}, i + j = n$ has no torsion, then the direct image $R^n f_*(\mathcal{O}_Y, d)$ of the constant filtered Frobenius logarithmic crystal of Y is an object in the category $\mathcal{MF}^{\nabla}_{[0,n]}(X)$.

One needs also a logarithmic version of the Taylor formula for the same purpose as in the smooth case. For that we refer the reader to [7, Formula (6.7.1)]. In the semistable case we make it more explicitly as follows. For $U = \mathrm{Spec}\, R$ étale over $\mathrm{Spec}\, W[t_1, \ldots, t_{d+1}]/(\prod_{1 \le i \le d+1} t_i - p)$, choose a surjection $R' \twoheadrightarrow R$ of W-algebras with R' log smooth over W and a logarithmic Frobenius lifting F' on the p-adic completion $\hat{R}'$. Assume $\{d \log x_1, \ldots, d \log x_r\}$ forms a basis for $\Omega^1_{\log}(R')$. For another choice $R'' \twoheadrightarrow R$ with the following commutative diagram

$$\begin{array}{ccc} R' & \longrightarrow & R\,, \\ {\scriptstyle \iota}\downarrow & \nearrow & \\ R'' & & \end{array}$$

and $F'' : \hat{R}'' \to \hat{R}''$ a logarithmic Frobenius lifting, we let

$$u_i = F'' \circ \iota(x_i)/\iota \circ F'(x_i), \quad 1 \leq i \leq r$$

and $\nabla^{\log}_{\partial_{x_i}}$ be the differential operator defined in [7, Theorem 6.2(iii)]. Then $\alpha : F''^*(\iota^* H) \to \iota^* F'^* H$ given by the Taylor formula

$$\alpha(e \otimes 1) = \sum_{\underline{i}=(i_1,\ldots,i_l,\ldots,i_r)\in\mathbb{N}^r} \left(\prod_{1\leq i\leq r, 0\leq j\leq i_l} (\nabla^{\log}_{\partial_{x_i}} - j)(e) \right) \otimes \left(\prod_{1\leq i\leq r} \frac{(u_i - 1)^{i_l}}{i_l!} \right)$$

is an isomorphism.

Remark 1.2. The analogue of the category $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ exists when X is smooth over a truncated Witt ring. In this case H is of p-torsion. So the formulation of strong divisibility as stated in (iii) has to be modified. See [2, Sec. 2(c)(d)]. In the logarithmic case one finds in [18, Sec. 2.3] the corresponding modification. Other conditions of the category can be obtained by taking reduction directly. In the following we shall abuse the notions of $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ in the case of (log) smooth X over W for the corresponding category in the case of (log) smooth over a truncated Witt ring.

2. The Construction

Let X be a smooth scheme over W and $(H, \mathrm{Fil}^{\cdot}, \nabla, \Phi)$ an object in $\mathcal{MF}^{\nabla}_{[0,n]}(X)$. Let $(E, \theta) = \mathrm{Gr}_{\mathrm{Fil}}(H, \nabla)$ be the associated Higgs bundle and $(G, \theta) \subset (E, \theta)_0$ a Higgs subbundle. We start with a description of a construction of the de Rham subbundle $(H_{(G,\theta)}, \nabla) \subset (H, \nabla)_0$. We first notice that there is a natural isomorphism of $\mathcal{O}_{X_0}$-modules:

$$\frac{1}{[p^i]} : p^i H/p^{i+1}H \to H_0.$$

This follows from the snake lemma applied to the commutative diagram of $\mathcal{O}_{X_i}$-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & pH/p^{i+1}H & \longrightarrow & H/p^{i+1}H & \xrightarrow{p^i} & p^iH/p^{i+1}H & \longrightarrow & 0 \\
 & & \| & & \| & & \uparrow{\scriptstyle p^i} & & \\
0 & \longrightarrow & pH/p^{i+1}H & \longrightarrow & H/p^{i+1}H & \longrightarrow & H/pH & \longrightarrow & 0
\end{array}$$

2.1. *Local construction*

Take $U \in \mathcal{U}$ and a Frobenius lifting $F_{\hat{U}} : \hat{U} \to \hat{U}$. So we get an object $(H_U, \mathrm{Fil}^{\cdot}_U, \nabla_U, \Phi_{F_{\hat{U}}}) \in \mathcal{MF}^{\nabla}_{[0,n]}(\hat{U})$ by the restriction. For simplicity of notation, we omit the appearance of U in this paragraph. Consider the composite

$$\frac{\Phi_{F_{\hat{U}}}}{[p^i]} : F^*_{\hat{U}} \mathrm{Fil}^i H \xrightarrow{\Phi_{F_{\hat{U}}}} p^i H \twoheadrightarrow p^i H/p^{i+1}H \xrightarrow{\frac{1}{[p^i]}} H_0.$$

By the property that $\Phi_{F_{\hat{U}}}(F_{\hat{U}}^*\mathrm{Fil}^{i+1}) \subset p^{i+1}H$ the above map factors through the quotient

$$F_{\hat{U}}^*\mathrm{Fil}^i H \twoheadrightarrow F_{\hat{U}}^*\mathrm{Fil}^i H/F_{\hat{U}}^*(\mathrm{Fil}^{i+1}H + p\mathrm{Fil}^i H).$$

By the filtered-freeness in (i) one has $\mathrm{Fil}^{i+1}H \cap p\mathrm{Fil}^i H = p\mathrm{Fil}^{i+1}H$. So one obtains an isomorphism

$$\mathrm{Fil}^i H/\mathrm{Fil}^{i+1}H + p\mathrm{Fil}^i H \cong E^{i,n-i}/pE^{i,n-i},$$

hence an $\mathcal{O}_{U_0}$-morphism

$$\frac{\Phi_{F_{\hat{U}}}}{[p^i]} : F_{U_0}^*(E^{i,n-i})_0 \to H_0.$$

It follows from the strong p-divisibility (see Sec. 1(iii)) that the map

$$\tilde{\Phi}_{F_{\hat{U}}} := \sum_{i=0}^{n} \frac{\Phi_{F_{\hat{U}}}}{[p^i]} : F_{U_0}^* E_0 \to H_0$$

is an isomorphism. For another choice of Frobenius lifting $F'_{\hat{U}}$ over $\hat{U}$, write $z_i := F_{\hat{U}}(t_i) - F'_{\hat{U}}(t_i)$. We have the following.

Lemma 2.1. *For a multi-index $\underline{j} = (j_1, \ldots, j_d)$, write $|\underline{j}| = \sum_{l=1}^d j_l$ and $\theta_{\partial}^{\underline{j}} = \theta_{\partial_{t_1}}^{j_1} \cdots \theta_{\partial_{t_d}}^{j_d}$. Then for a local section $e \in (E^{i,n-i})_0(U_0)$, one has the formula*

$$\frac{\Phi_{F_{\hat{U}}}}{[p^i]}(e \otimes 1) - \frac{\Phi_{F'_{\hat{U}}}}{[p^i]}(e \otimes 1) = \sum_{|\underline{j}|=1}^{i} \frac{\Phi_{F'_{\hat{U}}}}{[p^{i-|\underline{j}|}]}(\theta_{\partial}^{\underline{j}}(e) \otimes 1) \otimes \frac{z^{\underline{j}}}{p^{|\underline{j}|}\underline{j}!}.$$

Proof. First of all, as each $z_j, 1 \le j \le d$ is divisible by p and $i \le n \le p-2$, $\frac{z^{\underline{j}}}{p^{|\underline{j}|}\underline{j}!}$ in the above formula is a well-defined element in $\mathcal{O}_{U_0}$. Let $\tilde{e} \in \mathrm{Fil}^i H$ be a lifting of e. Applying the Taylor formula over $\mathcal{O}_{\hat{U}}$ in the situation that $R' = R$ and $\iota = id$, we get

$$\Phi_{F_{\hat{U}}}(\tilde{e} \otimes 1) = \sum_{|\underline{j}|=0}^{\infty} \Phi_{F'_{\hat{U}}}(\nabla_{\partial}^{\underline{j}}(\tilde{e}) \otimes 1) \otimes \frac{z^{\underline{j}}}{\underline{j}!}.$$

We observe $\mathrm{ord}_p(\frac{p^{|\underline{j}|}}{\underline{j}!}) \ge p-1$ for $|\underline{j}| \ge p$ and $\mathrm{ord}_p(\frac{p^{|\underline{j}|}}{\underline{j}!}) = |\underline{j}|$ for $|\underline{j}| \le p-1$. The above formula can written as

$$\begin{aligned}\Phi_{F_{\hat{U}}}(\tilde{e} \otimes 1) - \Phi_{F'_{\hat{U}}}(\tilde{e} \otimes 1) = &\sum_{|\underline{j}|=1}^{i} \Phi_{F'_{\hat{U}}}(\nabla_{\partial}^{\underline{j}}(\tilde{e}) \otimes 1) \otimes \frac{z^{\underline{j}}}{\underline{j}!} \\ &+ \sum_{|\underline{j}| \ge i+1} \Phi_{F'_{\hat{U}}}(\nabla_{\partial}^{\underline{j}}(\tilde{e}) \otimes 1) \otimes \frac{z^{\underline{j}}}{\underline{j}!}.\end{aligned}$$

As $i+1 \le p-1$, the above estimation on the p-adic valuation implies that the second term in the right-hand side is an element in $p^{i+1}H$. By the Griffiths transversality,

$\nabla^{\underline{j}}_{\partial}(\tilde{e}) \in \mathrm{Fil}^{i-|\underline{j}|}H$. Write $\tilde{e}_0 = \tilde{e} \mod p$. Thus we have the following formula which takes value in $H_0(U_0)$:

$$\frac{\Phi_{F_{\hat{U}}}}{[p^i]}(e\otimes 1) - \frac{\Phi_{F'_{\hat{U}}}}{[p^i]}(e\otimes 1) = \sum_{|\underline{j}|=1}^{i} \frac{\Phi_{F'_{\hat{U}}}}{[p^{i-|\underline{j}|}]}(\nabla^{\underline{j}}_{\partial}(\tilde{e}_0)\otimes 1)\otimes \frac{z^{\underline{j}}}{p^{|\underline{j}|}\underline{j}!}.$$

Regarding $\frac{\Phi_{F'_{\hat{U}}}}{[p^{i-|\underline{j}|}]}$ as a morphism between sheaves of abelian groups

$$F_{U_0}^{-1}(\mathrm{Fil}^{i-|\underline{j}|}H)_0 \to H_0,$$

one has the following commutative diagram:

$$\begin{array}{ccccc}
F_{U_0}^{-1}(\mathrm{Fil}^{i}H)_0 & \xrightarrow{F_{U_0}^{-1}\nabla^{\underline{j}}_{\partial}} & F_{U_0}^{-1}(\mathrm{Fil}^{i-|\underline{j}|}H)_0 & \xrightarrow{\frac{\Phi_{F'_{\hat{U}}}}{[p^{i-|\underline{j}|}]}} & H_0 \\
\downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} & \nearrow{\scriptstyle \frac{\Phi_{F'_{\hat{U}}}}{[p^{i-|\underline{j}|}]}} & \\
F_{U_0}^{-1}(E^{i,n-i})_0 & \xrightarrow[F_{U_0}^{-1}\theta^{\underline{j}}_{\partial}]{} & F_{U_0}^{-1}(E^{i-|\underline{j}|,n-i+|\underline{j}|})_0 & &
\end{array}$$

It implies that in the previous formula the connection can be replaced by the Higgs field. Hence the lemma follows. □

Proposition 2.2. *Notation as above. For a local section e of $E_0(U_0)$, one has the following formula*:

$$\tilde{\Phi}_{F_{\hat{U}}}(e\otimes 1) - \tilde{\Phi}_{F'_{\hat{U}}}(e\otimes 1) = \sum_{|\underline{j}|=1}^{n} \tilde{\Phi}_{F'_{\hat{U}}}(\theta^{\underline{j}}_{\partial}(e)\otimes 1)\otimes \frac{z^{\underline{j}}}{p^{|\underline{j}|}\underline{j}!}.$$

Proof. Write $e = \sum_{i=0}^{n} e_i$ with $e_i \in (E^{i,n-i})_0$. Lemma 2.1 implies

$$\tilde{\Phi}_{F_{\hat{U}}}(e\otimes 1) - \tilde{\Phi}_{F'_{\hat{U}}}(e\otimes 1) = \sum_{i=0}^{n}\sum_{|\underline{j}|=1}^{i} \frac{\Phi_{F'_{\hat{U}}}}{[p^{i-|\underline{j}|}]}(\theta^{\underline{j}}_{\partial}(e_i)\otimes 1)\otimes \frac{z^{\underline{j}}}{p^{|\underline{j}|}\underline{j}!}.$$

As $\theta^{\underline{j}}_{\partial}(e) = \sum_{i=0}^{n} \theta^{\underline{j}}_{\partial}(e_i)$ and $\theta^{\underline{j}}_{\partial}(e_i) = 0$ for $|\underline{j}| \geq i+1$, the above summation is equal to

$$\sum_{|\underline{j}|=1}^{n}\left[\sum_{i=|\underline{j}|}^{n} \frac{\Phi_{F'_{\hat{U}}}}{[p^{i-|\underline{j}|}]}(\theta^{\underline{j}}_{\partial}(e_i)\otimes 1)\right]\otimes \frac{z^{\underline{j}}}{p^{|\underline{j}|}\underline{j}!} = \sum_{|\underline{j}|=1}^{n} \tilde{\Phi}_{F'_{\hat{U}}}(\theta^{\underline{j}}_{\partial}(e)\otimes 1)\otimes \frac{z^{\underline{j}}}{p^{|\underline{j}|}\underline{j}!}.$$ □

The above proposition justifies the following

Definition 2.3. For the Higgs subbundle $(G,\theta) \subset (E,\theta)_0$, the locally associated subbundle $S_{U_0}(G) \subset H_0$ over $U_0 \subset X_0$ is defined to be $\tilde{\Phi}_{F_{\hat{U}}}(F_{U_0}^{*}G_{U_0})$, where U is a small affine subset of X with the closed fiber U_0 and $F_{\hat{U}}$ is a Frobenius lifting over $\hat{U}$.

2.2. *Gluing*

Take $U, V \in \mathcal{U}$, and Frobenius liftings $F_{\hat{U}}, F_V, F_{\widehat{U\cap V}}$ on $\hat{U}, \hat{V}, \widehat{U\cap U}$ respectively. We are going to show the following equality of subbundles in $H_0|_{U_0\cap V_0}$:

$$S_{U_0}(G)|_{U_0\cap V_0} = S_{U_0\cap V_0}(G) = S_{V_0}(G)|_{U_0\cap V_0}.$$

The following lemma is a variant of Lemma 2.1.

Lemma 2.4. *Write $z_i = F_{\hat{U}} \circ \iota(t_i) - \iota \circ F_{\widehat{U\cap V}}(t_i)$, where $\iota : \widehat{U\cap V} \hookrightarrow \hat{U}$ is the natural inclusion. Then for a local section $e \in (E^{i,n-i})_0(U_0)$, one has the formula*

$$\iota_0^*\left[\frac{\Phi_{F_{\hat{U}}}}{[p^i]}(e\otimes 1)\right] - \frac{\Phi_{F_{\widehat{U\cap V}}}}{[p^i]}[\iota_0^*(e)\otimes 1] = \sum_{|\underline{j}|=1}^{i} \frac{\Phi_{F_{\widehat{U\cap V}}}}{[p^{i-|\underline{j}|}]}(\iota_0^*[\theta_{\partial}^{\underline{j}}(e)]\otimes 1)\otimes \frac{z^{\underline{j}}}{p^{|\underline{j}|}\underline{j}!}.$$

Proof. The proof is the same as in Lemma 2.1 except that we shall apply the Taylor formula in the situation that R' is the one with $\mathrm{Spf}(R') = \widehat{U\cap V}$, $F' = F_{\widehat{U\cap V}}$ and $\iota : R \to R'$ is the one induced by the natural inclusion. □

A formula similar to that of Proposition 2.2 shows that $S_{U_0}(G)|_{U_0\cap V_0} = S_{U_0\cap V_0}(G)$. By symmetry we have also the second half equality. The open covering $\mathcal{U}$ of X gives rise to an open covering $\mathcal{U}_0$ of X_0 by reduction modulo p. Thus we glue the locally associated bundles $\{S_{U_0}(G)\}_{U_0\in\mathcal{U}_0}$ into a subbundle $H_{(G,\theta)} \subset H_0$, which we call the *associated subbundle to* (G,θ). We remark that the construction is independent of the choice of a small affine open covering $\mathcal{U}$ of X as we can always refine such a covering and Lemma 2.4 shows the invariance of the construction under a refinement.

2.3. *Horizontal property*

We ought to show the associated subbundle $H_{(G,\theta)} \subset H$ is actually ∇-invariant. Let $F_{\hat{U}} : \hat{U} \to \hat{U}$ be a Frobenius lifting over $\hat{U}$. Then one can write $\frac{\partial F_{\hat{U}}}{\partial t_j} = pf_j$ for $f_j \in \mathcal{O}_{\hat{U}}$.

Lemma 2.5. *For a local section $e \in (E^{i,n-i})_0(U_0)$, one has the formula*

$$\nabla_{\partial_{t_j}}\left[\frac{\Phi_{F_{\hat{U}}}}{[p^i]}(e\otimes 1)\right] = \frac{\Phi_{F_{\hat{U}}}}{[p^{i-1}]}[\theta_{\partial_{t_j}}(e)\otimes f_{j,0}].$$

Proof. Let $\tilde{e} \in \mathrm{Fil}^i H_U$ be a lifting of e. The horizontal property (iv) yields the following commutation formula

$$\nabla_{\partial_{t_j}}[\Phi_{F_{\hat{U}}}(\tilde{e}\otimes 1)] = \Phi_{F_{\hat{U}}}[\nabla_{\partial_{t_j}}(\tilde{e})\otimes 1]\otimes \frac{\partial F_{\hat{U}}}{\partial t_j}.$$

Thus we have a formula in characteristic p:

$$\nabla_{\partial_{t_j}}\left[\frac{\Phi_{F_{\hat{U}}}}{[p^i]}(e\otimes 1)\right] = \frac{\Phi_{F_{\hat{U}}}}{[p^{i-1}]}[\nabla_{\partial_{t_j}}(e)\otimes 1]\otimes f_{j,0}.$$

Finally by the same reason as given in the proof of Lemma 2.1 we could replace the connection in the right-hand side by the Higgs field, and hence obtain the lemma. □

Proposition 2.6. *The associated subbundle $H_{(G,\theta)}$ to the Higgs subbundle $(G,\theta) \subset (E,\theta)_0$ is a de Rham subbundle of $(H,\nabla)_0$.*

Proof. As the question is local, it suffices to show the invariance property of the locally associated subbundle $S_{U_0}(G) \subset H_0|_{U_0}$. Lemma 2.5 implies that for a local section $e \in G(U_0)$,

$$\nabla_{\partial_{t_j}}[\tilde{\Phi}_{F_{\hat{U}}}(e \otimes 1)] = \tilde{\Phi}_{F_{\hat{U}}}[\theta_{\partial_{t_j}}(e) \otimes f_{j,0}],$$

which is again an element of $S_{U_0}(G)$ by the θ-invariance of G. As $\{\partial_{t_j}\}_{1\le j\le d}$ spans $\mathrm{Der}_k(\mathcal{O}_{U_0}, \mathcal{O}_{U_0})$, we have shown that $\nabla(S_{U_0}(G)) \subset S_{U_0}(G) \otimes \Omega^1_{U_0}$ as claimed. □

2.4. *Variants*

By examining the above construction, one finds immediately that it works as well for a smooth scheme X over W_{n+1}. Also it is immediate to see that the same construction applies for a coherent subobject in $(E,\theta)_0$. Namely, for a Higgs subsheaf of $(E,\theta)_0$, we obtain a de Rham subsheaf of $(H,\nabla)_0$ from the construction. A similar construction also works in the logarithmic case. In the case of having a divisor at infinity, one simply replaces the Frobenius liftings and the integrable connection in the above construction with the logarithmic Frobenius liftings and the logarithmic integrable connection. In the semistable case, for a closed embedding $U \hookrightarrow Z$ as in Sec. 1.2, we replace the local operators $\frac{\Phi_{F_{\hat{U}}}}{[p^i]}$ in the smooth case with the reduction of the operator $\frac{\Phi_F}{\widehat{D_U^{\log}(Z)}^{[p^i]}}$ modulo the PD-ideal J, and use the Taylor formula of Sec. 1.2 in the proofs. The resulting construction yields a logarithmic de Rham subsheaf of $(H,\nabla)_0$ for any logarithmic Higgs subsheaf of $(E,\theta)_0$.

2.5. *Basic properties*

Let X be a smooth (respectively, log smooth) scheme over W (respectively, W_{n+1}) as above, and $(H, \mathrm{Fil}^{\cdot}, \nabla, \Phi) \in \mathcal{MF}^{\nabla}_{[0,n]}(X)$. Let $(E,\theta) = \mathrm{Gr}_{\mathrm{Fil}}(H,\nabla)$ be the associated Higgs bundle, and for a Higgs subbundle $(G,\theta) \subset (E,\theta)_0$, $(H_{(G,\theta)},\nabla) \subset (H,\nabla)_0$ the associated de Rham subbundle by the previous construction. It is not difficult to check the following properties.

Proposition 2.7. *The following statements hold*:

(i) *The construction is compatible with pull-backs. Namely, for f a morphism between smooth (respectively, log smooth) schemes over W (respectively, W_{n+1}),*

one has

$$(H_{f^*(G,\theta)}, \nabla) = f^*(H_{(G,\theta)}, \nabla).$$

(ii) *The construction is compatible with direct sum and tensor product.*

One can make our construction into a functor. First of all, one makes a category as follows: An object in this category is a Higgs subbundle of an $(E,\theta)_0$, which is the modulo p reduction of $\mathrm{Gr}_{\mathrm{Fil}}(H,\nabla)$ where $(H, \mathrm{Fil}^{\cdot}, \nabla)$ comes from an object in $\mathcal{MF}^{\nabla}_{[0,n]}(X)$ for an $n \geq p-2$. The set of morphisms are required to be inclusions of Higgs subbundles in the same Higgs bundle $(E,\theta)_0$. One defines the parallel category on the de Rham side. These categories have direct sums and tensor products. Proposition 2.7(ii) says that the functor respects direct sum and tensor product. Summarizing the above discussions, we have the following.

Theorem 2.8. *Let X be as above and $(H, \mathrm{Fil}^{\cdot}, \nabla, \Phi)$ an object in $\mathcal{MF}^{\nabla}_{[0,n]}(X)$. Let $(E,\theta) = \mathrm{Gr}_{\mathrm{Fil}}(H,\nabla)$ be the associated Higgs bundle. Then one associates* naturally *a Higgs subbundle of $(E,\theta)_0$ to a de Rham subbundle of $(H,\nabla)_0$.*

In the following let X be a smooth scheme over W or W_{n+1}. The next result relates our construction in the zero Higgs field case with the Cartier descent (see [8, Theorem 5.1]).

Proposition 2.9. *If $(G,0) \subset (E,\theta)_0$ is a Higgs subbundle with zero Higgs field, then one has an isomorphism of vector bundles with integrable connection*

$$\tilde{\Phi} : (F^*_{X_0} G, \nabla_{\mathrm{can}}) \xrightarrow{\cong} (H_{(G,0)}, \nabla|_{H_{(G,0)}}),$$

where ∇_{can} is the canonical connection associated to a Frobenius pull-back vector bundle.

Proof. This is a direct consequence of the construction of the subbundle $H_{(G,0)}$ and the formula in Proposition 2.2 in the case of $\theta = 0$. Note also that $\{\tilde{\Phi}(e_i \otimes 1)\}$, where $\{e_i\}$ runs through a local basis of G, makes an integrable basis of $S_{U_0}(G)$, which follows directly from the formula in Lemma 2.5 in the case of $\theta = 0$. □

2.6. *Cartier–Katz descent*

Let $(H, \mathrm{Fil}^{\cdot}, \nabla, \Phi)$ be a geometric one, namely it comes from Example 1.1. Then H_0 is equipped with the conjugate filtration $0 = F^{n+1}_{\mathrm{con}} \subset F^n_{\mathrm{con}} \subset \cdots \subset F^0_{\mathrm{con}} = H_0$, which is horizontal with respect to the Gauß-Manin connection (see [8, Sec. 3]). For a subbundle $W \subset H_0$ we put $\mathrm{Gr}_{F_{\mathrm{con}}}(W) = \bigoplus_{q=0}^{n} \frac{W \cap F^q_{\mathrm{con}}}{W \cap F^{q+1}_{\mathrm{con}}}$. The p-curvature ψ_∇ of ∇ defines the F-Higgs bundle

$$\psi_\nabla : \mathrm{Gr}_{F_{\mathrm{con}}}(H_0) \to \mathrm{Gr}_{F_{\mathrm{con}}}(H_0) \otimes F^*_{X_0} \Omega_{X_0}.$$

As a reminder to the reader, we recall the definition of *F-Higgs bundle*: an F-Higgs bundle over a base C, which is defined over k, is a pair (E', θ') where E' is

a vector bundle over C, and θ' is a bundle morphism $E' \to E' \otimes F_C^* \Omega_C$ with the integral property $\theta' \wedge \theta' = 0$. The following lemma is a simple consequence of Katz's p-curvature formula (see [9, Theorem 3.2]).

Lemma 2.10 ([15, Lemma 7.2][c]). *Let (W, ∇) be a de Rham subbundle of $(H, \nabla)_0$. Then the F-Higgs subbundle $(\mathrm{Gr}_{F_{\mathrm{con}}}(W), \psi_\nabla|_{\mathrm{Gr}_{F_{\mathrm{con}}}(W)})$ defines a Higgs subbundle of $(E, \theta)_0$ by the Cartier descent.*

We call the above Higgs subbundle the *Cartier–Katz descent* of (W, ∇). Back to the discussion on the associated de Rham subbundle $(H_{(G,\theta)}, \nabla) \subset (H, \nabla)_0$ with $(G, \theta) \subset (E, \theta)_0$. After a terminology of Simpson (see [16]) we shall call a Higgs subbundle (G, θ) with the property $G = \bigoplus_{i=0}^n (G \cap (E^{i,n-i})_0)$ a *subsystem of Hodge bundles*. We have the following.

Proposition 2.11. *Let $(G, \theta) \subset (E, \theta)_0$ be a subsystem of Hodge bundles. Then the Cartier–Katz descent of $(H_{(G,\theta)}, \nabla)$ is equal to (G, θ).*

Proof. First we recall that the relative Cartier isomorphism defines an isomorphism

$$\mathcal{C} : \mathrm{Gr}_{F_{\mathrm{con}}} H_0 \xrightarrow{\cong} F_{X_0}^* E_0.$$

We need to show that it induces an isomorphism

$$\mathrm{Gr}_{F_{\mathrm{con}}}(H_{(G,\theta)}) \cong F_{X_0}^* G.$$

Write $G^{i,n-i} = G \cap (E^{i,n-i})_0$. Then $G = \bigoplus_{i=1}^n G^{i,n-i}$. Now that over U_0 the composite

$$F_{U_0}^* E_0^{i,n-i}|_{U_0} \xrightarrow{\frac{\Phi_{F_{\hat{U}}}}{[p^i]}} F_{\mathrm{con}}^{n-i} H_0|_{U_0} \twoheadrightarrow \mathrm{Gr}_{F_{\mathrm{con}}}^{n-i} H_0|_{U_0}$$

is the inverse relative Cartier isomorphism $\mathcal{C}^{-1}|_{U_0}$ over U_0, it follows from the local construction of $H_{(G,\theta)}$ that

$$\mathcal{C}^{-1}|_{U_0}(F_{U_0}^* G^{i,n-i}|_{U_0}) \xrightarrow{\cong} \mathrm{Gr}_{F_{\mathrm{con}}}^{n-i} H_{(G,\theta)}|_{U_0}.$$

This implies the result. □

The above proof implies also the equalities

$$H_{(G^{\leq i},\theta)} = H_{(G,\theta)} \cap F_{\mathrm{con}}^{n-i}, \quad 0 \leq i \leq n,$$

where $G^{\leq i}$ is the Higgs subbundle $\bigoplus_{q \leq i} G^{q,n-q}$ of G.

Remark 2.12. The grading of $(H_{(G,\theta)}, \nabla)$ with respect to the Hodge filtration defines a Higgs subbundle of $(E, \theta)_0$ which is in general not (G, θ). In the case that they are equal and X is proper over W, (G, θ) defines a p-torsion subrepresentation of $\pi_1^{\mathrm{arith}}(X^0)$, the étale fundamental group of the generic fiber X^0 of X, implied

[c]The quoted lemma deals only with the weight one situation, but the proof works for an arbitrary weight.

by a result of Faltings (see [2, Theorem 2.6*]). A similar remark has appeared in [13, Sec. 4.6].

3. Applications

3.1. *Higgs semistability*

In this paragraph X is assumed to be smooth and projective over W_{n+1} with connected closed fiber X_0 over k. Fix an ample divisor D on X. Recall that the μ-slope of a torsion free coherent sheaf Z on X_0 is defined to be

$$\mu(Z) = \frac{c_1(Z) \cdot D_0^{d-1}}{\operatorname{rank} Z}.$$

Proposition 3.1. *Let (E, θ) be the associated Higgs bundle in the geometric case, i.e. Example* 1.1*. Then the following statements hold*:

(i) *For any subsystem of Hodge bundle $(G, \theta) \subset (E, \theta)_0$, one has $\mu(G) \leq 0$.*
(ii) *For any Higgs subbundle $G \subset E_0$ with zero Higgs field, it holds that $\mu(G) \leq 0$.*

Proof. Assume that there exists a subsystem of Hodge bundles (G, θ) with positive μ-slope in $(E, \theta)_0$. Take such one with the largest slope. By the proof of Proposition 2.11, one has an isomorphism $\mathrm{Gr}_{F_{\mathrm{con}}} H_{(G,\theta)} \cong F_{X_0}^* G$, and consequently the equalities $\mu(H_{(G,\theta)}) = \mu(F_{X_0}^* G) = p\mu(G)$. Then the observation in Remark 2.12 says that $\mathrm{Gr}_{\mathrm{Fil}}(H_{(G,\theta)}, \nabla)$ gives a subsystem of Hodge bundles of $(E, \theta)_0$ of slope $p\mu(G) > \mu(G)$, a contradiction. Hence (i) follows. Now assume the existence of a Higgs subbundle $(G, 0)$ with positive μ-slope. By Corollary 2.9, the associated de Rham subbundle $H_{(G,0)} \subset H_0$ is isomorphic to $F_{X_0}^* G$, whose μ-slope is equal to $p\mu(G) > 0$. Then $\mathrm{Gr}_{\mathrm{Fil}}(H_{(G,0)}, \nabla)$ gives rise to a subsystem of Hodge bundles with positive μ-slope, which contradicts (i). □

Let $C \subset X$ be a smooth projective curve over W_{n+1}. For a coherent sheaf Z over X_0, the μ_{C_0}-slope of Z is defined to be $\frac{\deg(Z|_{C_0})}{\operatorname{rank} Z}$. Recall that a Higgs bundle (E, θ) over X_0 is said to be Higgs semistable with respect to the μ_{C_0}-slope if for any Higgs subbundle $(F, \theta) \subset (E, \theta)$ the inequality $\mu_{C_0}(F) \leq \mu_{C_0}(E)$ holds.

Lemma 3.2. *For any Higgs subbundle $(G, \theta) \subset (E, \theta)_0$, it holds that*

$$\det C_0^{-1}(G, \theta) \cong (F_X^* \det G, \nabla_{\mathrm{can}}).$$

Proof. The functor C_0^{-1} has been further studied in the paper [10]. It follows from Proposition 5 loc. cit. that $C_0^{-1}(G, \theta)$ is isomorphic to the exponential twisting of $F_X^* G$. Precisely, it is obtained by gluing the local flat bundles $(F_U^* G|_U, \nabla_{\mathrm{can}} + \frac{dF_{\hat{U}}}{p} F_U^* \theta|_U)$ via the gluing functions $\exp[h_{UV}(F_{U\cap V}^* \theta)] F_{U\cap V}^* M_{U\cap V}$. Here U, V are two open subsets of X, $M_{U\cap V}$ is the transition function of two local bases of G over $U \cap V$, h_{UV} is the derivation measuring the difference of two Frobenius liftings. Due

to the fact that θ is nilpotent, the determinant of the exponential twisting is simply the identity. Therefore $\det C_0^{-1}(G,\theta)$ is isomorphic to $(F_X^* \det G, \nabla_{\mathrm{can}})$. □

Proposition 3.3. *Let (E,θ) be the associated Higgs bundle to an object in $\mathcal{MF}^{\nabla}_{[0,n]}(X)$. Then the following statements hold*:

(i) *Let $g_0 : C_0 \to X_0$ be a morphism from a smooth projective curve C_0 to X_0 over k which is liftable to a morphism $g : C \to X$ over W_{n+1}. Then for any Higgs subbundle $(G,\theta) \subset (E,\theta)_0$, one has $\deg(g_0^* G) \leq 0$.*

(ii) *Let C be a smooth projective curve in X over W_{n+1}. The Higgs bundle $(E,\theta)_0$ is Higgs semistable with respect to the μ_{C_0}-slope.*

Proof. Let $(E,\theta)|_C$ be the Higgs bundle pulled back via g. We use similar notions for the pull-backs via g_0. The pull-back of H via g gives an object in $\mathcal{MF}^{\nabla}_{[0,n]}(C)$. Assume the existence of a Higgs subbundle $(G,\theta) \subset (E,\theta)_0$ satisfying $\deg(G|_{C_0}) > 0$. By Lemma 3.2, it holds that

$$\deg \mathrm{Gr}_{\mathrm{Fil}} C_0^{-1}(G,\theta)|_{C_0} = \deg C_0^{-1}(G,\theta)|_{C_0} = \deg F_{C_0}^* G|_{C_0} = p \deg G|_{C_0}.$$

Iterating this process, one obtains Higgs subbundles in $E_0|_{C_0}$ with arbitrary large degrees, which is impossible. Hence the result (i) follows. The proof of (ii) is similar by replacing the degree in the previous argument with the μ_{C_0}-slope. □

Remark 3.4. In the above result (i), it is natural to make the liftability assumption on C_0. The example of Moret-Bailly (see [11]) shows that over a W_2-unliftable curve in the moduli space of principal polarized abelian surfaces in characteristic p the Higgs bundle of the restricted universal family contains a Higgs line bundle with positive degree. In the case X being a curve, the assumption on p made in [13, Proposition 4.19] reads $n(\mathrm{rank}\, E - 1)\max\{2g-2, 1\} \leq p-2$ where g is the genus of X_0. The above result (ii) removes the dependence of p on the genus as well as the rank.

3.2. *W_2-unliftability*

Let F be a real quadratic field with the ring of integers $\mathcal{O}_F$. Assume p is inert in F. Fix an integer $m \geq 3$, coprime to p. Let M be the moduli scheme over W, and S_h the Hasse locus of M_0 as described in Sec. 0. Let $Z_0 \subset S_h$ be a curve with simple normal crossing. It is said to be W_2-liftable inside M_1 if there exists a semistable curve $Z_1 \subset M_1$ over W_2 such that its closed fiber is Z_0. In the following the W_2-liftability means always the W_2-liftability inside M_1. It is interesting to know whether the components in S_h are liftable to W_2. The W_2-liftability on the whole or part of the $\mathbb{P}^1$-configuration S_h is in fact a subtle problem. On the one hand, it shall be more or less well known that each component of S_h is W_2-unliftable. On the other hand, a result of Goren (see [5, Theorem 2.1]) implies that the whole configuration is W_2-liftable if the zeta value $\zeta_F(2-p)$ is a non-p-adic integer. Our partial result on this question is the following.

Proposition 3.5. *Let D be a component in S_h. Let $C = \sum C_i$ (may be empty) be a simple normal crossing divisor in M_0 such that $D + C$ is again a simple normal crossing divisor. If $D \cdot C \leq p - 1$, then the curve $D + C$ is W_2-unliftable.*

Before giving the proof, we introduce several notations. Let $f : X \to M$ be the universal abelian scheme. Let $(E = E^{1,0} \oplus E^{0,1}, \theta)$ be the Higgs bundle of f, where $E^{1,0} = f_*\Omega^1_{X|M}$ and $E^{0,1} = R^1 f_* \mathcal{O}_X$. It is known that (E, θ) has a decomposition under the $\mathcal{O}_F \otimes W$-action in the form

$$(E, \theta) = (E_1, \theta_1) \oplus (E_2, \theta_2),$$

where $E_i^{1,0} = \mathcal{L}_i$ and $E_i^{0,1} = \mathcal{L}_i^{-1}$ for $i = 1, 2$. It is also known that either $\mathcal{L}_1|_D \simeq \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{L}_2|_D \simeq \mathcal{O}_{\mathbb{P}^1}(p)$ or $\mathcal{L}_1|_D \simeq \mathcal{O}_{\mathbb{P}^1}(p),\ \mathcal{L}_2|_D \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ holds for any component D in S_h. One has a description of the Higgs field of the Higgs bundle associated to the restricted universal family to D: In the former case, $\theta_2 : \mathcal{L}_2|_D \to \mathcal{L}_2^{-1}|_D \otimes \Omega^1_D$ is zero for the reason of degree, and $\theta_1 : \mathcal{L}_1|_D \to \mathcal{L}_1^{-1}|_D \otimes \Omega^1_D$ can be shown to be an isomorphism (we shall not use this fact in the following argument). The properties of θ_1 and θ_2 are exactly exchanged in the latter case. Put the log structure on $Z_0 = D + C$ by its components and the trivial log structure on $\operatorname{Spec} k$. Let $\Omega^1_{\log}(Z_0/k)$ be the sheaf of log differentials (see [7, (1.7)]). It is locally free $\mathcal{O}_{Z_0}$-module of rank one. The family f_0 restricts to a family $f_0 : Y_0 \to Z_0$. With the pull-back logarithmic structure on Y_0, f_0 is log smooth. So one forms the logarithmic de Rham bundle (H, ∇) of f_0, which is by definition the first hypercohomology of the relative logarithmic de Rham complex. The relative Hodge filtration on the complex degenerates at E_1, thus one forms the logarithmic Higgs bundle over Z_0:

$$\eta : F \to F \otimes \Omega^1_{\log}(Z_0/k),$$

where $F = F^{1,0} \oplus F^{0,1}$ with $F^{1,0} = \mathcal{L}_1|_{Z_0} \oplus \mathcal{L}_2|_{Z_0}$ and $F^{0,1} = \mathcal{L}_1^{-1}|_{Z_0} \oplus \mathcal{L}_2^{-1}|_{Z_0}$.

Proof of Proposition 3.5. Now we assume Z_0 lifts a semistable curve $Z \subset M_1$ over W_2. We equip Z with the log structure determined by the divisor Z_0 and $\operatorname{Spec} W_2$ with the one by $\operatorname{Spec} k$. We can assume that

$$\mathcal{L}_1|_D \simeq \mathcal{O}_{\mathbb{P}^1}(-1), \quad \mathcal{L}_2|_D \simeq \mathcal{O}_{\mathbb{P}^1}(p).$$

Over the open subset $D - D \cap C$, $\eta|_D$ coincides with the Higgs bundle coming from the restricted universal family to D. So by the above discussion, $\eta|_D(\mathcal{L}_2|_D) = 0$. Consider the following coherent subsheaf in F: Take the subsheaf

$$\mathcal{L}_2|_D \otimes \mathcal{O}_D(-D \cap C) \subset \mathcal{L}_2|_D$$

over D and take the zero sheaf over C, considered as the subsheaf of $\mathcal{L}_2|_C$. They glue into a subsheaf $\mathcal{L}$ of $\mathcal{L}_2|_{Z_0} \subset F$ over Z_0 since over a small open neighborhood of any point $P \in D \cap C$, $\mathcal{L}_2|_D \otimes \mathcal{O}_D(-D \cap C)$ has a local basis vanishing at P. Note that $\mathcal{L}$ is a Higgs subsheaf of F. In fact the Higgs field η acts on $\mathcal{L}$ trivially by construction. Then the construction of Sec. 2 in the semistable case applies. So $(\mathcal{L}, 0) \subset (F, \eta)$ gives rise to a de Rham subsheaf $H_{(\mathcal{L},0)} \subset (H, \nabla)$. Note that $H_{(\mathcal{L},0)}|_D$

is isomorphic to $F_D^*(\mathcal{L}_2|_D \otimes \mathcal{O}_D(-D \cap C))$ and hence has degree $p(p - D \cap C)$. We have a short exact sequence from the Hodge filtration:

$$0 \to \mathcal{L}_1|_D \oplus \mathcal{L}_2|_D \to H|_D \to \mathcal{L}_1^{-1}|_D \oplus \mathcal{L}_2^{-1}|_D \to 0.$$

Then we first consider the composite

$$H_{(\mathcal{L},0)}|_D \subset H|_D \twoheadrightarrow \mathcal{L}_1^{-1}|_D \oplus \mathcal{L}_2^{-1}|_D.$$

As $\deg \mathcal{L}_1^{-1}|_D = 1$ and $\deg \mathcal{L}_2^{-1}|_D = -p$ are both smaller than $\deg H_{(\mathcal{L},0)}|_D$, one has the factorization

$$H_{(\mathcal{L},0)}|_D \subset \mathcal{L}_1|_D \oplus \mathcal{L}_2|_D \subset H|_D.$$

Case 1: $D \cap C \leq p - 2$. Again for the reason of degree, the above nontrivial map is impossible.

Case 2: $D \cap C = p - 1$. In this case, one obtains the equality

$$H_{(\mathcal{L},0)}|_D = \mathcal{L}_2|_D.$$

This is also impossible because of the semilinearity of the relative Frobenius. For a small affine $U \subset Z$ whose modulo p reduction is $U_0 \subset D - D \cap C$ and a Frobenius lifting $F_{\hat{U}}$, the local operator $\frac{\Phi_{F_{\hat{U}}}}{[p]}$ maps a local section in $\mathcal{L}_2|_D(U_0)$ to a local section in $\mathcal{L}_1|_D(U_0)$. As $\mathcal{L}_2|_D \otimes \mathcal{O}_D(-D \cap C) \subset \mathcal{L}_2|_D$, it is impossible to have $H_{(\mathcal{L},0)}(U_0) \subset \mathcal{L}_2|_D(U_0)$ by the construction of $H_{(\mathcal{L},0)}$. □

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References

[1] E. Bachmat and E. Z. Goren, On the non-ordinary locus in Hilbert–Blumenthal surfaces, *Math. Ann.* **313**(3) (1999) 475–506.

[2] G. Faltings, Crystalline cohomology and Galois representations, in *Algebraic Analysis, Geometry, and Number Theory*, ed. J. Igusa (Johns Hopkins University Press, Baltimore, MD, 1989), pp. 25–80.

[3] G. Faltings, F-isocrystals on open varieties: Results and conjectures, in *The Grothendieck Festschrift*, Vol. 2, Progress in Mathematics, Vol. 87 (Birkhäuser, Boston, MA, 1990), pp. 219–248.

[4] G. Faltings, Integral crystalline cohomology over very ramified valuation rings, *J. Amer. Math. Soc.* **12**(1) (1999) 117–144.

[5] E. Z. Goren, Hasse invariants for Hilbert modular varieties, *Israel J. Math.* **122** (2001) 157–174.

[6] E. Z. Goren and F. Oort, Stratifications of Hilbert modular varieties, *J. Algebraic Geom.* **9**(1) (2000) 111–154.

[7] K. Kato, Logarithmic structures of Fontaine–Illusie, in *Algebraic Analysis, Geometry, and Number Theory* (Johns Hopkins University Press, Baltimore, MD, 1989), pp. 191–224.

[8] N. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin, *Publ. Math. Inst. Hautes Études Sci.* **39** (1970) 175–232.
[9] N. Katz, Algebraic solutions of differential equations (p-curvature and the Hodge filtration), *Invent. Math.* **18** (1972) 1–118.
[10] G.-T. Lan, M. Sheng and K. Zuo, An inverse Cartier transform via exponential in positive characteristic, preprint (2012), arXiv:1205.6599.
[11] L. Moret–Bailly, Familles de courbes et de varietes Abeliennes sur $\mathbb{P}^1$, *Asterisque* **86** (1981) 125–140.
[12] A. Ogus, F-crystals, Griffiths transversality, and the Hodge decomposition, *Astérisque* **221** (1994).
[13] A. Ogus and V. Vologodsky, Nonabelian Hodge theory in characteristic p, *Publ. Math. Inst. Hautes Études Sci.* **106** (2007) 1–138.
[14] D. Schepler, Logarithmic nonabelian Hodge theory in characteristic p, Ph.D. thesis, University of California, Berkeley (2005).
[15] M. Sheng, J.-J. Zhang and K. Zuo, Higgs bundles over the good reduction of a quaternionic Shimura curve, *J. Reine Angew. Math.* **671** (2012) 223–248.
[16] C. Simpson, Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization, *J. Amer. Math. Soc.* **1**(4) (1988) 867–918.
[17] C. Simpson, Higgs bundles and local systems, *Publ. Math. Inst. Hautes Étud. Sci.* **75** (1992) 5–95.
[18] T. Tsuji, Syntomic complexes and p-adic vanishing cycles, *J. Reine Angew. Math.* **472** (1996) 69–138.