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DEURING'S MASS FORMULA OF A MUMFORD FAMILY

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ABSTRACT. We study the Newton polygon jumping locus of a Mumford family in char p. Our main result says that, under a mild assumption on p, the jumping locus consists of only supersingular points and its cardinality is equal to $(p^r-1)(g-1)$, where r is the degree of the defining field of the base curve of a Mumford family in char p and g is the genus of the curve. The underlying technique is the p-adic Hodge theory.

1. INTRODUCTION

Let k be a finite field of char p and $\bar{k}$ an algebraic closure of k. A basic result of M. Deuring [8] says that elliptic curves over $\bar{k}$ can be divided into two classes: ordinary and supersingular, and there are finitely many supersingular elliptic curves up to isomorphisms. The mass formula is a formula on the number of isomorphism classes of supersingular elliptic curves. For an odd prime p, it can be deduced from the fact that there are exactly $\frac{p-1}{2}$ supersingular elliptic curves in the Legendre family

$$y^2 = x(x-1)(x-t), \quad t \neq 0, 1.$$

The purpose of this paper is to give analogous results for a Mumford family. In [19], D. Mumford gave the *first* example of families of Hodge type, which is characterized by the Hodge group (called also the special Mumford-Tate group in the literature) but not by the endomorphism algebra. We briefly recall the construction as follows. Let F be a totally real cubic field with three real places τ_1, τ_2, τ_3, and D a quaternion division algebra over F such that D splits at one real place of F and its corestriction to $\mathbb{Q}$ splits, i.e.,

$$\mathrm{Cor}_{F|\mathbb{Q}} D := (D^{(1)} \otimes D^{(2)} \otimes D^{(3)})^{\mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q})} \cong M_8(\mathbb{Q}),$$

where $D^{(i)} := D \otimes_{F,\tau_i} \bar{\mathbb{Q}}$. It gives rise to families of abelian four-folds over smooth projective arithmetic quotients of the upper half plane, whose general fiber has only $\mathbb{Z}$ as its endomorphism ring. For some purposes, his construction has been generalized (and also characterized) in the work of Viehweg and the second named author (see [29], particularly Theorem 0.5).

Now let F be a totally real field of degree $d \geq 3$, whose ring of algebraic integers is denoted by $\mathcal{O}$, and D a quaternion division algebra over F, which is split only at one real place of F. The corestriction $\mathrm{Cor}_{F|\mathbb{Q}} D$ is a central simple $\mathbb{Q}$-algebra. Following

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the construction of Mumford [19], one is able to associate $\mathrm{Cor}_{F|\mathbb{Q}}D$ with a Shimura curve of Hodge type (see §2 for details). The universal family of abelian varieties over such a Shimura curve with a suitable level structure is called a Mumford family in this paper. In order to do reduction modulo p, we also need a natural integral model of a Mumford family. For that, we make the following

Assumption 1.1. *Assume $p \geq 3$ and does not divide the discriminants of F and D.*

After the work of M. Kisin [15], one is able to define the integral canonical model of the Shimura curve over any prime $\mathfrak{p}$ of F over p together with a universal abelian scheme over the integral model, which is defined over $\mathcal{O}_{(\mathfrak{p})}$. Fix such a universal abelian scheme and denote its completion at $\mathfrak{p}$ by $f : X \to M$, whose modulo $\mathfrak{p}$ reduction is denoted by $f_0 : X_0 \to M_0$. By the theorem of Grothendieck-Katz, the Newton polygon jumping locus $\mathcal{S} \subset M_0(\bar{k})$ consists of finitely many points. Our main result is stated as follows:

Theorem 1.2 (Theorem 3.17, Corollary 5.17)**.** *The Newton jumping locus $\mathcal{S}$ consists only of supersingular points. Assume additionally that $p \geq 5$. Then one has a mass formula for the cardinality of $\mathcal{S}$:*

$$|\mathcal{S}| = (p^r - 1)(g - 1),$$

where $r = [F_{\mathfrak{p}} : \mathbb{Q}_p]$ and g is the genus of M_0.

As M_0 may not be geometrically connected, the genus is defined to be one plus the half of the summation of the degree of the canonical class of each component in $M_0 \otimes \bar{k}$.

Remark 1.3. The generic Newton polygon in $M_0(\bar{k})$ is also determined (see Theorem 3.17). For the original example of Mumford, i.e., $d = 3$ and $\mathrm{Cor}_{F|\mathbb{Q}}(D)$ split, R. Noot [20]-[21] (see particularly Proposition 3.6 [20] and Proposition 2.2 [21]) classified the possible Newton polygons for the mod p reduction of an abelian variety defined over a number field and appearing as a closed fiber of a Mumford family. Compared with his method, the new point here is a natural decomposition of the p-adic Galois representation into a tensor product of two dimensional potentially crystalline $\mathbb{Q}_{p^r}$-representations after tensorizing with $\mathbb{Q}_{p^r}, r \leq 3$, and this is true for a general Mumford family. In our approach, the classification result becomes a simple consequence of the admissibility of a filtered ϕ-module associated with a crystalline representation.

The above result settles Conjecture 1.3 in [26] for Mumford families. In our previous work [26], we have studied a certain Shimura curve of PEL type: Deligne-Shimura's modèle étrange [6]. However, the old technique does not suffice for the current situation (see §1-§5 [26]). One main reason is that a Mumford family is characterized by extra Hodge cycles in the generic fiber which are not yet known to be algebraic in general. It makes impossible a direct proof of an expected direct-tensor decomposition of the universal filtered Dieudonné module into rank two filtered crystals over the global base with predicted relative Frobenius actions on factors. Instead, we have to work with both the category of étale local systems and the category of families of filtered Frobenius crystals and use various comparison results in the p-adic Hodge theory. What we have achieved is the following result.

Theorem 1.4 (Theorem 4.12). *Let x_0 be a closed point of M and $\hat{M}_{x_0}$ the completion of M at x_0. Then one has a direct-tensor decomposition in the category $\mathcal{MF}^{\nabla}_{[0,1]}(\hat{M}_{x_0})$ of the restriction to $\hat{M}_{x_0}$ of the universal filtered Dieudonné module attached to f:*

$$
\begin{aligned}
&(H^1_{dR}, F_{hod}, \nabla^{GM}, \phi)|_{\hat{M}_{x_0}} \\
&\qquad \cong \{[\bigotimes_{i=0}^{r-1}(\mathcal{N}_i, Fil^1_{\mathcal{N}_i}, \nabla_{\mathcal{N}_i}), \phi_{ten}] \otimes (M_{A_2}, Fil^1_{A_2}, d, \phi_{A_2})\}^{\oplus 2^{\epsilon(D)}},
\end{aligned}
$$

where $\{(\mathcal{N}_i, Fil^1_{\mathcal{N}_i}, \nabla_{\mathcal{N}_i})\}_{0\leq i\leq r-1}$ are eigen-components of the universal filtered Dieudonné module of a versal deformation of a Drinfel'd $\mathcal{O}_{\mathfrak{p}}$-divisible module, ϕ_{ten} is the tensor product of the ϕ_i's on eigen-components, and $(M_{A_2}, Fil^1_{A_2}, d, \phi_{A_2})$ is a constant unit crystal.

Here $\epsilon(D)$ is equal to 0 or 1 which depends on D only (see §2). The above result shows an intimate relation between the associated p-divisible groups to a Mumford family and Drinfel'd $\mathcal{O}_{\mathfrak{p}}$-divisible modules, which we intend to understand in more depth in the future.

The paper is structured as follows. In §2 we review briefly the construction of a Shimura curve of Hodge type arising from the corestriction of a quaternion division algebra and deduce an integral model of a Mumford family from the work of Kisin [15]. In §3 we first show a natural direct-tensor decomposition of the étale $\mathbb{Z}_{p^d}$-local system attached to a Mumford family into rank two factors, and then show that over each closed point each factor is potentially crystalline, from which the classification of the Newton polygons in $M_0(\bar{k})$ follows. Section 4 is a bridge between the classification and the mass formula, in which we study the universal abelian scheme over the formal neighborhood a k-rational point of the base curve by using the deformation theory of a p-divisible group with Tate cycles due to G. Faltings, §7 [11] (see also §4 [18] and §1.5 [15]), and prove the direct-tensor decomposition result, Theorem 1.4. In the final section we prove the mass formula for the supersingular locus in $M_0(\bar{k})$. For a technical reason, we consider instead a second tensor power of the universal filtered Dieudonné crystal and construct from a direct factor in the corresponding decomposition a nonzero morphism $\tilde{F}_{rel}: F^{*r}_{M_0}\mathcal{P}_0 \to \mathcal{P}_0$ in char p whose reduced zero divisor coincides with the supersingular locus, where $\mathcal{P}_0$ is a line bundle of negative degree over M_0. With the aid of Theorem 1.4, we apply the theory of display to compute that the multiplicity of the zero divisor is everywhere two and hence obtain the mass formula.

2. Corestriction of a quaternion division algebra and an integral model of a Mumford family

Let F be a totally real field of degree $d \geq 3$ and D a quaternion division algebra over F, which is split only at one real place of F. We denote the set of real embeddings of F by $\Psi := \{\tau = \tau_1, ..., \tau_d\}$, and assume that D is split over τ. Let $\bar{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and $\mathrm{Gal}_{\mathbb{Q}}$ the absolute Galois group of $\mathbb{Q}$. Recall (see §4 [19]) that the corestriction $\mathrm{Cor}_{F|\mathbb{Q}}D$ is defined as the subalgebra of $\mathrm{Gal}_{\mathbb{Q}}$-invariant elements of $\bigotimes_{i=1}^{d} D\otimes_{F,\tau_i}\bar{\mathbb{Q}}$. For it one has the following result.

Lemma 2.1 (Lemma 5.7 (a) [29]). *Let F and D be as above. It holds that either*

(i) $\mathrm{Cor}_{F|\mathbb{Q}}(D) \cong M_{2^d}(\mathbb{Q})$ *and d is an odd number ≥ 3 or*

(ii) $\mathrm{Cor}_{F|\mathbb{Q}}(D) \ncong M_{2^d}(\mathbb{Q})$. *Then*

$$\mathrm{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{b}) \cong M_{2^d}(\mathbb{Q}(\sqrt{b})),$$

where $\mathbb{Q}(\sqrt{b})$ is a quadratic field extension of $\mathbb{Q}$.

Both cases can be written uniformly into $\mathrm{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{b}) \cong M_{2^d}(\mathbb{Q}(\sqrt{b}))$ for a square free rational number $b \in \mathbb{Q}$, and such an isomorphism will be fixed in the following. We define a number $\epsilon(D)$ to be 0 in case (i) and 1 in case (ii). So we can fix an embedding $\mathrm{Cor}_{F|\mathbb{Q}}(D) \hookrightarrow M_{2^{d+\epsilon(D)}}(\mathbb{Q})$ of $\mathbb{Q}$-algebras. Note that the case $d = 3$ and $\epsilon(D) = 0$ is the original example considered by Mumford [19]. Recall also that one comes along with a natural morphism of $\mathbb{Q}$-groups:

$$\mathrm{Nm} : D^* \to \mathrm{Cor}_{F|\mathbb{Q}}(D)^*,\ d \mapsto (d \otimes 1) \otimes \cdots \otimes (d \otimes 1).$$

So one obtains a linear representation $\mathrm{Nm} : D^* \to \mathrm{GL}_{2^{d+\epsilon(D)},\mathbb{Q}}$ of the $\mathbb{Q}$-group D^*. It gives rise to a Shimura curve of Hodge type, which is not PEL type (see Construction 5.8, pages 269-273 [29] and §1.1 [21]). Put $\tilde{G'}_{\mathbb{Q}} := \{x \in D |\ \mathrm{Norm}(x) = 1\}$ and $\tilde{G}_{\mathbb{Q}} := \mathbb{G}_{m,\mathbb{Q}} \times \tilde{G'}_{\mathbb{Q}}$, and write $\mathrm{GL}_{\mathbb{Q}}$ for $\mathrm{GL}_{2^{d+\epsilon(D)},\mathbb{Q}}$. The $\mathbb{Q}$-group $G_{\mathbb{Q}}$ is defined to be the image of the morphism $\tilde{G}_{\mathbb{Q}} \to \mathrm{GL}_{\mathbb{Q}}$, which is the product of the natural morphism $\mathbb{G}_{m,\mathbb{Q}} \to \mathrm{GL}_{\mathbb{Q}}$ and $\mathrm{Nm}|_{\tilde{G'}_{\mathbb{Q}}}$. It is connected and reductive. The natural morphism $N : \tilde{G}_{\mathbb{Q}} \to G_{\mathbb{Q}}$ is a central isogeny. Let $G'_{\mathbb{Q}}$ be the image of $\tilde{G}'_{\mathbb{Q}}$ in $G_{\mathbb{Q}}$. The natural embedding $G_{\mathbb{Q}} \hookrightarrow \mathrm{GL}_{\mathbb{Q}}$ factors through $\mathrm{GSp}_{\mathbb{Q}} \subset \mathrm{GL}_{\mathbb{Q}}$, which can be seen as follows: let $H_{\mathbb{Q}} := \mathbb{Q}(\sqrt{b})^{2^d}$ be a $\mathbb{Q}$-vector space with the $\mathrm{Cor}_{F|\mathbb{Q}}(D)$ action by the left multiplication and $\mathbb{G}_{m,\mathbb{Q}}$ action by scalar multiplication. This induces a $G_{\mathbb{Q}}$-action on $H_{\mathbb{Q}}$. It is easy to verify that there exists a $\mathbb{Q}(\sqrt{b})$-valued symplectic form ω on $H_{\mathbb{Q}}$, unique up to scalar, which is invariant under the $G'_{\mathbb{Q}}$-action. Then $G_{\mathbb{Q}} = \mathbb{G}_{m,\mathbb{Q}} \cdot \tilde{G}'_{\mathbb{Q}} \subset \mathrm{GL}_{\mathbb{Q}}$ acts on the $\mathbb{Q}$-valued symplectic form $\psi := \mathrm{tr}_{\mathbb{Q}(\sqrt{b})|\mathbb{Q}}\omega$ by similitude. Let S^1 be the real group $\{z \in \mathbb{C} |\ z\bar{z} = 1\}$. One defines

$$u_0 : S^1 \to \tilde{G'}_{\mathbb{R}}(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SU}(2)^{\times d-1},\ e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \times id^{\times d-1}.$$

The morphism $\tilde{h}_0 = id \times u_0 : \mathbb{R}^* \times S^1 \to \tilde{G}_{\mathbb{R}}$ descends to a morphism of real groups:

$$h_0 : \mathbb{S} = \mathrm{Res}_{\mathbb{C}|\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}.$$

Let X be the $G(\mathbb{R})$-conjugacy class of h_0 and $(\mathrm{GSp}(H_{\mathbb{Q}}, \psi), X(\psi))$ the Siegel space defined by $(H_{\mathbb{Q}}, \psi)$. One verifies that $(G_{\mathbb{Q}}, X) \hookrightarrow (\mathrm{GSp}(H_{\mathbb{Q}}, \psi), X(\psi))$ is a morphism of Shimura datum and therefore defines a Shimura curve of Hodge type. Now let $K \subset G(\mathbb{A}_f)$ be a compact open subgroup, and one defines the Shimura curve as the double coset

$$Sh_K(G, X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

where

$$q(x, a)b = (qx, qab),\ q \in G(\mathbb{Q}), x \in X, a \in G(\mathbb{A}_f), b \in K.$$

By the theory of canonical model, $M_K := Sh_K(G, X)$ is naturally defined over the reflex field of (G, X), that is, $\tau(F) \subset \mathbb{C}$ in this current case. It is not difficult to show that M_K is proper over F.

Now let p be a rational prime satisfying Assumption 1.1 and let $p\mathcal{O} = \prod_{i=1}^n \mathfrak{p}_i$ be the prime decomposition of p in F. By choosing an embedding $\iota : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ (which is fixed once and for all), one gets an identification of Ψ with

$$\mathrm{Hom}_{\mathbb{Q}}(F, \bar{\mathbb{Q}}_p) = \coprod_{i=1}^{n} \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \bar{\mathbb{Q}}_p).$$

Write F_i for $F_{\mathfrak{p}_i}$ and put $r_i := [F_i : \mathbb{Q}_p]$. We assume that $\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(F_1, \bar{\mathbb{Q}}_p)$. Set $\mathfrak{p} = \mathfrak{p}_1$ and $r = r_1$. The condition on p implies that $G_{\mathbb{Q}_p}$ is quasi-split and split over an unramified extension of $\mathbb{Q}_p$. Hence hyperspecial subgroups exist in $G(\mathbb{Q}_p)$ (see 1.10 [27]). Recall that we have a central isogeny $\tilde{G}_{\mathbb{Q}} \to G_{\mathbb{Q}} \subset \mathrm{GL}_{\mathbb{Q}}$ over $\mathbb{Q}$ with $\tilde{G}_{\mathbb{Q}} = \mathbb{G}_{m,\mathbb{Q}} \times \tilde{G}'_{\mathbb{Q}}$, where $\tilde{G}'_{\mathbb{Q}} = \ker(\mathrm{Norm} : D^* \to F^*)$. The assumption on p implies that $D^*(\mathbb{Q}_p) \cong \prod_{i=1}^n \mathrm{GL}_2(F_i)$. It is clear that $\mathrm{Norm} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ becomes a product of the determinants under the isomorphism. So this implies that $\tilde{G}'(\mathbb{Q}_p) \cong \prod_{i=1}^n \mathrm{SL}_2(F_i)$, and hence $\tilde{G}(\mathbb{Q}_p) \cong \mathbb{Q}_p^* \times \prod_{i=1}^n \mathrm{SL}_2(F_i)$. Thus a hyperspecial subgroup of $G(\mathbb{Q}_p)$ is conjugate to the image of $\mathbb{Z}_p^* \times \prod_{i=1}^n \mathrm{SL}_2(\mathcal{O}_{F_i}) \subset \tilde{G}(\mathbb{Q}_p)$ under the isogeny $\tilde{G}(\mathbb{Q}_p) \to G(\mathbb{Q}_p)$. In what follows the p-component $K_p \subset G(\mathbb{Q}_p)$ of the level structure $K(= K_p K^p \subset G(\mathbb{Q}_p) G(\mathbb{A}_f^p))$ is always taken to be hyperspecial. The main result of Kisin [15] asserts then that, for our chosen prime $\mathfrak{p}|p$, *the* integral canonical model $\mathcal{M}_K$ of M_K exists, which is a smooth $\mathcal{O}_{(\mathfrak{p})}$-scheme for K^p sufficiently small. The construction of $\mathcal{M}_K$ (see §2.3 [15]) provides a universal abelian scheme over $\mathcal{M}_K$ as well, once the coprime to p-component K^p is chosen small enough: take a suitable maximal order $\mathcal{O}_D$ of the F-algebra D and consider

$$\mathrm{Cor}_{F|\mathbb{Q}} \mathcal{O}_D := (\bigotimes_{i=1}^{d} \mathcal{O}_D \otimes_{\mathcal{O}_F, \tau_i} \bar{\mathbb{Z}})^{\mathrm{Gal}_{\mathbb{Q}}} \subset \mathrm{Cor}_{F|\mathbb{Q}} D.$$

There exists a lattice $H_{\mathbb{Z}} \subset H_{\mathbb{Q}}$, which is stabilized by $\mathrm{Cor}_{F|\mathbb{Q}} \mathcal{O}_D \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Q}(\sqrt{b})}$, such that there is a closed embedding $G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(H_{\mathbb{Z}_p})$ (where $G_{\mathbb{Z}_p}$ is the reductive group scheme over $\mathbb{Z}_p$ associated with K_p) whose generic fiber is the base change to $\mathbb{Q}_p$ of $G_{\mathbb{Q}} \hookrightarrow \mathrm{GL}_{\mathbb{Q}}$. Let $K'_p \subset \mathrm{GSp}(\mathbb{Q}_p)$ be the stabilizer of $H_{\mathbb{Z}_p}$. One can choose a $K'^p \subset \mathrm{GSp}(\mathbb{A}_f^p)$ such that for $K' = K'_p K'^p$ one has an embedding of Shimura varieties $M_K \hookrightarrow Sh_{K'}(\mathrm{GSp}(\psi), X(\psi))$ and $Sh_{K'}(\mathrm{GSp}(\psi), X(\psi))$ has an integral model $\mathcal{S}_{K'} := \mathcal{S}_{K'}(\mathrm{GSp}(\psi), X(\psi))$ over $\mathbb{Z}_{(p)}$ (which is not necessarily smooth) representing a moduli functor over $\mathbb{Z}_{(p)}$ (see §2.3.3 [15]). As K^p is required to be small enough, one may further assume that K'^p is taken so small that there exists an abelian scheme $\mathcal{A}_{K'} \to \mathcal{S}_{K'}$ over $\mathcal{S}_{K'}$. Recall (see Theorem 2.3.8 [15]) that $\mathcal{M}_K$ is defined as the normalization of the closure of the composite

$$M_K \hookrightarrow Sh_{K'}(\mathrm{GSp}(\psi), X(\psi)) \hookrightarrow \mathcal{S}_{K'} \times_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\mathfrak{p})}.$$

Now we define our abelian scheme $f_K : \mathcal{X}_K \to \mathcal{M}_K$ to be the morphism sitting in the Cartesian diagram:

$$\begin{tikzcd}
\mathcal{X}_K \arrow[r] \arrow[d, "f_K"'] & \mathcal{A}_{K'} \times_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\mathfrak{p})} \arrow[d] \\
\mathcal{M}_K \arrow[r] & \mathcal{S}_{K'} \times_{\mathbb{Z}_{(p)}} \mathcal{O}_{(\mathfrak{p})}.
\end{tikzcd}$$

For the sake of convenience, we shall change our foregoing notation as follows: let M (resp. $f : X \to M$) be the completion of the integral canonical model $\mathcal{M}_K$

(resp. $f_K : \mathcal{X}_K \to \mathcal{M}_K$) at $\mathfrak{p}$. The schemes X and M are defined over the discrete valuation ring $\mathcal{O}_{\mathfrak{p}}$, the completion of $\mathcal{O}_{(\mathfrak{p})}$ at the maximal ideal, and f is an $\mathcal{O}_{\mathfrak{p}}$-morphism. The superscript (resp. subscript) zero on an object means the base change of the object to the generic (resp. closed) fiber of $\mathcal{O}_{\mathfrak{p}}$.

To the universal abelian scheme $f : X \to M$ we attach the étale $\mathbb{Z}_p$-local system $\mathbb{H} := R^1 f^0_*(\mathbb{Z}_p)_{\bar{X}^0_{et}}$ over M^0 and the universal filtered Dieudonné module $(H, F, \nabla, \phi) := (H^1_{dR}, F_{hod}, \nabla^{GM}, \phi)$, which is an object in the category $\mathcal{MF}^{\nabla}_{[0,1]}(M)$ introduced by Faltings (see Ch. II [10] and §3 [11]). To distinguish the notation, the p-torsion analogue of the previous category will be denoted by $\mathcal{MF}^{\nabla}_{[0,a]}(M)_{tor}$. In [10] Faltings constructed a fully faithful functor $\mathbf{D}$ from $\mathcal{MF}^{\nabla}_{[0,p-2]}(M)$ (resp. $\mathcal{MF}^{\nabla}_{[0,p-2]}(M)_{tor}$) to the category of étale $\mathbb{Z}_p$ (resp. p-torsion) local systems over M^0. By the Remark after Theorem 2.6* in [10], one has $\mathbf{D}(\mathcal{O}_X/p^n, d) = \mathbb{Z}/p^n$ for each $n \in \mathbb{N}$. Applying Theorem 6.2 and the Remark after the theorem [10] on the compatibility of the direct image with the functor $\mathbf{D}$, one gets $\mathbf{D}(H/p^n, F, \nabla, \phi) = \mathbb{H}^{\vee}/p^n$. By taking the inverse limit, one obtains then $\mathbf{D}(H, F, \nabla, \phi) = \mathbb{H}^{\vee}$. One notices that the information on the Newton jumping locus of $f_0 : X_0 \to M_0$ is encoded in the attached universal filtered Dieudonné module, while the defining information of a Mumford family is basically contained in the étale local system over M^0.

3. Two dimensional potentially crystalline $\mathbb{Q}_{p^r}$-representations and classification of the Newton polygons

For a k-rational point x_0 of M_0, the closed fiber of f_0 at x_0 is denoted by A_{x_0}. The Newton polygon of A_{x_0} is defined to be the Newton polygon of its associated p-divisible group. The aim of this section is to determine the possible Newton polygons of A_{x_0} when x_0 varies in $M_0(\bar{k})$.

3.1. **Tensor decomposition of the Galois representation.** Let x_0 be as above. Because M is smooth over $\mathcal{O}_{\mathfrak{p}}$, there exists an $\mathcal{O}_{W(k)}$-valued point x of M which lifts x_0. Let A_x be the corresponding abelian scheme over $\mathcal{O}_{W(k)}$ whose reduction is equal to A_{x_0}. The aim of this subsection is to show a certain direct-tensor decomposition of the p-adic Galois representation associated with the generic fiber A_{x^0} of A_x.

Lemma 3.1. *One has a natural isomorphism of $\mathbb{Q}_p$-algebras:*

$$\mathrm{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \bigotimes_{i=1}^{n} \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D \otimes_F F_i).$$

Proof. Put $D_i = D \otimes_{F,\tau_i} \bar{\mathbb{Q}}, 1 \le i \le d$. For an element $a \otimes \lambda \in D_i$ and $g \in \mathrm{Gal}_{\mathbb{Q}}$,

$$g(a \otimes_{\tau_i} \lambda) = a \otimes_{g(\tau_i)} g(\lambda).$$

By the definition of the corestriction, one has a natural isomorphism of $\mathrm{Gal}_{\mathbb{Q}}$-modules:

$$\mathrm{Cor}_{F|\mathbb{Q}}(D) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \cong \bigotimes_{i=1}^{d} D_i.$$

Let D_ι be the decomposition group of ι in $\mathrm{Gal}_{\mathbb{Q}}$, which is isomorphic to the local Galois group $\mathrm{Gal}_{\mathbb{Q}_p}$. Now we consider the D_ι-invariants of two sides of the above isomorphism after tensorizing with $\bar{\mathbb{Q}}_p$ via ι. Obviously one obtains $\mathrm{Cor}_{F|\mathbb{Q}}(D)\otimes_{\mathbb{Q}}\mathbb{Q}_p$ from the left side. Let

$$O_1 := \{\tau_1=\tau,\cdots,\tau_{r_1}=\tau_r\},\cdots,O_n=\{\tau_{r_1+\cdots+r_{d-1}+1},\cdots,\tau_{r_1+\cdots+r_{d-1}+r_d}=\tau_n\}$$

be the n-orbits of D_ι-action on Ψ. Note that there is a natural isomorphism

$$\Big(\bigotimes_{\tau_j\in O_i} D_j\otimes_{\bar{\mathbb{Q}},\iota}\bar{\mathbb{Q}}_p\Big)^{D_\iota}\cong \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D\otimes_F F_i).$$

As the tensor product $\bigotimes_{i=1}^n(\bigotimes_{\tau_j\in O_i} D_j\otimes_{\bar{\mathbb{Q}},\iota}\bar{\mathbb{Q}}_p)^{D_\iota}$ over $\mathbb{Q}_p$ is clearly a subspace of the D_ι-invariants on the right side, it must be the whole invariant space for the dimension reason. So the lemma follows. $\square$

Consider the base change to $\mathbb{Q}_p$ of the $\mathbb{Q}$-morphism $\mathrm{Nm}: D^*\to\mathrm{Cor}_{F|\mathbb{Q}}(D)^*$. The following statement is clear from the proof of the last lemma.

Lemma 3.2. *The morphism* $\mathrm{Nm}_{\mathbb{Q}_p}: D^*(\mathbb{Q}_p)\to\mathrm{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p)$ *factors through the natural morphism*

$$\prod_{i=1}^n \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D\otimes_F F_i)^*\to\mathrm{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p).$$

Moreover, under the natural decomposition $D^*(\mathbb{Q}_p)=\prod_{i=1}^n(D\otimes_F F_i)^*$, $\mathrm{Nm}_{\mathbb{Q}_p}$ *is written as a product* $\prod_{i=1}^n\mathrm{Nm}_i$ *where for each i the morphism*

$$\mathrm{Nm}_i:(D\otimes_F F_i)^*\to\mathrm{Cor}_{F_i|\mathbb{Q}_p}(D\otimes_F F_i)^*$$

is the natural diagonal morphism for the corestriction.

As a consequence, the representation of $D^*(\mathbb{Q}_p)$ on $H_{\mathbb{Q}_p}$ admits a natural tensor decomposition: by Schur's lemma the representation decomposes as a tensor product. In the current situation, this can be seen in a direct way: by Assumption 1.1, $D\otimes_F F_i$ splits for each i, and so does $\mathrm{Cor}_{F_i|\mathbb{Q}_p}(D\otimes_F F_i)$, which is isomorphic to $M_{2^{r_i}}(\mathbb{Q}_p)$. In the case $\epsilon(D)=0$, the morphism

$$\prod_{i=1}^n \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D\otimes_F F_i)^*\to\mathrm{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p)=\mathrm{GL}(H_{\mathbb{Q}_p})$$

is isomorphic to the tensor product morphism

$$\prod_{i=1}^n\mathrm{GL}_{2^{r_i}}(\mathbb{Q}_p)\longrightarrow\mathrm{GL}_{2^d}(\mathbb{Q}_p),\ (g_1,\cdots,g_n)\mapsto g_1\otimes\cdots\otimes g_n.$$

In the case $\epsilon(D)=1$, $\mathrm{Cor}_{F|\mathbb{Q}}(D)$ is nonsplit, and it splits after tensorizing with $\mathbb{Q}(\sqrt{b})$. Consider the composite

$$\begin{aligned}\prod_{i=1}^n \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D\otimes_F F_i)^* &\to \mathrm{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p)\subset(\mathrm{Cor}_{F|\mathbb{Q}}(D)\otimes_{\mathbb{Q}}\mathbb{Q}(\sqrt{b}))^*(\mathbb{Q}_p)\\ &= \mathrm{GL}_{\mathbb{Q}(\sqrt{b})}(H_{\mathbb{Q}})(\mathbb{Q}_p)\subset\mathrm{GL}(H_{\mathbb{Q}})(\mathbb{Q}_p).\end{aligned}$$

The above inclusion $\mathrm{Cor}_{F|\mathbb{Q}}(D)^* \subset (\mathrm{Cor}_{F|\mathbb{Q}}(D)\otimes_{\mathbb{Q}}\mathbb{Q}(\sqrt{b}))^*$ is given by $a\mapsto a\otimes 1$. One has the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p)\times\mathbb{Q}(\sqrt{b})^*(\mathbb{Q}_p) \arrow[r] \arrow[d, "\cap"'] & \mathrm{GL}_{\mathbb{Q}(\sqrt{b})}(H_{\mathbb{Q}})(\mathbb{Q}_p) \arrow[d, "\cap"] \\
\mathrm{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p)\times \mathrm{GL}_2(\mathbb{Q}_p) \arrow[r] & \mathrm{GL}(H_{\mathbb{Q}})(\mathbb{Q}_p).
\end{tikzcd}$$

The image of $\prod_{i=1}^n \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D\otimes_F F_i)^*$ in the left-up element of the above diagram is contained in $\mathrm{Cor}_{F|\mathbb{Q}}(D)^*(\mathbb{Q}_p)\times\{1\}$. Thus the morphism

$$\prod_{i=1}^n \mathrm{Cor}_{F_i|\mathbb{Q}_p}(D\otimes_F F_i)^* \to \mathrm{GL}(H_{\mathbb{Q}_p})$$

is isomorphic to the composite of the obvious morphisms

$$\prod_{i=1}^n \mathrm{GL}_{2^{r_i}}(\mathbb{Q}_p) \hookrightarrow \prod_{i=1}^n \mathrm{GL}_{2^{r_i}}(\mathbb{Q}_p)\times \mathrm{GL}_2(\mathbb{Q}_p) \xrightarrow{\otimes} \mathrm{GL}_{2^{d+1}}(\mathbb{Q}_p).$$

For $s\in\mathbb{N}$, one denotes by $\sigma\in \mathrm{Gal}(\mathbb{Q}_{p^s}|\mathbb{Q}_p)$ the Frobenius automorphism. For a topological group P with a continuous $\mathbb{Z}_{p^s}$ linear representation W, the σ^i-conjugate W_{σ^i} of W for $0\le i\le s-1$ is defined to be the tensor product $W\otimes_{\mathbb{Z}_{p^s},\sigma^i}\mathbb{Z}_{p^s}$, where P acts on $\mathbb{Z}_{p^s}$ trivially. Similarly define the $\sigma^{\cdot}$-conjugations of a $\mathbb{Q}_{p^s}$-representation. The symbol $\otimes_{\sigma^i}$ signifies the equalities of two tensors:

$$\lambda(x\otimes\mu)=x\otimes\lambda\mu,\ \ \lambda x\otimes\mu = x\otimes\lambda^{\sigma^i}\mu,\ \text{ for } \lambda,\mu\in\mathbb{Z}_{p^s}, x\in W.$$

Consider the morphism

$$\mathrm{Nm}_1 : (D\otimes_F F_1)^* \longrightarrow \mathrm{Cor}_{F_1|\mathbb{Q}_p}(D\otimes_F F_1)^*,\ a\mapsto (a\otimes_{F_1,\tau_1}1)\otimes\cdots\otimes(a\otimes_{F_1,\tau_r}1).$$

Note that for $1\le i\le r$, $\tau_i(F_1)=\mathbb{Q}_{p^r}$, the unique unramified extension of $\mathbb{Q}_p$ of degree r in $\bar{\mathbb{Q}}_p$. Then one has a natural isomorphism

$$\mathrm{Cor}_{F_1|\mathbb{Q}_p}(D\otimes_F F_1)\otimes_{\mathbb{Q}_p} F_1 \cong \bigotimes_{i=0}^{r-1}(D\otimes_F F_1\otimes_{F_1,\sigma^i} F_1).$$

This implies that the natural morphism $(D\otimes_F F_1)^* \longrightarrow \mathrm{Cor}_{F_1|\mathbb{Q}_p}(D\otimes_F F_1)^*(F_1)$ is isomorphic to the composite of

$$\mathrm{GL}_2(\mathbb{Q}_{p^r}) \hookrightarrow \prod_{i=0}^{r-1}\mathrm{GL}_2(\mathbb{Q}_{p^r}),\ g\mapsto (g,\cdots,\sigma^i(g),\cdots,\sigma^{r-1}(g))$$

with the tensor product morphism $\prod_{i=0}^{r-1}\mathrm{GL}_2(\mathbb{Q}_{p^r})\longrightarrow \mathrm{GL}_{2^r}(\mathbb{Q}_{p^r})$. Summarizing the above discussions, we derive the following:

Lemma 3.3. *The representation of $D^*(\mathbb{Q}_p)$ on $H_{\mathbb{Q}_p}$ admits a natural tensor decomposition*

$$H_{\mathbb{Q}_p} = (V_{\mathbb{Q}_p}\otimes U_{1,\mathbb{Q}_p}\otimes\cdots\otimes U_{n-1,\mathbb{Q}_p})^{\oplus 2^{\epsilon(D)}}.$$

Moreover, the representation $V_{\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}\mathbb{Q}_{p^r}$ decomposes further into a tensor product $V_1\otimes_{\mathbb{Q}_{p^r}}V_{1,\sigma}\otimes_{\mathbb{Q}_{p^r}}\cdots\otimes_{\mathbb{Q}_{p^r}}V_{1,\sigma^{r-1}}$ with $\dim_{\mathbb{Q}_{p^r}}V_1=2$. For $1\le i\le n-1$, $U_{i,\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}\mathbb{Q}_{p^{r_{i+1}}}$ decomposes into a tensor product in a similar manner.

Lemma 3.4. *Let P be a topological group together with a continuous linear representation on a finite dimensional $\mathbb{Q}_{p^s}$-vector space W. Assume the following two conditions hold:*

(i) *The representation factors as*

$$P \to \mathrm{GL}(W_1) \times \mathrm{SL}(W_2) \xrightarrow{\otimes} \mathrm{GL}(W),$$

where $W_i, i = 1, 2$, are two $\mathbb{Q}_{p^s}$-vector spaces.

(ii) *There is a $\mathbb{Z}_{p^s}$-lattice $W_{\mathbb{Z}_{p^s}}$ in W which is stable under the P-action and admits a lattice tensor decomposition*

$$W_{\mathbb{Z}_{p^s}} = W_{1,\mathbb{Z}_{p^s}} \otimes_{\mathbb{Z}_{p^s}} W_{2,\mathbb{Z}_{p^s}},$$

where $W_{i,\mathbb{Z}_{p^s}}$ is a $\mathbb{Z}_{p^s}$-lattice of W_i for $i = 1, 2$.

Then in the factorization of (i)*, the lattice $W_{i,\mathbb{Z}_{p^s}}$ for $i = 1, 2$ is stable under the P-action on W_i.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{GL}(W_{1,\mathbb{Z}_{p^s}}) \times \mathrm{SL}(W_{2,\mathbb{Z}_{p^s}}) \arrow[r, "\otimes"] \arrow[d, "\cap"'] & \mathrm{GL}(W_{\mathbb{Z}_{p^s}}) \arrow[d, "\cap"] \\
P \arrow[r] & \mathrm{GL}(W_1) \times \mathrm{SL}(W_2) \arrow[r, "\otimes"] & \mathrm{GL}(W).
\end{tikzcd}$$

It suffices to show that the representation $P \to \mathrm{GL}(W_{\mathbb{Z}_{p^s}}) \subset \mathrm{GL}(W)$ factors through

$$\mathrm{GL}(W_{1,\mathbb{Z}_{p^s}}) \times \mathrm{SL}(W_{2,\mathbb{Z}_{p^s}}) \to \mathrm{GL}(W_{\mathbb{Z}_{p^s}}).$$

Note that $\mathrm{GL}(W_{\mathbb{Z}_{p^s}})$ is a compact subgroup of $\mathrm{GL}(W)$. As the morphism $\otimes$ has a finite kernel,

$$T := \otimes^{-1}(\mathrm{GL}(W_{\mathbb{Z}_{p^s}}) \cap \otimes(\mathrm{GL}(W_1) \times \mathrm{SL}(W_2)))$$

is a compact subgroup of $\mathrm{GL}(W_1) \times \mathrm{SL}(W_2)$. Since T contains $\mathrm{GL}(W_{1,\mathbb{Z}_{p^s}}) \times \mathrm{SL}(W_{2,\mathbb{Z}_{p^s}})$, which is maximal compact, it holds that $T = \mathrm{GL}(W_{1,\mathbb{Z}_{p^s}}) \times \mathrm{SL}(W_{2,\mathbb{Z}_{p^s}})$. Since the image of P in $\mathrm{GL}(W_1) \times \mathrm{SL}(W_2)$ is contained in T by assumption, the morphism $P \to \mathrm{GL}(W_{\mathbb{Z}_{p^s}})$ factors through $\mathrm{GL}(W_{1,\mathbb{Z}_{p^s}}) \times \mathrm{SL}(W_{2,\mathbb{Z}_{p^s}}) \to \mathrm{GL}(W_{\mathbb{Z}_{p^s}})$. This proves the lemma. ◻

Proposition 3.5. *The K_p-representation $H_{\mathbb{Z}_p}$ admits a natural direct-tensor decomposition*

$$H_{\mathbb{Z}_p} = (V \otimes U)^{\oplus 2^{\epsilon(D)}},$$

where U decomposes into $U = U_1 \otimes \cdots \otimes U_{n-1}$. The tensor factor V after tensorizing with $\mathbb{Z}_{p^r}$ decomposes further into

$$V \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = V_1 \otimes_{\mathbb{Z}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Z}_{p^r}} \cdots \otimes_{\mathbb{Z}_{p^r}} V_{1,\sigma^{r-1}}.$$

Similarly for other tensor factors $U_i, 1 \le i \le n-1$, after tensorizing with $\mathbb{Z}_{p^{r_{i+1}}}$.

Proof. Recall that K_p is conjugate to the image of

$$\tilde{K}_p := (\mathbb{Z}_p^* \times \mathrm{SL}_2(\mathcal{O}_{F_\mathfrak{p}})) \times \prod_{i=2}^{n} \mathrm{SL}_2(\mathcal{O}_{F_i}) \subset \tilde{G}(\mathbb{Q}_p)$$

under the map $N_{\mathbb{Q}_p} : \tilde{G}(\mathbb{Q}_p) \to G(\mathbb{Q}_p)$. The direct-tensor decomposition of $H_{\mathbb{Q}_p}$ as $D^*(\mathbb{Q}_p)$-representation in Lemma 3.3 induces a direct-tensor decomposition of $H_{\mathbb{Q}_p}$

as $\tilde{K}_p$-representation. Since the $\tilde{K}_p$-action on $H_{\mathbb{Q}_p}$ factors through the K_p-action on $H_{\mathbb{Q}_p}$ by definition, one obtains the direct-tensor decomposition of $H_{\mathbb{Q}_p}$ for the K_p-action as well. By the definition of the lattice $H_{\mathbb{Z}}$, it is easy to see that $H_{\mathbb{Z}_p}$ decomposes into a direct-tensor product of $\mathbb{Z}_p$-lattices. Then it is also a direct-tensor decomposition as K_p-representation by Lemma 3.4. The proofs of the tensor decompositions for the factors V and U are analogous. □

By Proposition 2.2.4 [15], each geometrically connected component of M^0 is defined over an unramified extension of $F_{\mathfrak{p}}$. Since there is a finite number of them, we can fix a finite extension L of $F_{\mathfrak{p}}$ inside the maximal unramified extension $\mathbb{Q}_p^{ur}$ such that each component defines and admits an L-rational point.

Corollary 3.6. *One has a direct-tensor decomposition of étale local systems over $M^0 \times_{F_{\mathfrak{p}}} L$:*

$$\mathbb{H} = (\mathbb{V} \otimes \mathbb{U})^{\oplus 2^{\epsilon(D)}}, \mathbb{U} = \mathbb{U}_1 \otimes \cdots \otimes \mathbb{U}_{n-1}$$

and

$$\mathbb{V} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} \cong \mathbb{V}_1 \otimes_{\mathbb{Z}_{p^r}} \mathbb{V}_{1,\sigma} \otimes_{\mathbb{Z}_{p^r}} \cdots \otimes_{\mathbb{Z}_{p^r}} \mathbb{V}_{1,\sigma^{r-1}},$$

where for $0 \leq i \leq r-1$, $\mathbb{V}_{1,\sigma^i}$ is the σ^i-conjugate of $\mathbb{V}_1$. Similarly for $\mathbb{U}_i, 1 \leq i \leq n-1$, one has

$$\mathbb{U}_i \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^{r_{i+1}}} \cong \mathbb{U}_{i,1} \otimes_{\mathbb{Z}_{p^{r_{i+1}}}} \mathbb{U}_{i,\sigma} \otimes_{\mathbb{Z}_{p^{r_{i+1}}}} \cdots \otimes_{\mathbb{Z}_{p^{r_{i+1}}}} \mathbb{U}_{1,\sigma^{r_{i+1}-1}}.$$

Proof. Let $M^0 \times_{F_{\mathfrak{p}}} L = \coprod_i M_i^0$ be the disjoint union of its geometrically connected components, and let M_1^0 be the component which is represented by the double coset $[1] \in G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$. It suffices to show the tensor decomposition of the restriction $\mathbb{H}$ to M_1^0. Consider the short exact sequence of étale fundamental groups:

$$1 \to \pi_1^{geo}(M_1^0) \to \pi_1^{arith}(M_1^0) \to \mathrm{Gal}(\bar{\mathbb{Q}}_p|L) \to 1.$$

An L-rational point of M^0 induces a splitting of the exact sequence, and one writes

$$\pi_1^{arith}(M_1^0) = \pi_1^{geo}(M_1^0) \cdot \mathrm{Gal}(\bar{\mathbb{Q}}_p|L).$$

To show the tensor decomposition of $\mathbb{H}|_{M_1^0}$, it suffices to show the factorization of $\pi_1^{geo}(M_1^0) \to K_p$ and $\mathrm{Gal}(\bar{\mathbb{Q}}_p|L) \to K_p$. The latter follows from the proof of Lemma 2.2.1 [15]. The former goes as follows: it is known that $\pi_1^{top}(M_1^0(\mathbb{C}))$ is equal to $K \cap G(\mathbb{Q})_+$, and the representation $\pi_1^{geo}(M_1^0) \to \mathrm{GL}(H_{\mathbb{Z}_p})$ is the composite of

$$\pi_1^{geo}(M_1^0) \cong \hat{\pi}_1^{top}(M_1^0) \xrightarrow{\wedge} \mathrm{GL}(H_{\hat{\mathbb{Z}}}) \twoheadrightarrow \mathrm{GL}(H_{\mathbb{Z}_p}).$$

Obviously the representation $\pi_1^{top}(M_1^0) = K \cap G(\mathbb{Q})_+ \to \mathrm{GL}(H_{\mathbb{Z}})$ factors through $K \subset \mathrm{GL}(H_{\mathbb{Z}})$. Hence the result follows from Proposition 3.5. □

Specializing the above tensor decompositions of étale local systems into a closed point, one obtains the following.

Corollary 3.7. *Let E be a finite extension of L and x^0 an E-rational point of M^0. Let $H_{\mathbb{Z}_p} = H^1_{et}(\bar{A}_{x^0}, \mathbb{Z}_p)$ and $\rho : \mathrm{Gal}_E \to \mathrm{GL}(H_{\mathbb{Z}_p})$ be the associated Galois representation. Then one has a direct-tensor decomposition of Gal_E-modules,*

$$H_{\mathbb{Z}_p} = (V_{\mathbb{Z}_p} \otimes U_{\mathbb{Z}_p})^{\oplus 2^{\epsilon(D)}},$$

and a further tensor decomposition of $V_{\mathbb{Z}_p}$ after tensorizing with $\mathbb{Z}_{p^r}$,

$$V_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = V_1 \otimes_{\mathbb{Z}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Z}_{p^r}} \cdots \otimes_{\mathbb{Z}_{p^r}} V_{1,\sigma^{r-1}}.$$

3.2. **Each tensor factor is potentially crystalline.** It is standard that $H_{\mathbb{Q}_p}$ is a polarizable crystalline representation of Hodge-Tate weights $\{0,1\}$. In the following we will show that each factor appearing in the direct-tensor decomposition of Corollary 3.7 is potentially crystalline.

Proposition 3.8. *$U_{\mathbb{Q}_p}$ is a potentially unramified representation. As a consequence, both $V_{\mathbb{Q}_p}$ and $U_{\mathbb{Q}_p}$ are potentially crystalline.*

Proof. Let $I_E \subset \mathrm{Gal}_E$ be the inertia group. We claim that the image of I_E in $\mathrm{GL}(U_{\mathbb{Q}_p})$ is finite. Assuming the claim, one sees that $U_{\mathbb{Q}_p}$ is potentially unramified and hence potentially crystalline. Clearly $V_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p}$, as a direct factor of $H_{\mathbb{Q}_p}$, is crystalline. Therefore $V_{\mathbb{Q}_p}$, that is, a subrepresentation of $V_{\mathbb{Q}_p} \otimes U_{\mathbb{Q}_p} \otimes U^{\vee}_{\mathbb{Q}_p}$, is also potentially crystalline. To show the claim, we introduce the Hodge-Tate cocharacter $\mu_{HT} : \mathbb{G}_m(\mathbb{C}_p) \to G(\mathbb{C}_p) \subset \mathrm{GL}(H_{\mathbb{C}_p})$ induced by the Hodge-Tate decomposition of $H_{\mathbb{Q}_p}$ and the Hodge-de Rham cocharacter $\mu_{HdR} : \mathbb{G}_m(\mathbb{C}) \to G(\mathbb{C}) \subset \mathrm{GL}(H_{\mathbb{C}})$ induced by the Hodge decomposition of $H^1_B(A(\mathbb{C}), \mathbb{Q})$. Let C_{HdR} (resp. C_{HT}) be the $G(\mathbb{C})$ (resp. $G(\mathbb{C}_p)$)-conjugacy class of μ_{HdR} (resp. μ_{HT}). Then C_{HdR} is defined over the reflex field $\tau(F) \subset \mathbb{C}$ of (G, X). It follows from a result of Blasius and Wintenberger (see Theorem 0.3 [2] and Proposition 7 [30]; see also Theorem 4.2 [24]) that

$$C_{HT} = C_{HdR} \otimes_{F,\tau} \mathbb{C}_p,$$

where $\tau : F \to \mathbb{C}_p$ is the composite $F \xrightarrow{\tau} \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p \subset \hat{\bar{\mathbb{Q}}}_p = \mathbb{C}_p$. Since $\tilde{G} \to G$ is a central isogeny, there is a natural number a such that the a-th power ${\mu_{HdR}}^a$ (resp. ${\mu_{HT}}^a$) lifts to a cocharacter into $\tilde{G}(\mathbb{C})$ (resp. $\tilde{G}(\mathbb{C}_p)$). Consider the projection of ${\mu_{HdR}}^a$ to an SL_2-factor in the decomposition $\tilde{G}(\mathbb{C}) = \mathbb{C}^* \times \mathrm{SL}_2(\mathbb{C}) \times \cdots \times \mathrm{SL}_2(\mathbb{C})$, where the order of SL_2-factors is arranged according to Ψ. By the definition of $\tilde{h}_0$ in §2, one sees that *only* the projection to the first SL_2-factor (corresponding to τ) is nontrivial. By the above identification, the same situation holds for the projections of ${\mu_{HT}}^a$ to SL_2-factors in the decomposition

$$\tilde{G}(\mathbb{C}_p) = \mathbb{C}_p^* \times \mathrm{SL}_2(\mathbb{C}_p) \times \cdots \times \mathrm{SL}_2(\mathbb{C}_p),$$

where the order of SL_2-factors is arranged according to $\mathrm{Hom}_{\mathbb{Q}}(F, \bar{\mathbb{Q}}_p)$, which has been identified with Ψ. By the construction of the U-factor, the projection of ${\mu_{HT}}^a$ to the factor $\mathrm{GL}(U_{\mathbb{C}_p})$ is trivial. So the projection of μ_{HT} to $\mathrm{GL}(U_{\mathbb{C}_p})$ is finite. By S. Sen's theorem (see [25]), the Zariski closure of $\rho(I_E) \subset G(\mathbb{Q}_p)$ is equal to the $\mathbb{Q}_p$-Zariski closure of μ_{HT}. So the image of I_E in $\mathrm{GL}(U_{\mathbb{Q}_p})$ is finite. □

For a finite extension E of $\mathbb{Q}_p$ let $E_0 \subset E$ be the maximal unramified subextension. Recall that

Definition 3.9. A $\mathbb{Q}_{p^r}$-representation of Gal_E is a finite dimensional $\mathbb{Q}_{p^r}$-vector space V equipped with a continuous action $\mathrm{Gal}_E \times V \to V$ satisfying

$$g(v_1 + v_2) = g(v_1) + g(v_2),\ \ g(\lambda v) = g(\lambda) g(v)$$

for $g \in \mathrm{Gal}_E$, $\lambda \in \mathbb{Q}_{p^r}$ and $v, v_1, v_2 \in V$. It is called a *Hodge-Tate* (resp. *de-Rham*, *crystalline*) $\mathbb{Q}_{p^r}$-representation if it is the case considered as a $\mathbb{Q}_p$-representation.

The following result is known among experts. A variant of it was communicated by L. Berger to the first named author during the p-adic Hodge theory workshop at ICTP, 2009. The first official proof should appear in the PhD thesis of G. Di Matteo (see the recent preprint [17]). Another proof has been communicated to us by L. Xiao (see [28]).

Theorem 3.10. *Let V and W be two $\mathbb{Q}_{p^r}$-representations of* Gal_E. *If $V \otimes_{\mathbb{Q}_{p^r}} W$ is de Rham and one of the tensor factors is Hodge-Tate, then each tensor factor is de Rham.*

Applying Theorem 3.10 to the tensor factor $V_{\mathbb{Q}_p}$ in Proposition 3.8, one obtains the following.

Proposition 3.11. *Making an additional finite field extension $E' \subset E''$ if necessary, one has a further decomposition of* $\mathrm{Gal}_{E''}$*-representations:*

$$V_{\mathbb{Q}_p} \otimes \mathbb{Q}_{p^r} \cong V_1 \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^{r-1}},$$

where $\mathrm{Gal}_{E''}$ *acts on $\mathbb{Q}_{p^r}$ trivially and V_{1,σ^i} is the σ^i-conjugate of V_1. Then V_1 is potentially crystalline.*

Each σ-conjugate V_{1,σ^i} is isomorphic to V_1 as a $\mathbb{Q}_p$-representation. Thus each tensor factor in the above decomposition is potentially crystalline as well.

Proof. Assume $r = 2$ for simplicity. The above tensor decomposition implies the tensor decomposition of $\mathbb{C}_p$-representations:

$$V_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong (V_1 \otimes_{\mathbb{Q}_{p^2}} \mathbb{C}_p) \otimes_{\mathbb{C}_p} (V_{1,\sigma} \otimes_{\mathbb{Q}_{p^2}} \mathbb{C}_p).$$

Since $V_{\mathbb{Q}_p}$ is crystalline, it is Hodge-Tate. This implies that Sen's operator Θ_V of $V_{\mathbb{Q}_p}$ is diagonalizable over $\mathbb{C}_p$. Let Θ_{V_1} be Sen's operator of V_1. It can be written naturally as $\Theta_1 \oplus \Theta_{1,\sigma}$, where Θ_1 is associated with $V_1 \otimes_{\mathbb{Q}_{p^2}} \mathbb{C}_p$ and $\Theta_{1,\sigma}$ to $V_{1,\sigma} \otimes_{\mathbb{Q}_{p^2}} \mathbb{C}_p$. Thus one has

$$\Theta_V = \Theta_1 \otimes id + id \otimes \Theta_{1,\sigma}.$$

This implies that Θ_1 and $\Theta_{1,\sigma}$ are diagonalizable. Now consider the eigenvalues of them. For that we use the relation between the Hodge-Tate cocharacter and the eigenvalues of Sen's operator: they are related by the maps log and exp. Continue the argument about Hodge-Tate cocharacter in Proposition 3.8, so let $\{\tau = \tau_1, \tau_2\}$ be the $\mathrm{Gal}_{\mathbb{Q}_p}$-orbit of τ. We can assume that in the above decomposition the V_1-factor corresponds to τ. It follows that the projection of μ_{HT} to the $V_{1,\sigma}$-factor is trivial. This implies that the eigenvalues of $\Theta_{1,\sigma}$ are zero. Particularly they are integral. So are those of Θ_1. Hence Θ_{V_1} is diagonalizable with integral eigenvalues. So V_1 is Hodge-Tate, and by Theorem 3.10 it is de Rham. By the p-adic monodromy theorem, conjectured by Fontaine and first proved by Berger (see [1]), it is potentially log crystalline. One shows further that it is potentially crystalline. Let N_V (resp. N_{V_1}) be the monodromy operator of V (resp. V_1). Then one has the formulas:

$$N_{V_1} = N_1 + N_{1,\sigma}, \ N_V = N_1 \otimes id + id \otimes N_{1,\sigma}.$$

Since V is crystalline, $N_V = 0$. This implies that $N_1 = N_{1,\sigma} = 0$. Hence $N_{V_1} = 0$ and V_1 is potentially crystalline. □

3.3. **Consequence on the possibilities of the Newton polygon.** We show that the admissibility of a filtered ϕ-module associated with a crystalline representation yields a classification of the Newton polygons in $M_0(\bar{k})$. The following functors were introduced by Fontaine in order to study a $\mathbb{Q}_{p^r}$-representation:

Definition 3.12. Let V be a $\mathbb{Q}_{p^r}$-representation. For each $0 \leq m \leq r-1$, one defines

$$D^{(m)}_{crys,r}(V) := (V \otimes_{\mathbb{Q}_{p^r},\sigma^m} B_{crys})^{\mathrm{Gal}_E}.$$

One defines similarly the functors $\{D^{(m)}_{dR,r}(V)\}_{0\leq m\leq r-1}$ by replacing B_{crys} with B_{dR}. The following lemma is obvious:

Lemma 3.13. *Let V be a $\mathbb{Q}_{p^r}$-representation. Then there is a natural isomorphism of Gal_E-representations*

$$V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \cong \bigoplus_{m=1}^{r-1} V \otimes_{\mathbb{Q}_{p^r},\sigma^m} \mathbb{Q}_{p^r}.$$

By the lemma there is a natural direct decomposition

$$\begin{aligned} V \otimes_{\mathbb{Q}_p} B_{crys} &\cong V \otimes_{\mathbb{Q}_p} (\mathbb{Q}_{p^r} \otimes_{\mathbb{Q}_{p^r}} B_{crys}) \\ &\cong (V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}) \otimes_{\mathbb{Q}_{p^r}} B_{crys} \\ &\cong \bigoplus_{m=0}^{r-1} V \otimes_{\mathbb{Q}_{p^r},\sigma^m} B_{crys}, \end{aligned}$$

which implies in particular a direct decomposition of E_0-vector spaces

$$D_{crys}(V) = \bigoplus_{m=0}^{r-1} D^{(m)}_{crys,r}(V).$$

It is clear that V is crystalline iff $\dim_{E_0} D^{(m)}_{crys,r}(V) = \dim_{\mathbb{Q}_{p^r}} V$ for either m holds. Let V be a crystalline $\mathbb{Q}_{p^r}$-representation. Over $D_{crys}(V)$ there is a natural σ-linear map ϕ, and over $D_{dR}(V) = D_{crys}(V) \otimes_{E_0} E$ there is a natural filtration Fil. We want to study some properties of the restrictions of them to a direct factor.

Lemma 3.14. *The map ϕ permutes the direct factors $\{D^{(m)}_{crys,r}(V)\}$ cyclically. Consequently, one has the decomposition of ϕ^r-modules*

$$(D_{crys}(V), \phi^r) = \bigoplus_{m=0}^{r-1} (D^{(m)}_{crys,r}(V), \phi^r|_{D^{(m)}_{crys,r}(V)}).$$

Moreover, each ϕ^r-submodule $(D^{(m)}_{crys,r}(V), \phi^r|_{D^{(m)}_{crys,r}(V)})$ has the same Newton slopes.

Proof. For $d = v \otimes_{\sigma^m} b \in D^{(m)}_{crys,r}(V)$, it follows from the formula $\phi(d) = v \otimes_{\sigma^{m+1 \bmod r}} \phi(b)$ that $\phi(d) \in D^{(m+1 \bmod r)}_{crys,r}(V)$. So ϕ permutes the direct factors in a cyclic way. Thus ϕ^r is preserved under direct decomposition. The last statement can be seen as follows: take a basis e of $D^{(0)}_{crys,r}(V)$. Then as ϕ is a semilinear isomorphism, $e_m = \phi^m(e)$ is a basis of $D^{(m)}_{crys,r}(V)$. Let A be the matrix satisfying

$$\phi(e_{r-1}) = Ae_0.$$

Then under the basis $\{e_0, e_1, \cdots, e_{r-1}\}$ of $D_{crys}(V)$, the representation matrix of ϕ^r reads:

$$\phi^r \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{r-1} \end{pmatrix} = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A^\sigma & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{\sigma^{r-1}} \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{r-1} \end{pmatrix}.$$

From here, one sees the equality of Newton slopes on each factor clearly. □

As a consequence, one can define on the tensor product $\bigotimes_{m=0}^{r-1} D^{(m)}_{crys,r}(V)$ a ϕ-module structure: for a vector of form $v_0 \otimes \cdots \otimes v_{r-1}$, define

$$\phi_{ten}(v_0 \otimes \cdots \otimes v_{r-1}) := \phi(v_{r-1}) \otimes \phi(v_0) \otimes \cdots \otimes \phi(v_{r-2}).$$

It is easily seen that the σ^r-linear map ϕ^r_{ten} is the tensor product of $\phi^r|_{D^{(m)}_{crys,r}(V)}$'s.

Next we consider the induced filtration $Fil^i_m := Fil^i \cap D^{(m)}_{dR,r}(V)$ on each direct factor $D^{(m)}_{dR,r}(V)$ from $D_{dR}(V)$. As filtered modules it holds that

$$(D_{dR}(V), Fil) = \bigoplus_{m=0}^{r-1} (D^{(m)}_{dR,r}(V), Fil_m).$$

The tensor product $\bigotimes_{m=0}^{r-1} D^{(m)}_{dR,r}(V)$ is equipped with the filtration Fil_{ten}, which is the tensor product of Fil_m's.

Proposition 3.15. *Let V_1 be a crystalline $\mathbb{Q}_{p^r}$-representation and V a $\mathbb{Q}_p$-representation such that there is an isomorphism of $\mathbb{Q}_{p^r}$-representations:*

$$V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \cong V_1 \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^{r-1}}.$$

Then there is an isomorphism of filtered ϕ-modules:

$$D_{crys}(V) \cong \bigotimes_{m=0}^{r-1} D^{(m)}_{crys,r}(V_1),$$

where the filtered ϕ-module structure on $\bigotimes_{m=0}^{r-1} D^{(m)}_{crys,r}(V_1)$ is given by Fil_{ten} and ϕ_{ten}.

Proof. The original proof is lengthy. The current argument was suggested by the referee. It is simpler and clearer. One observes the following natural isomorphisms of filtered ϕ-modules:

$$\begin{aligned}
D_{crys}(V) &= [V \otimes_{\mathbb{Q}_p} B_{crys}]^{\mathrm{Gal}_E} \\
&\cong [V \otimes_{\mathbb{Q}_p} (\mathbb{Q}_{p^r} \otimes_{\mathbb{Q}_{p^r}} B_{crys})]^{\mathrm{Gal}_E} \\
&\cong [(V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r}) \otimes_{\mathbb{Q}_{p^r}} B_{crys}]^{\mathrm{Gal}_E} \\
&\cong [(V_1 \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^{r-1}}) \otimes_{\mathbb{Q}_{p^r}} B_{crys}]^{\mathrm{Gal}_E} \\
&\cong [\bigotimes_{m=0}^{r-1} (V_{1,\sigma^m} \otimes_{\mathbb{Q}_{p^r}} B_{crys})]^{\mathrm{Gal}_E} \\
&\cong [\bigotimes_{m=0}^{r-1} (V_1 \otimes_{\mathbb{Q}_{p^r},\sigma^m} B_{crys})]^{\mathrm{Gal}_E} \\
&= \bigotimes_{m=0}^{r-1} D^{(m)}_{crys,r}(V_1).
\end{aligned}$$

An extra explanation is possibly necessary: note first that the ϕ action on B_{crys} preserves $\mathbb{Q}_{p^r}$ and acts on it by σ. So it permutes the terms in the first tensor product in the fourth line of the above isomorphisms. This implies the resulting ϕ-structure on the tensor product in the last line as given by ϕ_{ten}. As V_1 is crystalline, the subspace $\bigotimes_{m=0}^{r-1} D^{(m)}_{crys,r}(V_1)$ in the Gal_E-invariant space has the same dimension as $D_{crys}(V)$. Therefore, the equality in the last line follows. □

In the previous proposition we consider the case where V is polarizable and of Hodge-Tate weights $\{0,1\}$. Here V being polarizable means that there is a perfect Gal_E-pairing $V \otimes V \to \mathbb{Q}_p(-1)$. This condition implies that if λ is a Newton (resp. Hodge) slope of V, then $1-\lambda$ is also a Newton (resp. Hodge) slope of V with the *same* multiplicity.

Proposition 3.16. *Let V be a polarizable crystalline representation with Hodge-Tate weights $\{0,1\}$. If there exists a two dimensional crystalline $\mathbb{Q}_{p^r}$-representation V_1 such that*

$$V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r} \cong V_1 \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma} \otimes_{\mathbb{Q}_{p^r}} \cdots \otimes_{\mathbb{Q}_{p^r}} V_{1,\sigma^{r-1}}$$

holds, then it holds that

(i) *the Hodge slopes of V_1 are $\{2r-1 \times 0, 1 \times 1\}$,*
(ii) *the Newton slopes of V_1 are either $\{2r \times \frac{1}{2r}\}$ or $\{r \times 0, r \times \frac{1}{r}\}$.*

Consequently, there are only two possible Newton slopes for V: $\{2^r \times \frac{1}{2}\}$ or $\{1 \times 0, \cdots, \binom{r}{i} \times \frac{i}{r}, \cdots, 1 \times 1\}$.

Proof. Since the Hodge slopes of V are $\{n \times 0, n \times 1\}$, by Proposition 3.15, there exists a unique factor $D^{(i)}_{crys,r}(V_1)$ with two distinct Hodge slopes $\{0,1\}$ and the other factors have all Hodge slopes zero. Without loss of generality one can assume that $D^{(0)}_{crys,r}(V_1)$ has Hodge slopes $\{1 \times 0, 1 \times 1\}$ (and any other factor has $\{2 \times 0\}$). Summing up the Hodge slopes of all factors, one obtains the Hodge polygon of $D_{crys}(V_1)$ as claimed. By the admissibility of the filtered ϕ-module structure on $D_{crys}(V_1)$, one finds that its Newton slopes must be of form $\{m_1 \times 0, m_2 \times \lambda\}$ where

$m_1 + m_2 = 2r$ holds and $\lambda \in \mathbb{Q}$ satisfies $\lambda m_2 = 1$. By Lemma 3.14, one finds that $r|m_i,\ i = 1, 2$. So $\frac{m_1}{r} + \frac{m_2}{r} = 2$ and $m_2 \neq 0$. There are only two possible cases:

Case 1: $m_1 = 0$. This implies that $m_2 = 2r$ and $\lambda = \frac{1}{2r}$.

Case 2: $m_1 \neq 0$. This implies that $m_1 = m_2 = r$ and $\lambda = \frac{1}{r}$. □

Now we can prove the following.

Theorem 3.17. *Notation as above. Then there are two possible Newton polygons in $M_0(\bar{k})$. Precisely it is either $\{2^{d+\epsilon(D)} \times \frac{1}{2}\}$ or*

$$\{2^{d-r+\epsilon(D)} \times 0, \cdots, 2^{d-r+\epsilon(D)} \cdot \binom{r}{i} \times \frac{i}{r}, \cdots, 2^{d-r+\epsilon(D)} \times 1\}.$$

Proof. Let $x_0 \in M_0(\bar{k})$, A_{x_0} and A_x as above. The question is to determine the possible Newton polygons of the filtered ϕ-module $D_{crys}(H^1_{et}(\bar{A}_{x^0}, \mathbb{Q}_p))$. For a suitable large finite extension E of $\mathbb{Q}_p$, we can assume the direct-tensor decomposition of the Gal_E-module $H^1_{et}(\bar{A}_{x^0}, \mathbb{Q}_p)$ as given in Corollary 3.7. Now Propositions 3.8 and 3.11 imply that we can assume that each tensor factor in the tensor decomposition is crystalline. An unramified factor contributes only to a multiplicity in the Newton polygon. Hence the theorem follows directly from Proposition 3.16. □

Remark 3.18. In this remark we would like to discuss the existence of each Newton polygon in the classification. To this end, it suffices to realize that the method of Noot for the original example of Mumford (see §§3-5 [21]) can be generalized directly: Noot studied the reductions of CM points of a Mumford family. The set of CM points can be divided into two types: let $F \subset J$ be a maximal subfield of D. Then J can either be written in a form $N \otimes_{\mathbb{Q}} F$ (N is necessarily an imaginary quadratic extension of $\mathbb{Q}$) or not in such a form. To our purpose one finds that the second case generalizes, and the resulting generalization gives the necessary existence result. More precisely, Proposition 5.2 [21] provides the maximal subfields in D of the second type with the following freedom: Let $\mathfrak{p}$ be a prime of F over p. Then J can be chosen so that $\mathfrak{p}$ is split or inert in J. Secondly, Lemma 3.5 and Proposition 3.7 [21] work verbatim for a general D except that in the case $\epsilon(D) = 1$ one adds the multiplicity two to the constructions appearing therein. This step gives us an isogeny class of CM abelian varieties which appear as $\bar{\mathbb{Q}}$-points of M_K, and also as $\bar{\mathbb{Z}}_p$-points of M and hence in $M_0(\bar{k})$. Finally the proof of Proposition 4.4 [21], namely the method of computing the Newton polygon for a CM abelian variety modulo p, works in general. Thus one can also conclude the existence result for the general case.

4. A direct-tensor decomposition of the universal filtered Dieudonné module over a formal neighborhood

Let x_0 be a k-rational point of M and $\hat{M}_{x_0}$ the completion of M at x_0. The aim of this section is to show a direct-tensor decomposition of the restriction of (H, F, ∇, ϕ) to the formal neighborhood of x_0. Let E be a finite extension of L and x^0 an E-rational point of M which specializes into x_0. Corollary 3.7 gives a decomposition of $H_{\mathbb{Z}_p}$ into direct-tensor product of Gal_E-lattices. By Propositions 3.11 and 3.16, $V_1 \subset V_1 \otimes \mathbb{Q}_p$ is a Gal_E-lattice of a two dimensional potentially crystalline $\mathbb{Q}_{p^r}$-representation with Hodge-Tate weights $\{2r-1 \times 0, 1 \times 1\}$. By making a possible finite extension of E, we can assume that $V_1 \otimes \mathbb{Q}_p$ is already crystalline as a Gal_E-module.

4.1. **Drinfel'd's $\mathcal{O}_\mathfrak{p}$-divisible module and versal deformation.** Recall the following notion of $\mathcal{O}_\mathfrak{p}$-divisible modules due to Drinfel'd (see Appendix [5]):

Definition 4.1. Let $S = \mathrm{Spec}R$ be an $\mathcal{O}_\mathfrak{p}$-scheme. An $\mathcal{O}_\mathfrak{p}$-divisible module over S is a pair (G, f) consisting of a p-divisible group G over S and an action of $\mathcal{O}_\mathfrak{p}$ on G:

$$f : \mathcal{O}_\mathfrak{p} \to \mathrm{End}(G)$$

satisfying

(a) $f(1)$ is the identity,
(b) G^0 (the connected part of G) is of dimension 1,
(c) the derivation of f, $f' : \mathcal{O}_\mathfrak{p} \to \mathrm{End}(Lie(G)) = R$ coincides with the structural morphism $\mathcal{O}_\mathfrak{p} \to R$.

By the fundamental theorem of C. Breuil (see Corollary 3.2.4, Theorem 3.2.5 [4]), the Gal_E-lattice V_1 corresponds to a p-divisible group B over $\mathcal{O}_E$.

Lemma 4.2. *The corresponding p-divisible group B is an $\mathcal{O}_\mathfrak{p}$-divisible module over $\mathcal{O}_E$ of height* 2.

Proof. By construction, one has the inclusion $\mathbb{Z}_{p^r} \subset \mathrm{End}_{\mathrm{Gal}_E}(V_1)$. Since the functor of Breuil is an anti-equivalence of categories, one obtains an inclusion as well, $\mathcal{O}_\mathfrak{p} \cong \mathbb{Z}_{p^r} \subset \mathrm{End}(B)$. The condition (a) is obvious. The condition (b) on the dimension of G and the assertion on the height of B follow from the Hodge-Tate weights of $V_1 \otimes \mathbb{Q}_p$ given in Proposition 3.16. By taking the derivation of the inclusion, one obtains an inclusion of $\mathbb{Z}_p$-algebras $\mathcal{O}_\mathfrak{p} \subset \mathcal{O}_E$, which ought to be the structural morphism by the naturalness of the functor. $\square$

Let M_B be the filtered Dieudonné module associated with B. Denote by L_B the previous Gal_E-lattice V_1. Fix a generator s of $\mathbb{Z}_{p^r}$ as a $\mathbb{Z}_p$-algebra. The image of s in $\mathrm{End}_{\mathrm{Gal}_E}(L_B) \subset \mathrm{End}_{\mathbb{Z}_p}(L_B) \subset L_B^{\otimes}$ is an étale Tate cycle of L_B and is denoted by $s_{B,et}$. By the p-adic comparison theorem, one has a natural isomorphism respecting Gal_E-action, filtrations and ϕ's:

$$L_B \otimes_{\mathbb{Z}_p} B_{crys} \cong M_B \otimes_{W(k)} B_{crys}.$$

By one of the main technical results of [15] (see Proposition 0.2 [15]), the crystalline Tate cycle s_B, which corresponds to $s_{B,et} \otimes 1 \in L_B \otimes_{\mathbb{Z}_p} B_{crys}$ in the above comparison, lies actually in $M_B^{\otimes}$. Let $G_B \subset \mathrm{GL}_{\mathbb{Z}_p}(L_B)$ (resp. $\mathcal{G}_B \subset \mathrm{GL}_{W(k)}(M_B)$) be the subgroup defined by $s_{B,et}$ (resp. s_B). By Corollary 1.4.3 (4) [15], the filtration $Fil^1 \otimes k$ on $M_B \otimes k$ is $\mathcal{G}_B \otimes_{W(k)} k$-split. Choose a cocharacter $\mu_0 : \mathbb{G}_m \to \mathcal{G}_B \otimes k$ inducing this filtration and further choose a cocharacter $\mu : \mathbb{G}_m \to \mathcal{G}_B \subset \mathrm{GL}(M_B)$ lifting μ_0 (see 1.5.4 [15]). The cocharacter μ defines the opposite unipotent subgroups:

$$\begin{tikzcd} U_{\mathcal{G}_B} \arrow[r, "\subset"] \arrow[d, "\cap"'] & \mathcal{G}_B \arrow[d, "\cap"] \\ U_B \arrow[r, "\subset"] & \mathrm{GL}(M_B) \end{tikzcd}$$

By construction, one has $U_{\mathcal{G}_B} = U_B \cap \mathcal{G}_B$. Let $\hat{U}_{\mathcal{G}_B}$ (resp. $\hat{U}_B$) be the completion of $U_{\mathcal{G}_B}$ at the identity section of $U_{\mathcal{G}_B}$ (resp. U_B), whose corresponding complete local rings are denoted by $R_{\mathcal{G}_B}$ and R respectively. The filtration on M_B defined by μ corresponds to a p-divisible group B' over $W(k)$ whose closed fiber $B' \otimes k$ is isomorphic to $B \otimes k$ as a p-divisible group over k. For later use we denote this μ also by $\mu_{B'}$. Write $(M'_B, Fil^1_{M'_B}, \phi_{M'_B})$ for the tuple defined by the filtered crystal structure on M'_B, and fix $\phi_{\hat{U}_{\mathcal{G}_B}} : R_{\mathcal{G}_B} \to R_{\mathcal{G}_B}$ as a lifting of the absolute Frobenius. Following Faltings Remarks, §7 [11] (see also §1.5 [15]), one defines a filtered F-crystal over $\hat{U}_{\mathcal{G}_B}$ by the tuple

$$(\mathcal{N}_B = M'_B \otimes_{W(k)} R_{\mathcal{G}_B}, Fil^1_{\mathcal{N}_B} = Fil^1_{M'_B} \otimes_{W(k)} R_{\mathcal{G}_B}, \phi_{\mathcal{N}_B} = u \circ (\phi_{M'_B} \otimes \phi_{\hat{U}_{\mathcal{G}_B}})),$$

where $u \in U_{\mathcal{G}_B}(R_{\mathcal{G}_B})$ is the tautological $R_{\mathcal{G}_B}$-point of $U_{\mathcal{G}_B}$. Then by Faltings [11], there is a unique integrable connection $\nabla_{\mathcal{N}_B}$ over $\mathcal{N}_B$ such that the quadruple $(\mathcal{N}_B, Fil^1_{\mathcal{N}_B}, \phi_{\mathcal{N}_B}, \nabla_{\mathcal{N}_B})$ defines an object in $\mathcal{MF}^{\nabla}_{[0,1]}(\hat{U}_{\mathcal{G}_B})$. By Faltings Theorem 7.1 [10], there is a p-divisible group $\mathcal{B}$ over $R_{\mathcal{G}_B}$, unique up to isomorphism, such that the attached filtered Dieudonné module to $\mathcal{B}$ is isomorphic to the above quadruple. If we replace everything of $U_{\mathcal{G}_B}$ with that of U_B, the above discussion gives then a versal deformation of $B \otimes k$ over $\hat{U}_B$, which by abuse of notation is denoted again by $\mathcal{B}$. In this context, the sublocus $\mathrm{Spf}(R_{\mathcal{G}_B}) \hookrightarrow \mathrm{Spf}(R)$ has an interpretation as the versal deformation respecting the Tate cycles which are stabilized by $\mathcal{G}_B$ (see §7 [11] and Corollary 1.5.5 [15]). By Corollary 1.5.11 [15], B is isomorphic to the pull-back of $\mathcal{B}$ along a $W(k)$-algebra morphism $R_{\mathcal{G}_B} \to \mathcal{O}_E$.

We proceed to study the natural $\mathcal{G}_B$-action on M_B via the inclusion $\mathcal{G}_B \subset \mathrm{GL}_{W(k)}(M_B)$. Recall that we have fixed an element $s \in \mathbb{Z}_{p^r}$. Let $\{s_i := s^{\sigma^i}\}_{0\leq i\leq r-1} \subset \mathbb{Z}_{p^r}$ be the Galois conjugates of s. The minimal polynomial of $s_B \in \mathrm{End}_{W(k)}(M_B)$ is that of $s \in \mathbb{Q}_{p^r}$ over $\mathbb{Q}_p$. As $\mathbb{Z}_{p^r} \subset W(k)$, the minimal polynomial of s_B splits into linear factors and one has then the relation in $\mathrm{End}_{W(k)}(M_B)$:

$$(s_B - s_0) \cdots (s_B - s_{r-1}) = 0.$$

Let $M_i \subset M_B$ be the eigenspace of s_B corresponding to the eigenvalue s_i. Recall that we have shown in §3 a direct sum decomposition of $M_B \otimes \mathrm{Frac}(W(k)) = D_{crys}(L_B \otimes \mathbb{Q}_p)$ by using the functors $D^{(i)}_{crys,r}$ of Fontaine. Now we have the following.

Lemma 4.3. *The eigen-decomposition $M_B = \bigoplus_{i=0}^{r-1} M_i$ is a lattice decomposition of*

$$M_B \otimes \mathrm{Frac}(W(k)) = \bigoplus_{i=0}^{r-1} D^{(i)}_{crys,r}(L_B \otimes \mathbb{Q}_p).$$

Namely, M_i is a lattice in $D^{(i)}_{crys,r}(L_B \otimes \mathbb{Q}_p)$ for each i.

Proof. In the comparison isomorphism $L_B \otimes_{\mathbb{Z}_p} B_{crys} \cong M_B \otimes_{W(k)} B_{crys}$, the endomorphism $s_{B,et} \otimes 1$ corresponds to $s_B \otimes 1$. Also the endomorphisms commute with the Gal_E-actions on both sides. The endomorphism $s_{B,et} \otimes 1$ on $L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$ decomposes into eigenspaces, and so we obtain a $\mathbb{Z}_{p^r}[\mathrm{Gal}_E]$-module decomposition:

$$L_B \otimes_{\mathbb{Z}_p} B_{crys} = (L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}) \otimes_{\mathbb{Z}_{p^r}} B_{crys} = \bigoplus_{i=0}^{r-1} L_i \otimes_{\mathbb{Z}_{p^r}} B_{crys},$$

where L_i is the $\mathbb{Z}_{p^r}$-submodule of $L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$ corresponding to the eigenvalue s_i. Under the comparison isomorphism it corresponds to the decomposition of a $W(k)[\mathrm{Gal}_E]$-module:

$$M_B \otimes_{W(k)} B_{crys} = \bigoplus_{i=0}^{r-1} M_i \otimes_{W(k)} B_{crys}.$$

Taking the Gal_E-invariants, we obtain

$$(L_i \otimes_{\mathbb{Z}_{p^r}} B_{crys})^{\mathrm{Gal}_E} = M_i \otimes_{W(k)} \mathrm{Frac}(W(k)).$$

Finally we notice that the two indexed sets of Gal_E-modules $\{L_i \otimes_{\mathbb{Z}_{p^r}} B_{crys}\}_{0\le i\le r-1}$ and $\{L_B[\frac{1}{p}] \otimes_{\mathbb{Q}_{p^r},\sigma^i} B_{crys}\}_{0\le i\le r-1}$ are actually equal. The lemma follows. □

Proposition 4.4. *The tensor product $\bigotimes_{i=0}^{r-1} M_i$ is a lattice of the admissible filtered ϕ-module $\bigotimes_{i=0}^{r-1} D^{(i)}_{crys,r}(L_B \otimes \mathbb{Q}_p)$ in Proposition* 3.15*.*

Proof. The filtration Fil^1_B on $M_B \otimes_{W(k)} \mathcal{O}_E$ is filtered free and restricts to the filtration Fil^1_i on each direct factor $M_i \otimes \mathcal{O}_E$. Also one sees from the proof of Proposition 3.16 that there is a unique factor M_i with nontrivial Fil^1_i. Since $\phi_{M'_B}$ is σ-linear, it permutes the eigen-factors $\{M_i\}_{0\le i\le r-1}$ cyclically. The proposition is now clear. □

For later use, we denote the above lattice by $(\bigotimes_{i=0}^{r-1} M_i, Fil^1_{ten}, \phi_{ten})$.

Lemma 4.5. *The eigen-decomposition of M_B is also a decomposition as a $\mathcal{G}_B$-module. In fact, the $W(k)$-group $\mathcal{G}_B$ is naturally isomorphic to $\prod_{i=0}^{r-1} \mathrm{GL}_2(W(k))$, and the $\mathcal{G}_B$-module M_B is isomorphic to the $\prod_{i=0}^{r-1} \mathrm{GL}_2(W(k))$-module $\bigoplus_{i=0}^{r-1}(W(k)^{\oplus 2})_i$, in which the i-th factor $(W(k)^{\oplus 2})_i$ is the tensor product of the standard representation of the i-th factor $\mathrm{GL}_2(W(k))$ and the trivial representations of the j-th factors with $j \neq i$.*

Proof. Because the $\mathcal{G}_B$-action on M_B commutes with the s_B-action by definition, the eigen-decomposition of M_B with respect to s_B is preserved by the $\mathcal{G}_B$-action. This can be seen more clearly if we go to the étale side: first of all, it is easy to see that the commutant subalgebra of $\mathbb{Z}_{p^r} \subset \mathrm{End}_{\mathbb{Z}_p}(L_B) \cong M_{2r}(\mathbb{Z}_p)$ is $\mathrm{End}_{\mathbb{Z}_{p^r}}(L_B) \cong M_2(\mathbb{Z}_{p^r})$. So the group $G_B \subset \mathrm{GL}_{\mathbb{Z}_p}(L_B) \cong \mathrm{GL}_{2r}(\mathbb{Z}_p)$ is isomorphic to $\mathrm{GL}_2(\mathbb{Z}_{p^r})$, and particularly it is connected. Next, by Corollary 1.4.3 (3) [15], there is a $W(k)$-linear isomorphism $L_B \otimes_{\mathbb{Z}_p} W(k) \cong M_B$ which induces an isomorphism $G_B \times_{\mathbb{Z}_p} W(k) \cong \mathcal{G}_B$. As $\mathbb{Z}_{p^r} \subset W(k)$, one has isomorphisms

$$\mathcal{G}_B \cong \mathrm{GL}_2(\mathbb{Z}_{p^r} \otimes_{\mathbb{Z}_p} W(k)) \cong \mathrm{GL}_2(\prod_{i=0}^{r-1} W(k)) = \prod_{i=0}^{r-1} \mathrm{GL}_2(W(k)).$$

Under the above isomorphism, the $\mathcal{G}_B$-module M_B is isomorphic to the G_B-module L_B tensorizing with $W(k)$. Moreover the isomorphism preserves the eigen-decompositions of both sides. As said above, the G_B-action on L_B is isomorphic to the standard representation of $\mathrm{GL}_2(\mathbb{Z}_{p^r})$ on $\mathbb{Z}_{p^r}^{\oplus 2}$, which is considered as a $\mathbb{Z}_p$-group acting on a $\mathbb{Z}_p$-module by restriction of scalar. Thus, the action after tensorizing with $\mathbb{Z}_{p^r}$, that is, the standard $\mathrm{GL}_2(\mathbb{Z}_{p^r} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})$-action on $(\mathbb{Z}_{p^r} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^{\oplus 2}$, splits; write

$$\mathrm{GL}_2(\mathbb{Z}_{p^r} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}) \cong \prod_{i=0}^{r-1} \mathrm{GL}_2(\mathbb{Z}_{p^r}),\ g \mapsto (g_0, \cdots, g_{r-1})$$

and

$$(\mathbb{Z}_{p^r}\otimes_{\mathbb{Z}_p}\mathbb{Z}_{p^r})^{\oplus 2}\cong\prod_{i=0}^{r-1}\mathbb{Z}_{p^r}^{\oplus 2},\ v\mapsto(v_0,\cdots,v_{r-1}).$$

Then $g(v)$ is mapped to $(g_0v_0,\cdots,g_{r-1}v_{r-1})$, and so also the action after tensorizing with the larger ring $W(k)$. Hence the lemma follows. $\square$

Proposition 4.6. *The sublocus $\hat{U}_{\mathcal{G}_B}$ is a versal deformation of the p-divisible group $B\otimes k$ as an $\mathcal{O}_{\mathfrak{p}}$-divisible module.*

Proof. We calculate first the dimension of $\hat{U}_{\mathcal{G}_B}$. It is equal to $\mathrm{rank}_{W(k)}\frac{\mathfrak{g}}{Fil^0\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}_B$ and the filtration is the restriction of the tensor filtration on $\mathrm{End}_{W(k)}(M_B)=M_B^{\vee}\otimes M_B$ via the inclusion $\mathfrak{g}\subset\mathrm{End}_{W(k)}(M_B)$. We claim that it is one dimensional. By the discussion on the filtration in the proof of Proposition 4.4 and Lemma 4.5, one has an isomorphism of Lie algebras over $W(k)$ $\mathfrak{g}\cong\bigoplus_{i=0}^{r-1}\mathfrak{gl}_2$, such that there is a unique factor with the nontrivial induced filtration through the isomorphism. This shows the claim. Now let $\tilde{B}\to Z$ be a versal deformation of $B\otimes k$ as $\mathcal{O}_{\mathfrak{p}}$-divisible modules. Thus one has a map $\tilde{f}:\mathcal{O}_{\mathfrak{p}}\to\mathrm{End}(\tilde{B})$ which makes the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathfrak{p}} \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & \mathrm{End}(\tilde{B}) \arrow[d, "\otimes k"] \\
 & \mathrm{End}(B)
\end{tikzcd}$$

Let $s_{cycle}\in\mathrm{End}(B)$ and $\tilde{s}_{cycle}\in\mathrm{End}(\tilde{B})$ be the images of $s\in\mathcal{O}_{\mathfrak{p}}$ in the endomorphism $\mathbb{Z}_p$-algebras. The element s_{cycle} corresponds to s_B under the Dieudonné functor, and by Faltings Theorem 7.1 [10], the corresponding element $\tilde{s}_B$ to $\tilde{s}_{cycle}$, as an endomorphism of the filtered Dieudonné crystal attached to $\tilde{B}$, is a crystalline Tate cycle and is the parallel continuation of s_B over Z. By the universal property Proposition 4.9 [18], the inclusion $Z\subset\hat{U}_B$ factors $Z\subset\hat{U}_{\mathcal{G}_B}\subset\hat{U}_B$. As both Z and $\hat{U}_{\mathcal{G}_B}$ are formally smooth of dimension one, it follows that $Z=\hat{U}_{\mathcal{G}_B}$. $\square$

Let $(\mathcal{N}_B,Fil^1_{\mathcal{N}_B},\phi_{\mathcal{N}_B},\nabla_{\mathcal{N}_B})$ be the universal filtered Dieudonné module attached to $\mathcal{B}$ over $\hat{U}_{\mathcal{G}_B}$. Put $\mathcal{N}_i=M_i\otimes R_{\mathcal{G}_B},0\leq i\leq r-1$. Then the Tate cycle $s_B\in\mathrm{End}_{R_{\mathcal{G}_B}}(\mathcal{N}_B)$ induces the eigen-decomposition

$$(\mathcal{N}_B,Fil^1_{\mathcal{N}_B},\nabla_{\mathcal{N}_B})=\bigoplus_{i=0}^{r-1}(\mathcal{N}_i,Fil^1_{\mathcal{N}_i},\nabla_{\mathcal{N}_i}),$$

where $Fil^1_{\mathcal{N}_i}$ (resp. $\nabla_{\mathcal{N}_i}$) is the restriction of $Fil^1_{\mathcal{N}_B}$ (resp. $\nabla_{\mathcal{N}_B}$) to $\mathcal{N}_i$. However the eigen-decomposition is not preserved by $\phi_{\mathcal{N}_B}$: recall that $\phi_{\mathcal{N}_B}=u\circ(\phi_{M'_B}\otimes\phi_{\hat{U}_{\mathcal{G}_B}})$. As $U_{\mathcal{G}_B}\subset\mathcal{G}_B$, u preserves the eigen-decomposition by Lemma 4.5. So $\phi_{\mathcal{N}_B}$ permutes the factors in the eigen-decomposition in a cyclic way. In order to state the following decomposition result, we need to introduce the category $\mathcal{MF}^{\nabla}_{big,r}$, which is analogous to the category $\mathcal{MF}^{\nabla}_{big}$ introduced by Faltings (see c)-d), Ch. II [10]). The category $\mathcal{MF}^{\nabla}_{big,r}(R_{\mathcal{G}_B})$ consists of four tuples (N,Fil,ϕ_r,∇), where N is a free $R_{\mathcal{G}_B}$-module, Fil is a sequence of $R_{\mathcal{G}_B}$-submodules with $Gr_{Fil}(N)$ torsion free,

$$\phi_r:N\otimes_{R_{\mathcal{G}_B},\phi^r_{\hat{U}_{\mathcal{G}_B}}}R_{\mathcal{G}_B}\to N$$

satisfies the divisibility condition $\phi_r(Fil^i) \subset p^i N$, ∇ is an integrable connection satisfying the Griffiths transversality, and finally ϕ_r is parallel with respect to ∇. Summarizing the above discussions, we have shown that

Proposition 4.7. *The object $(\mathcal{N}_B, Fil^1_{\mathcal{N}_B}, \nabla_{\mathcal{N}_B})$ has a decomposition*

$$(\mathcal{N}_B, Fil^1_{\mathcal{N}_B}, \nabla_{\mathcal{N}_B}) = \bigoplus_{i=0}^{r-1} (\mathcal{N}_i, Fil^1_{\mathcal{N}_i}, \nabla_{\mathcal{N}_i}),$$

such that $\phi_{\mathcal{N}_B}$ permutes the factors cyclically. Consequently, one has a direct sum decomposition in the category $\mathcal{MF}^{\nabla}_{big,r}(\hat{U}_{\mathcal{G}_B})$:

$$(\mathcal{N}_B, Fil^1_{\mathcal{N}_B}, \phi^r_{\mathcal{N}_B}, \nabla_{\mathcal{N}_B}) = \bigoplus_{i=0}^{r-1} (\mathcal{N}_i, Fil^1_{\mathcal{N}_i}, \phi_{\mathcal{N}_i}, \nabla_{\mathcal{N}_i}),$$

where $\phi_{\mathcal{N}_i}$ is the restriction of $\phi_{\mathcal{N}_B}$ to $\mathcal{N}_i$.

As a consequence, one can define an object in $\mathcal{MF}^{\nabla}_{[0,1]}(\hat{U}_{\mathcal{G}_B})$ by equipping the tensor product $\bigotimes_{i=0}^{r-1}(\mathcal{N}_B, Fil^1_{\mathcal{N}_B}, \nabla_{\mathcal{N}_B})$ with the Frobenius ϕ_{ten}, a construction mimicing Proposition 4.4.

4.2. **Tensor decomposition of the universal filtered Dieudonné module over a formal neighborhood.** Notation is as in the introductory part of this section. Let A be the abelian scheme over $\mathcal{O}_E$ with the closed fiber (resp. generic fiber) A_0 (resp. A^0) given by x_0 (resp x^0). Put $L_A = H_{\mathbb{Z}_p} = H^1_{et}(\bar{A}^0, \mathbb{Z}_p)$. For simplicity of notation, we use the same letters A, etc., to mean the associated p-divisible groups. Recall that Corollary 3.7 gives a Gal_E-lattice decomposition $L_A = (V_{\mathbb{Z}_p} \otimes U_{\mathbb{Z}_p})^{\oplus 2^{\epsilon(D)}}$. Let A_1 and A_2 be the two p-divisible groups over $\mathcal{O}_E$ corresponding to the lattice $V_{\mathbb{Z}_p}$ and $U_{\mathbb{Z}_p}$ respectively by the theorem of Breuil (see [4]). Write $L_{A_1} = V_{\mathbb{Z}_p}$ and $L_{A_2} = U_{\mathbb{Z}_p}$. Let (M_A, Fil^1_A, ϕ_A) be the filtered Dieudonné module attached to A. We use similar notation for A_1 and A_2.

Proposition 4.8. *One has a natural isomorphism of filtered ϕ-modules:*

$$(M_A, Fil^1_A, \phi_A) \cong [(M_{A_1}, Fil^1_{A_1}, \phi_{A_1}) \otimes (M_{A_2}, Fil^1_{A_2}, \phi_{A_2})]^{\oplus 2^{\epsilon(D)}},$$

where the factor $(M_{A_1}, Fil^1_{A_1}, \phi_{A_1})$ is naturally isomorphic to $(\bigotimes_{i=0}^{r-1} M_i, Fil^1_{ten}, \phi_{ten})$ in Proposition 4.4 and the factor $(M_{A_2}, Fil^1_{A_2}, \phi_{A_2})$ is a unit crystal.

Proof. We have shown the above isomorphisms after inverting p of both sides: the first isomorphism is a consequence of Propositions 3.8 and 3.11 as the functor D_{crys} commutes with tensor product. The second isomorphism is Proposition 3.15. Also since $U_{\mathbb{Q}_p}$ is an unramified Gal_E-representation, (M_{A_2}, ϕ_{A_2}) is a unit crystal and the filtration $Fil^1_{A_2}$ is trivial. To show that the isomorphisms hold without inverting p, we shall apply the theory of $\mathfrak{S}$-modules of Kisin developed in [14] and in §1.2, §1.4 [15]. Consider the first isomorphism: apply first the functor $\mathfrak{M}$ to the Gal_E-lattice decomposition $L_A = (L_{A_1} \oplus L_{A_2})^{\oplus 2^{\epsilon(D)}}$. From the proof of Theorem 1.2.2 [15] (see 1.2.2 [15]), one sees that the functor $\mathfrak{M}$ respects the tensor product. So after this step one obtains a corresponding decomposition of $\mathfrak{S}$-modules. To get the decomposition of the filtered Dieudonné modules as claimed in the first isomorphism, one applies next Theorem 1.4.2 and Corollary 1.4.3 (i) [15] to each factor in the previous decomposition of $\mathfrak{S}$-modules. Consider then the second isomorphism: by Corollary 3.7, one has a tensor decomposition of $\mathbb{Z}_{p^r}[\mathrm{Gal}_E]$-modules: $L_{A_1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = \bigotimes_{i=0}^{r-1} L_{B,\sigma^i}$, where $L_{B,\sigma^i} = L_B \otimes_{\mathbb{Z}_{p^r},\sigma^i} \mathbb{Z}_{p^r}$, which is also equal to L_i in the eigen-decomposition of $L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$ with respect to $s_{B,et}$ in the proof of Lemma 4.3. Taking the r-th tensor power of $L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = \bigoplus_{i=0}^{r-1} L_i$, and then the $\mathrm{Gal}(\mathbb{Z}_{p^r}|\mathbb{Z}_p)$-invariants of both sides, one gets a natural direct decomposition $L_B^{\otimes r} = L_{A_1} \oplus L'_{A_1}$ of $\mathbb{Z}_p[\mathrm{Gal}_E]$-modules. Thus L_{A_1} is naturally isomorphic to a Gal_E-sublattice of $L_B^{\otimes r}$. Proposition 3.15 shows that this sublattice is in fact of Hodge-Tate weights $\{0, 1\}$ with the induced filtered ϕ-module structure given by Proposition 4.4. This implies the second isomorphism. □

Put $G_A = G_{\mathbb{Z}_p}$ and $\mathcal{G}_A = G_A \times_{\mathbb{Z}_p} W(k) \subset \mathrm{GL}_{W(k)}(M_A)$, the subgroup defined by the corresponding crystalline Tate cycles. Recall that after a conjugation by an element in $G(\mathbb{Q}_p)$, there is a central isogeny $\mathbb{Z}_p^* \times \prod_{i=1}^n \mathrm{SL}_2(\mathcal{O}_{F_{\mathfrak{p}_i}}) \to G_A$. Let G_{A_1} (resp. G_{A_2}) be the image of $\mathbb{Z}_p^* \times \mathrm{SL}_2(\mathcal{O}_{F_{\mathfrak{p}_1}})$ (resp. $\prod_{i=2}^n \mathrm{SL}_2(\mathcal{O}_{F_{\mathfrak{p}_i}})$) in $\mathrm{GL}(L_{A_1})$ (resp. $\mathrm{SL}(L_{A_2})$). By the construction of the tensor decomposition, one has the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Gal}_E \arrow[r] \arrow[dr] & G_A \arrow[r, "\twoheadrightarrow"] \arrow[d] & G_{A_1} \times G_{A_2} \arrow[dl, "\hookleftarrow"'] \arrow[d] \\
& \mathrm{GL}(L_{A_1}) \times \mathrm{SL}(L_{A_2}) \arrow[r, "\otimes"] & \mathrm{GL}(L_{A_1} \otimes L_{A_2})
\end{tikzcd}$$

Consider the group homomorphism

$$\otimes^r : \mathrm{GL}(L_B) \to \mathrm{GL}(L_B^{\otimes r}), g \mapsto (g^{\otimes r} : v_1 \otimes \cdots \otimes v_r \mapsto g(v_1) \otimes \cdots \otimes g(v_r)).$$

It is a central isogeny over the image.

Lemma 4.9. *The restriction of* $\otimes^r$ *to the subgroup* G_B *factors*

$$\otimes^r|_{G_B} : G_B \to \mathrm{GL}(L_{A_1}) \times \mathrm{GL}(L'_{A_1}) \subset \mathrm{GL}(L_B^{\otimes r}).$$

Proof. Recall that for a $g \in \mathrm{GL}(L_B)$, $g \in G_B$ iff $g(s_B) = s_B$ up to a scalar. This implies that $g \otimes 1$ preserves the eigen-decomposition of $L_B \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$. So $\otimes^r(g \otimes 1)$ respects the direct sum decomposition

$$L_B^{\otimes r} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = L_{A_1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} \oplus L'_{A_1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}.$$

Thus $\otimes^r(g)$ preserves the decomposition $L_B^{\otimes r} = L_{A_1} \oplus L'_{A_1}$. Hence the lemma follows. □

Let $\xi_{et} : G_B \to \mathrm{GL}(L_{A_1})$ be the composite of $\otimes^r|_{G_B}$ with the projection to the first factor in the above lemma. The reductive subgroup $G_{A_1} \subset \mathrm{GL}(L_{A_1})$ is defined by a finite set of tensors in $L_{A_1}^{\otimes}$.

Lemma 4.10. *A tensor in* $L_{A_1}^{\otimes n}$ *is fixed by* G_{A_1} *only if* $n = 2a$ *is even, and it must be of form* $\det(L_{A_1})^{\otimes a} \subset L_{A_1}^{\otimes n}$.

Proof. Assume n is positive. The G_{A_1}-action respects the tensor decomposition $L_{A_1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = \bigotimes_{i=0}^{r-1} L_i$. A tensor in $L_{A_1}^{\otimes n}$ fixed by G_{A_1} is by definition a rank one $\mathbb{Z}_p$-subrepresentation of G_{A_1}. So it gives rise to a rank one $\mathbb{Z}_{p^r}$-subrepresentation of G_{A_1} in $[\bigotimes_{i=0}^{r-1} L_i]^{\otimes n}$. Recall that the G_{A_1}-action on L_i is isomorphic to the i-th σ-conjugate of the standard action of $\mathrm{GL}_2(\mathbb{Z}_{p^r})$ on $\mathbb{Z}_{p^r}^{\oplus 2}$. Then we study the $\mathrm{GL}_2(\mathbb{Q}_{p^r})$-invariant lines in $[\bigotimes_{i=0}^{r-1} (\mathbb{Q}_{p^r}^{\oplus 2})_i]^{\otimes n}$, where $(\mathbb{Q}_{p^r}^{\oplus 2})_i := \mathbb{Q}_{p^r}^{\oplus 2} \otimes_{\mathbb{Q}_{p^r},\sigma^i} \mathbb{Q}_{p^r}$. For that we apply the standard finite dimensional representation theory of complex Lie groups (see [12] Ch. 6). For a partition λ of n, one has the irreducible decomposition of $\prod_{i=0}^{r-1}\mathrm{GL}_2(\mathbb{Q}_{p^r})$-modules:

$$\mathbb{S}_\lambda[\bigotimes_{i=0}^{r-1}(\mathbb{Q}_{p^r}^{\oplus 2})_i]=\bigoplus_{\lambda_0,\cdots,\lambda_{r-1}}C_{\lambda_0\cdots\lambda_{r-1}\lambda}\cdot\mathbb{S}_{\lambda_0}(\mathbb{Q}_{p^r}^{\oplus 2})_0\otimes\cdots\otimes\mathbb{S}_{\lambda_{r-1}}(\mathbb{Q}_{p^r}^{\oplus 2})_{r-1},$$

where λ_i in the summation runs through all possible partitions of n. As $\dim(\mathbb{Q}_{p^r}^{\oplus 2})_i = 2$, the only possible λ_is are of the form $\{n-a,a\}$ for $a\le\frac{n}{2}$, and

$$\mathbb{S}_{\{n-a,a\}}(\mathbb{Q}_{p^r}^{\oplus 2})_i=\begin{cases}\mathbb{S}_{\{n-2a\}}(\mathbb{Q}_{p^r}^{\oplus 2})_i=\mathrm{Sym}^{n-2a}(\mathbb{Q}_{p^r}^{\oplus 2})_i\otimes[\det(\mathbb{Q}_{p^r}^{\oplus 2})_i]^a & \text{if } 2a<n;\\ \mathbb{S}_{\{a,a\}}(\mathbb{Q}_{p^r}^{\oplus 2})_i=[\det(\mathbb{Q}_{p^r}^{\oplus 2})_i]^a & \text{if } 2a=n.\end{cases}$$

Since $\mathrm{GL}_2(\mathbb{Q}_{p^r})$ is embedded into $\prod_{i=0}^{r-1}\mathrm{GL}_2(\mathbb{Q}_{p^r})$ via $g\mapsto(g,\sigma g,\cdots,\sigma^{r-1}g)$, the above decomposition is also irreducible with respect to the $\mathrm{GL}_2(\mathbb{Q}_{p^r})$-action. Summarizing these discussions, we conclude that there exists a G_{A_1}-invariant tensor s_α in $L_{A_1}^{\otimes n}$ only if $n=2a$ is even and $s_\alpha\otimes 1\in[\bigotimes_{i=0}^{r-1}L_i]^{\otimes n}$ is of the form $\bigotimes_{i=0}^{r-1}[\det(L_i)]^a$, which implies that $s_\alpha\in\det(L_{A_1})^{\otimes a}$. □

Proposition 4.11. *The morphism ξ_{et} factors*

$$\xi_{et}:G_B\to G_{A_1}\subset\mathrm{GL}(L_{A_1}),$$

and the induced morphism $\xi_{et}:G_B\to G_{A_1}$ is a central isogeny.

Proof. Fix an even natural number n. Let $s_\alpha\in L_{A_1}^{\otimes n}$ be a tensor for G_{A_1}. It is to show that the image of G_B under ξ_{et} fixes s_α. By Lemma 4.10, $s_\alpha\otimes 1=\bigotimes_{i=0}^{r-1}[\det(L_i)]^{\frac{n}{2}}$. It is clear that for a $g\in G_B$, $\otimes^r(g)\otimes 1$ stabilizes the line $\bigotimes_{i=0}^{r-1}[\det(L_i)]^{\frac{n}{2}}$. This implies that $\otimes^r(g)$ stabilizes s_α. As ξ_{et} is a central isogeny over its image and both G_B and G_{A_1} are isomorphic to $\mathrm{GL}_2(\mathbb{Z}_{p^r})$, ξ_{et} induces a central isogeny from G_B to G_{A_1}. □

So we have the central isogenies $G_B\times G_{A_2}\twoheadrightarrow G_{A_1}\times G_{A_2}\twoheadleftarrow G_A$ of groups over $\mathbb{Z}_p$. Put $\mathcal{G}_{A_i}=G_{A_i}\times_{\mathbb{Z}_p}W(k)$. Taking the base change to $W(k)$ one obtains central isogenies of groups over $W(k)$:

$$\mathcal{G}_B\times\mathcal{G}_{A_2}\overset{\xi_1}{\twoheadrightarrow}\mathcal{G}_{A_1}\times\mathcal{G}_{A_2}\overset{\xi_2}{\twoheadleftarrow}\mathcal{G}_A.$$

For a certain natural number l, the cocharacter $\mathbb{G}_m\to\mathcal{G}_{A_1}\times\mathcal{G}_{A_2}$, which is the composite

$$\mathbb{G}_m\xrightarrow{x\mapsto x^l}\mathbb{G}_m\xrightarrow{\mu_{B'}\times\mathrm{id}}\mathcal{G}_B\times\mathcal{G}_{A_2}\xrightarrow{\xi_1}\mathcal{G}_{A_1}\times\mathcal{G}_{A_2},$$

lifts to a cocharacter $\nu:\mathbb{G}_m\to\mathcal{G}_A$. By Proposition 4.8, the reduction of ν modulo p induces the same filtration as given by $Fil_A^1\otimes k$ on $M_A\otimes k$. Then the filtration on M_A defined by ν corresponds to a p-divisible group A' over $W(k)$ lifting the p-divisible $A\otimes k$ over k. We call ν in the following by $\mu_{A'}$. One discusses the cocharacter $\xi_1\circ\mu_{B'}$ similarly and obtains then a p-divisible group A_1' over $W(k)$ lifting $A_1\otimes k$. It follows that one has an isomorphism of filtered ϕ modules similar to that in Proposition 4.8 for the filtered Dieudonné module of A by replacing A_1 with A_1' and B in Proposition 4.4 with B'. Consider the opposite unipotents $U_{\mathcal{G}_B}\times id$ (resp. $U_{\mathcal{G}_{A_1}}\times id$ and $U_{\mathcal{G}_A}$) induced by the cocharacter $\mu_{B'}\times id$ (resp. $\xi_1\circ(\mu_{B'}\times id)$ and $\mu_{A'}$). By the construction, ξ_1 (resp. ξ_2) restricts to an isogeny from $U_{\mathcal{G}_B}\times id$

to $U_{\mathcal{G}_{A_1}} \times id$. (resp. from $U_{\mathcal{G}_A}$ to $U_{\mathcal{G}_{A_1}} \times id$). Thus taking the completion along the identity section, one obtains an isomorphism

$$\hat{\xi}_{cris} = \hat{\xi}_2^{-1} \circ \hat{\xi}_1 : \hat{U}_{\mathcal{G}_B} \xrightarrow{\cong} \hat{U}_{\mathcal{G}_A}.$$

Let $(\mathcal{N}_A, Fil^1_{\mathcal{N}_A}, \nabla_{\mathcal{N}_A}, \phi_{\mathcal{N}_A})$ be the following filtered Dieudonné module over $\hat{U}_{\mathcal{G}_A}$: let $R_{\mathcal{G}_A}$ be the complete local ring of $\hat{U}_{\mathcal{G}_A}$ and $\phi_{\hat{U}_{\mathcal{G}_A}} : R_{\mathcal{G}_A} \to R_{\mathcal{G}_A}$ the lifting of the absolute Frobenius obtained by pulling back the $\phi_{\hat{U}_{\mathcal{G}_B}}$ via $\hat{\xi}_{cris}^{-1}$. The triple

$$(\mathcal{N}_A = M'_A \otimes_{W(k)} R_{\mathcal{G}_A}, Fil^1_{\mathcal{N}_A} = Fil^1_{M'_A} \otimes_{W(k)} R_{\mathcal{G}_A}, \phi_{\mathcal{N}_A} = u \circ (\phi_{M'_A} \otimes \phi_{\hat{U}_{\mathcal{G}_A}})),$$

where u is the tautological $R_{\mathcal{G}_A}$-point of $U_{\mathcal{G}_A}$, together with the connection $\nabla_{\mathcal{N}_A}$ deduced from Theorem 10 [11], makes the quadruple $(\mathcal{N}_A, Fil^1_{\mathcal{N}_A}, \nabla_{\mathcal{N}_A}, \phi_{\mathcal{N}_A})$ an object in $\mathcal{MF}^{\nabla}_{[0,1]}(R_{\mathcal{G}_A})$. We denote again by $\hat{\xi}_{cris}$ the equivalence of categories from $\mathcal{MF}^{\nabla}_{[0,1]}(\hat{U}_{\mathcal{G}_B})$ to $\mathcal{MF}^{\nabla}_{[0,1]}(\hat{U}_{\mathcal{G}_A})$ induced by the isomorphism $\hat{\xi}_{cris}$.

Theorem 4.12. *One has a natural isomorphism in the category $\mathcal{MF}^{\nabla}_{[0,1]}(\hat{M}_{x_0})$:*

$$(H, F, \phi, \nabla)|_{\hat{M}_{x_0}} \cong \{\hat{\xi}_{cris}[\bigotimes_{i=0}^{r-1}(\mathcal{N}_i, Fil^1_{\mathcal{N}_i}, \nabla_{\mathcal{N}_i}), \phi_{ten}] \otimes (M_{A_2}, Fil^1_{A_2}, \phi_{A_2}, d)\}^{\oplus 2^{\epsilon(D)}},$$

where $[\bigotimes_{i=0}^{r-1}(\mathcal{N}_i, Fil^1_{\mathcal{N}_i}, \nabla_{\mathcal{N}_i}), \phi_{ten}] \in \mathcal{MF}^{\nabla}_{[0,1]}(\hat{U}_{\mathcal{G}_B})$ is the one introduced after Proposition 4.7 *and $(M_{A_2}, Fil^1_{A_2}, \phi_{A_2}, d)$ is a constant unit crystal with the trivial connection.*

Proof. From Proposition 2.3.5 [15] and its proof, one knows that $\hat{M}_{x_0} = \hat{U}_{\mathcal{G}_A}$ is the deformation space of the p-divisible group A_0 with Tate cycles $\subset M_A^{\otimes}$ fixed by the group $\mathcal{G}_A \subset \mathrm{GL}_{W(k)}(M_A)$. By the remarks of Faltings, §7 [11], the above quadruple $(\mathcal{N}_A, Fil^1_{\mathcal{N}_A}, \nabla_{\mathcal{N}_A}, \phi_{\mathcal{N}_A})$ gives an explicit description in the category $\mathcal{MF}^{\nabla}_{[0,1]}(\hat{M}_{x_0})$ of the restriction $(H, F, \phi, \nabla)|_{\hat{M}_{x_0}}$. The decomposition of the triple $(\mathcal{N}_A, Fil^1_{\mathcal{N}_A}, \phi_{\mathcal{N}_A})$ follows from the above description of the universal filtered Dieudonné module and the corresponding statement of Proposition 4.8. Then the connection decomposes accordingly: we equip the decomposition with the connection $\nabla_{dec} := (\bigotimes_{i=0}^{r-1} \nabla_{\mathcal{N}_i} \otimes d)^{\oplus 2^{\epsilon(D)}}$. Then the decomposition of $\phi_{\mathcal{N}_A}$ shows that it is horizontal with respect to both ∇_{dec} and $\nabla_{\mathcal{N}_A}$. By the uniqueness of such a connection (see proof of Theorem 10 [11]), $\nabla_{\mathcal{N}_A}$ is isomorphic to ∇_{dec} as claimed. □

The following consequence of the previous result will be used in the next section.

Corollary 4.13. *One has an isomorphism in the category $\mathcal{MF}^{\nabla}_{big,r}(\hat{M}_{x_0})$:*

$$(H, F, \phi^r, \nabla)|_{\hat{M}_{x_0}} \cong \{\hat{\xi}_{cris} \bigotimes_{i=0}^{r-1}(\mathcal{N}_i, Fil^1_{\mathcal{N}_i}, \phi_{\mathcal{N}_i}, \nabla_{\mathcal{N}_i}) \otimes (M_{A_2}, Fil^1_{A_2}, \phi^r_{A_2}, d)\}^{\oplus 2^{\epsilon(D)}}.$$

5. Second tensor power of the universal filtered Dieudonné module and a mass formula

Let $f_0 : X_0 \to M_0$ be the reduction of the universal abelian scheme modulo $\mathfrak{p}$. In this section we construct a pair $(\mathcal{P}_0, \tilde{F}_{rel})$ over $M_0 \otimes \bar{k}$, where $\mathcal{P}_0$ is a line bundle of negative degree and $\tilde{F}_{rel} : F^{*r}_{M_0}\mathcal{P}_0 \to \mathcal{P}_0$ is a nonzero morphism. We show that the reduced zero divisor of $\tilde{F}_{rel}$ is equal to the supersingular locus and the multiplicity at each supersingular point is two.

5.1. **Preliminary discussion.** In this subsection, we collect Faltings's results ([10], [11]) into a form which we can apply in the following conveniently. Note also that M in the following discussion could be relaxed to be an arbitrary smooth proper scheme over $W(k)$. Let $U = \mathrm{Spec}R \subset M$ be a small affine subset, which means that there is an étale map $W(k)[T^{\pm}] \to R$. Let $\bar{R}$ be the maximal extension of R which is étale in characteristic zero (see Ch. II a) [10]) and $\Gamma_R = \mathrm{Gal}(\bar{R}|R)$ be the Galois group. Let $\mathcal{MF}^{\nabla}_{[0,p-2]}(R)$ be the category introduced in §3 [11], and $\mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_R)$ the category of continuous representations of Γ_R on free $\mathbb{Z}_p$-modules of finite rank. By the fundamental theorem (Theorem 5* [11]), there is a fully faithful contravariant functor

$$\mathbf{D} : \mathcal{MF}^{\nabla}_{[0,p-2]}(R) \to \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_R).$$

An object lying in the image of the functor $\mathbf{D}$ is called a *dual crystalline representation.*[1] For our convenience, we shall also consider the covariant functor $\mathbf{D}^{\vee}$, which maps an object $H \in \mathcal{MF}^{\nabla}_{[0,p-2]}(R)$ to the dual of $\mathbf{D}(H)$ in $\mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_R)$, and call an object in the image of $\mathbf{D}^{\vee}$ a crystalline representation. The p-torsion analogue of the above theorem is established in [10]. For clarity of exposition, we use the subscript *tor* to distinguish the torsion analogues. So there is also a fully faithful functor (Theorem 2.6 [10])

$$\mathbf{D}_{tor} : \mathcal{MF}^{\nabla}_{[0,p-2]}(R)_{tor} \to \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_R)_{tor}.$$

It follows from the construction that for an object $H \in \mathcal{MF}^{\nabla}_{[0,p-2]}(R)$, one has $\mathbf{D}(H) = \lim_{\infty \leftarrow n} \mathbf{D}_{tor}(\frac{H}{p^n H})$. Faltings has defined an adjoint functor $\mathbf{E}_{tor}$ of $\mathbf{D}_{tor}$ (see Ch. II, f)-g) [10]). For an object $\mathbb{L} \in \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_R)$, one defines

$$\mathbf{E}(\mathbb{L}) := [\lim_{\infty \leftarrow n} \mathbf{E}_{tor}(\frac{\mathbb{L}}{p^n \mathbb{L}})]/\text{torsion}.$$

Clearly, for $\mathbb{L} = \mathbf{D}(H)$, it holds that

$$\mathbf{E}(\mathbb{L}) = \lim_{\infty \leftarrow n} \mathbf{E}_{tor}(\frac{\mathbb{L}}{p^n \mathbb{L}}) = \lim_{\infty \leftarrow n} \frac{H}{p^n H} = H.$$

Finally define $\mathbf{E}^{\vee}(\mathbb{L}) := \mathbf{E}(\mathbb{L}^{\vee})$.

Lemma 5.1. *Suppose $\mathbb{W}, \mathbb{W}_1, \mathbb{W}_2 \in \mathrm{Rep}_{\mathbb{Z}_p}(\Gamma_R)$. The following basic properties hold:*

(i) *Suppose $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2$. Then $\mathbb{W}$ is crystalline if and only if each $\mathbb{W}_i$ is also.*
(ii) *Suppose $\mathbb{W}$ is crystalline with Hodge-Tate weight n and a Schur functor $\mathbb{S}_\lambda$ with λ a partition of $m \le p-1$ satisfying $mn \le p-2$. Then $\mathbb{S}_\lambda(\mathbb{W})$ is still crystalline, and there is a natural isomorphism $\mathbf{E}^{\vee}(\mathbb{S}_\lambda \mathbb{W}) \cong \mathbb{S}_\lambda \mathbf{E}^{\vee}(\mathbb{W})$.*
(iii) *Suppose $\mathbb{W}_i, i = 1, 2,$ is crystalline with Hodge-Tate weights n_i satisfying $n_1 n_2 \le p-2$. Then $\mathbb{W}_1 \otimes \mathbb{W}_2$ is crystalline, and there is a natural isomorphism*

$$\mathbf{E}^{\vee}(\mathbb{W}_1 \otimes \mathbb{W}_2) \cong \mathbf{E}^{\vee}(\mathbb{W}_1) \otimes \mathbf{E}^{\vee}(\mathbb{W}_2).$$

[1] It is said to be dual because the functor $\mathbf{D}$ maps the first crystalline cohomology of an abelian variety to the dual of the first étale cohomology. See Theorem 7 [11].

Proof. Consider

$$\mathbf{E}_{tor}(\mathbb{W}/p^n) = \mathbf{E}_{tor}(\mathbb{W}_1/p^n) \oplus \mathbf{E}_{tor}(\mathbb{W}_2/p^n).$$

By Ch. II (g) [10], one has always that $l(\mathbf{E}_{tor}(\mathbb{W}_i/p^n)) \leq l(\mathbb{W}_i/p^n)$, and the equality holds iff $\mathbb{W}_i/p^n$ lies in the image of $\mathbf{D}_{tor}$. Now assume $\mathbb{W}$ to be dual crystalline, that is, $\mathbb{W} = \mathbf{D}(H)$. So $\frac{\mathbb{W}}{p^n\mathbb{W}} = \frac{\mathbf{D}(H)}{\mathbf{D}(p^nH)} = \mathbf{D}_{tor}(\frac{H}{p^nH})$. Hence from

$$l(\mathbb{W}/p^n) = \sum_i l(\mathbb{W}_i/p^n) \geq \sum_i l(\mathbf{E}_{tor}(\mathbb{W}_i/p^n)) = l(\mathbf{E}_{tor}(\mathbb{W}/p^n)),$$

it follows that there are $H_{i,n} \in \mathcal{MF}^{\nabla}_{[0,p-2]}(R)_{tor}, i = 1, 2$, such that $\mathbf{D}_{tor}(H_{i,n}) = \mathbb{W}_i/p^n$ and by the faithfulness of $\mathbf{D}_{tor}$, $H_{1,n} \oplus H_{2,n} = H/p^n$. Taking the inverse limit, one obtains $H_i = \lim_{\infty\leftarrow n} H_{i,n}$ with the equality $H_1 \oplus H_2 = H$, which implies that H_i is torsion free and is an object in $\mathcal{MF}^{\nabla}_{[0,p-2]}(R)$. Thus it follows that

$$\mathbf{D}(H_i) = \lim_{\infty\leftarrow n} \mathbf{D}_{tor}(H_i/p^n) = \lim_{\infty\leftarrow n} \mathbb{W}_i/p^n = \mathbb{W}_i,$$

and thereby $\mathbb{W}_i$ is dual crystalline. The other direction of (i) is obvious. Clearly (ii) follows from (iii). To show (iii), it is to show that for $H_i \in \mathcal{MF}^{\nabla}_{[0,p-2]}(R), i = 1, 2$, there is a natural isomorphism $\mathbf{D}(H_1) \otimes \mathbf{D}(H_2) \cong \mathbf{D}(H_1 \otimes H_2)$. Taking an element $f_i \in \mathbf{D}(H_i)$, which is an $\hat{R}$-linear map from H_i to $B^+(R)$ respecting the filtrations and the ϕ's, one forms the $\hat{R}$-linear map $f_1 \otimes f_2 : H_1 \otimes H_2 \to B^+(R)$. It respects the filtrations and the ϕ's and therefore gives an element in $\mathbf{D}(H_1 \otimes H_2)$. So one has a natural map $\mathbf{D}(H_1) \otimes \mathbf{D}(H_2) \to \mathbf{D}(H_1 \otimes H_2)$, which is obviously injective. Because both sides have the same $\mathbb{Z}_p$-rank, it remains to show that the quotient $\mathbf{D}(H_1 \otimes H_2)/\mathbf{D}(H_1) \otimes \mathbf{D}(H_2)$ has no torsion. For that we pass to modulo p reduction and use the functor $\mathbf{D}_{tor}$. The same argument as above applied to H_i/p shows that the $\mathbb{F}_p$-linear map $\mathbf{D}_{tor}(H_1/p) \otimes \mathbf{D}_{tor}(H_2/p) \to \mathbf{D}_{tor}(H_1 \otimes H_2/p)$ is injective and therefore is bijective. This shows the non-p-torsionness. $\square$

Let $\mathcal{U} = \{U\}$ be a small affine open covering of M. Theorem 2.3 [10] shows that one can define the global category $\mathcal{MF}^{\nabla}_{[0,p-2]}(M)$. Furthermore Faltings explained that these various local functors $\mathbf{D}_{tor}$ glue to a global one from $\mathcal{MF}^{\nabla}_{[0,p-2]}(M)_{tor}$ to $\mathrm{Rep}_{\mathbb{Z}_p}(\pi_1(M^0))_{tor}$ (see page 42 [10]). By passing to a limit, one obtains a global functor $\mathbf{D} : \mathcal{MF}^{\nabla}_{[0,p-2]}(M) \to \mathrm{Rep}_{\mathbb{Z}_p}(\pi_1(M^0))$. An object in the image of $\mathbf{D}$ is called a *dual crystalline sheaf*. Similarly, one defines $\mathbf{D}^{\vee}$ and $\mathbf{E}^{\vee}$ in the global setting and calls an object in the image of $\mathbf{D}^{\vee}$ a crystalline sheaf. Now let $\mathbb{W}$ be a crystalline sheaf of M^0 and H the corresponding filtered Frobenius crystal to $\mathbb{W}$ (i.e., $\mathbf{D}^{\vee}(H) = \mathbb{W}$). Let x be a $W(k)$-valued point of M. Consider the specialization of both objects into the point x: via the splitting of the short exact sequence

$$1 \to \pi_1(\bar{M}^0) \to \pi_1(M^0) \to \mathrm{Gal}_{\mathrm{Frac}(W(k))} \to 1$$

induced by the point $x^0 : \mathrm{Frac}(W(k)) \to M^0$, $\mathbb{W}_{x^0}$ is a representation of $\mathrm{Gal}_{\mathrm{Frac}(W(k))}$. On the other hand, H_x is obviously an object in $\mathcal{MF}_{[0,p-2]}(W(k))$.

Lemma 5.2. *Notation as above. Then the following statements hold:*

(i) *The Galois representation $\mathbb{W}_{x^0} \otimes \mathbb{Q}_p$ is crystalline in the sense of Fontaine.*
(ii) *H_x is naturally a strong divisible lattice of the* $\mathrm{Frac}(W(k))$*-vector space $D_{crys}(\mathbb{W}_{x^0} \otimes \mathbb{Q}_p)$ in the sense of Fontaine-Laffaille* ([9]).

(iii) *There is a natural isomorphism of* $\mathbb{Z}_p[\mathrm{Gal}_{\mathrm{Frac}(W(k))}]$*-modules:*

$$\mathbf{D}^\vee(H_x) \cong \mathbf{D}^\vee(H)_{x^0}.$$

Consequently, there is a natural isomorphism in $\mathcal{MF}_{[0,p-2]}(W(k))$*:*

$$\mathbf{E}^\vee(\mathbb{W})_x \cong \mathbf{E}^\vee(\mathbb{W}_{x^0}).$$

Proof. Clearly we can reduce the problem to a small affine subset $U = \mathrm{Spec}R \subset M$. Choose a local coordinate T of R such that the $W(k)$-point x of R is given by $T = 1$ (i.e., the composite $W(k)[T^{\pm}] \to R \to W(k)$ is the $W(k)$-morphism determined by $T \mapsto 1$). Fix an isomorphism $\bar{R} \otimes_R W(k) \cong \overline{W(k)}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Spec}\overline{W(k)} \arrow[r, "\bar{x}"] \arrow[d] & \mathrm{Spec}\bar{R} \arrow[d] \\
\mathrm{Spec}W(k) \arrow[r, "x"] & \mathrm{Spec}R
\end{tikzcd}$$

Note that the subgroup $\Gamma_{R,x} \subset \Gamma_R$ preserving the prime ideal $\ker(\bar{R} \to \overline{W(k)})$ of $\bar{x}$ is naturally isomorphic to $\mathrm{Gal}_{\mathrm{Frac}(W(k))}$, and it is equal to the image of the splitting $\Gamma_R \twoheadrightarrow \mathrm{Gal}_{\mathrm{Frac}(W(k))}$ induced by the $W(k)$-point x. Fix a Frobenius lifting ϕ of $\hat{R}$ which fixes the point x (e.g. the one determined by $T \mapsto T^p$). Note also that the point $\bar{x} : \bar{R} \to \overline{W(k)}$ induces a surjection of $B^+(W(k))$-algebras $B^+(\hat{R}) \to B^+(W(k))$, which preserves the filtration and the Frobenius. Recall that

$$\mathbf{D}(H) = \mathrm{Hom}_{\hat{R},Fil,\phi}(H, B^+(\hat{R})) = (H^\vee \otimes_{\hat{R}} B^+(\hat{R}))^{Fil=0,\phi=1}$$

and

$$\mathbf{D}(H_x) = \mathrm{Hom}_{W(k),Fil,\phi}(H_x, B^+(W(k))) = (H_x^\vee \otimes_{W(k)} B^+(W(k)))^{Fil=0,\phi=1}.$$

The above free $\mathbb{Z}_p$-modules (say of rank n) are basically obtained by solving certain equations (see pages 127-128 [11] and pages 37-38 [10]). There are also natural surjections

$$B^+(\hat{R}) \twoheadrightarrow B^+(\hat{R})/p \cdot B^+(\hat{R}) \twoheadrightarrow \bar{R}/p \cdot \bar{R},$$

and similarly for $B^+(W(k))$. These make the following diagrams commute:

$$\begin{tikzcd}
B^+(\hat{R}) \arrow[r] \arrow[d] & B^+(\hat{R})/p \cdot B^+(\hat{R}) \arrow[r] \arrow[d] & \bar{R}/p \cdot \bar{R} \arrow[d] \\
B^+(W(k)) \arrow[r] & B^+(W(k))/p \cdot B^+(W(k)) \arrow[r] & \overline{W(k)}/p \cdot \overline{W(k)},
\end{tikzcd}$$

where the vertical arrows are induced by the point $\bar{x}$. Faltings showed [10] that it suffices to solve the equations over the quotient $\bar{R}/p$ (resp. $\overline{W(k)}/p$) because each solution over the quotient can be uniquely lifted. Now choose a filtered basis $\{h_i\}$ of H, which restricts to a filtered basis of H_x. An element of $\mathbf{D}(H)$ is then given by an n-tuple in $B^+(\hat{R})$ satisfying a system of equations coming from the condition on ϕ's. For each such n-tuple, we obtain an n-tuple in $B^+(W(k))$ by projecting each component to $B^+(W(k))$ (the projection $B^+(\hat{R}) \twoheadrightarrow B^+(W(k))$ induced by the point $\bar{x}$). As the projection preserves the filtration and the Frobenius, and as the filtration and the Frobenius on H_x are the ones of H by restriction, any so-obtained n-tuple satisfies the equations required for $\mathbf{D}(H_x)$. So we have a $\mathbb{Z}_p$-linear map

$$ev_x : \mathbf{D}(H) \to \mathbf{D}(H_x), f \mapsto f(x).$$

Consider first the Galois action. Recall that $\mathrm{Gal}_{\mathrm{Frac}(W(k))}$ acts on $\mathbf{D}(H_x) \subset H_x^\vee \otimes_{W(k)} B^+(W(k))$ on the second tensor factor. But the Γ_R-action on $\mathbf{D}(H) \subset H^\vee \otimes_{\hat{R}} B^+(\hat{R})$ must also be intertwined with the connection ∇ on the first factor. However the restriction to the subgroup $\Gamma_{R,x}$ does not involve the connection (see Ch. II e) [10] for the p-torsion situation which we can also assume in the argument). So the above map ev_x is equivariant with respect to $\Gamma_{R,x}$-action on $\mathbf{D}(H)$ and $\mathrm{Gal}_{\mathrm{Frac(W(k))}}$ action on $\mathbf{D}(H_x)$.

Next we claim that ev_x is a $\mathbb{Z}_p$-isomorphism. For that we consider the base change $ev_x \otimes \mathbb{Q}_p$ and then the reduction $ev_x \otimes \mathbb{F}_p$. By Ch. II (h) [10], one has a natural isomorphism

$$H \otimes_{\hat{R}} B(\hat{R}) \cong \mathbf{D}^\vee(H) \otimes_{\mathbb{Z}_p} B(\hat{R}),$$

which respects the Γ_R-actions, filtrations and ϕ's. Tensorizing the above isomorphism with $B(W(k))$ as $B(\hat{R})$-modules (the morphism $B(\hat{R}) \twoheadrightarrow B(W(k))$ induced by $\bar{x}$) and taking the $\Gamma_{R,x}$-invariance of both sides, we obtain an isomorphism of $\mathrm{Gal}_{\mathrm{Frac}(W(k))}$-representations:

$$V_{crys}(H_x \otimes \mathrm{Frac}(W(k))) \cong \mathbf{D}^\vee(H)_{x^0} \otimes \mathbb{Q}_p.$$

That is, there is a natural isomorphism $\mathbf{D}(H)_{x^0} \otimes \mathbb{Q}_p \cong V_{crys}^\vee(H_x \otimes \mathrm{Frac}(W(k)))$. By Fontaine-Laffaille (see §§7-8 in [9]; see also §2 [4]), $\mathbf{D}(H_x)$ is a Galois lattice of $V_{crys}^\vee(H_x \otimes \mathrm{Frac}(W(k)))$ by the isomorphism, and H_x is a strong divisible lattice of $D_{crys}(\mathbf{D}^\vee(H)_{x^0} \otimes \mathbb{Q}_p)$. This shows (i) and (ii). Also it implies that the map $ev_x \otimes \mathbb{Q}_p$ is an isomorphism. In particular the map ev_x is injective. Using the fact that the composite $B^+(W(k)) \to B^+(\hat{R}) \to B^+(W(k))$ is the identity, one sees that $ev_x(\mathbf{D}(H)) \cap p\mathbf{D}(H_x) = ev_x(p\mathbf{D}(H))$, and the map $ev_x \otimes \mathbb{F}_p : \mathbf{D}(H)/p\mathbf{D}(H) \to \mathbf{D}(H_x)/p\mathbf{D}(H_x)$ is therefore injective. Now that the $\mathbb{F}_p$-vector spaces $\mathbf{D}(H)/p\mathbf{D}(H)$ and $\mathbf{D}(H_x)/p\mathbf{D}(H_x)$ have the same dimension n, $ev_x \otimes \mathbb{F}_p$ is an isomorphism. Thus ev_x is an isomorphism. This proves (iii). □

Let $r \in \mathbb{N}$ be a natural number. Let $\mathrm{Rep}_{\mathbb{Z}_{p^r}}(\pi_1(M^0)) \subset \mathrm{Rep}_{\mathbb{Z}_p}(\pi_1(M^0))$ be the full subcategory of $\mathbb{Z}_{p^r}[\pi_1(M^0)]$-modules. An object which lies in both $\mathrm{Rep}_{\mathbb{Z}_{p^r}}(\pi_1(M^0))$ and the image of $\mathbf{D}^\vee$ is called a $\mathbb{Z}_{p^r}$-crystalline sheaf. One notes that the proof of Theorem 2.3 [10] works verbatim to show that the local categories $\{\mathcal{MF}^\nabla_{big,r}(U)\}_{U\in\mathcal{U}}$ (see §4.1) glue into a global category $\mathcal{MF}^\nabla_{big,r}(M)$. A typical object in this category is obtained by replacing the Frobenius of an object in $\mathcal{MF}^\nabla_{[0,p-2]}(M)$ with its r-th power.

Lemma 5.3. *Let $\mathbb{W}$ be a $\mathbb{Z}_{p^r}$-crystalline sheaf. Assume that $\mathbb{Z}_{p^r} \subset \mathcal{O}_M$. Then there is a natural decomposition $\mathbf{E}^\vee(\mathbb{W}) = \bigoplus_{i=0}^{r-1} \mathbf{E}^\vee(\mathbb{W})_i$ in the category $\mathcal{MF}^\nabla_{big,r}(M)$.*

Proof. The multiplication by $s \in \mathbb{Z}_{p^r}$ on $\mathbb{W}$ commutes with $\pi_1(M^0)$-action. Hence it gives rise to an endomorphism $s_{\mathcal{MF}}$ of $\mathbf{E}^\vee(\mathbb{W})$ in the category $\mathcal{MF}^\nabla_{[0,p-2]}(M)$. By assumption $\mathcal{O}_M$ contains the eigenvalues of $s_{\mathcal{MF}}$. The eigen-decomposition of $\mathbf{E}^\vee(\mathbb{W})$ with respect to $s_{\mathcal{MF}}$ gives rise to a decomposition of form $\bigoplus_{i=0}^{r-1} \mathbf{E}^\vee(\mathbb{W})_i$ such that the direct factors are preserved by ∇ and permute cyclically by ϕ. Hence the lemma follows. □

5.2. **Second wedge/symmetric power of the universal filtered Dieudonné module.** From now on the rational prime number p is assumed to be ≥ 5 in addition to Assumption 1.1. The aim of this subsection is to show a direct sum decomposition of the second wedge (resp. symmetric) of the universal filtered Dieudonné crystal for d an even (resp. odd) number. Recall that by Corollary 3.6 we have a direct-tensor decomposition of étale local systems $\mathbb{H} = (\mathbb{V} \otimes \mathbb{U})^{\oplus 2^{\epsilon(D)}}$. It follows from Lemma 5.1 that the direct summand $\mathbb{H}' := \mathbb{V} \otimes \mathbb{U}$ of $\mathbb{H}$ is crystalline. As $\det(\mathbb{H}) \cong \mathbb{Z}_p(-2^{d+\epsilon(D)-1})$, it follows that

$$\det \mathbb{H}' \cong \mathbb{Z}_p(-2^{d-1}) \otimes \chi,$$

where χ is a 2-torsion crystalline sheaf (which is trivial when $\epsilon(D) = 0$). Consider the following $\mathbb{Z}_{p^d}$-étale local system:

$$\tilde{\mathbb{H}}' := (\mathbb{V} \otimes \det(\mathbb{V})^{-\frac{1}{2}} \otimes \mathbb{U} \otimes \det(\mathbb{U})^{-\frac{1}{2}}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^d}.$$

Because of the equality

$$\bigwedge^2(\tilde{\mathbb{H}}') = [\bigwedge^2(\mathbb{H}') \otimes \det(\mathbb{H}')^{-1}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^d},$$

$\bigwedge^2(\tilde{\mathbb{H}}')$ is a $\mathbb{Z}_{p^d}$-crystalline sheaf. For $1 \leq i \leq r$, put

$$\tilde{\mathbb{V}}_i = \mathbb{V}_{1,\sigma^{i-1}} \otimes \det(\mathbb{V}_1)^{-\frac{1}{2}}, \ \tilde{\mathbb{V}}'_i = \tilde{\mathbb{V}}_i \otimes_{\mathbb{Z}_{p^r}} \mathbb{Z}_{p^d},$$

and for $r + 1 = r_1 + 1 \leq i \leq r_1 + r_2$, put

$$\tilde{\mathbb{V}}'_i = (\mathbb{U}_{1,\sigma^{i-1}} \otimes \det(\mathbb{U}_1)^{-\frac{1}{2}}) \otimes_{\mathbb{Z}_{p^{r_2}}} \mathbb{Z}_{p^d},$$

and so on. Then by Corollary 3.6, we have a decomposition of $\tilde{\mathbb{H}}'$ into tensor product of rank two $\mathbb{Z}_{p^d}$-étale local systems: $\tilde{\mathbb{H}}' = \bigotimes_{i=1}^d \tilde{\mathbb{V}}'_i$. In the tensor decomposition, we assume that the factor $\tilde{\mathbb{V}}_1$ corresponds to the place τ (see Lemma 3.1).

Remark 5.4. We conjecture that each tensor factor $\tilde{\mathbb{V}}'_i$ in the above decomposition is a $\mathbb{Z}_{p^d}$-crystalline sheaf. The next lemma shows that $\mathrm{Sym}^2\tilde{\mathbb{V}}'_i$ is a direct factor of a crystalline sheaf and therefore crystalline by Lemma 5.1 (i).

The following lemma is proved by induction on d:

Lemma 5.5. *For $I = (i_1, \cdots, i_l)$ a multi-index in $\{1, \cdots, d\}$, put* $\mathrm{Sym}^2(\tilde{\mathbb{V}}')_I := \bigotimes_{j=1}^l \mathrm{Sym}^2\tilde{\mathbb{V}}'_{i_j}$. *One has a direct sum decomposition of $\mathbb{Z}_{p^d}$-étale local systems:*

(i) *for d even,*

$$\bigwedge^2(\tilde{\mathbb{H}}') = \bigoplus_{I,|I| \text{ odd}} \mathrm{Sym}^2(\tilde{\mathbb{V}}')_I, \quad \mathrm{Sym}^2(\tilde{\mathbb{H}}') = \bigoplus_{I,|I| \text{ even}} \mathrm{Sym}^2(\tilde{\mathbb{V}}')_I;$$

(ii) *for d odd,*

$$\bigwedge^2(\tilde{\mathbb{H}}') = \bigoplus_{I,|I| \text{ even}} \mathrm{Sym}^2(\tilde{\mathbb{V}}')_I, \quad \mathrm{Sym}^2(\tilde{\mathbb{H}}') = \bigoplus_{I,|I| \text{ odd}} \mathrm{Sym}^2(\tilde{\mathbb{V}}')_I.$$

In the following we shall focus on the direct summand $\mathrm{Sym}^2(\tilde{\mathbb{V}}_1)$ in the decomposition since it is, so to speak, the (rank three) uniformizing direct factor of the weight two integral p-adic variation of Hodge structures of the universal family. Also one notices that this factor is actually defined over $\mathbb{Z}_{p^r}$. So by taking the $\mathrm{Gal}(\mathbb{Z}_{p^d}|\mathbb{Z}_{p^r})$-invariants, one obtains a direct decomposition into $\mathbb{Z}_{p^r}$-dual crystalline sheaves with

$\mathrm{Sym}^2\tilde{\mathbb{V}}_1$ as a direct factor for the second wedge (resp. symmetric) power for n even (resp. odd).

Proposition 5.6. *Let $(H', F, \phi, \nabla) \in \mathcal{MF}^{\nabla}_{[0,1]}(M)$ be the subfiltered F-crystal corresponding to the factor $\mathbb{H}' \subseteq \mathbb{H}$. One has a direct sum decomposition in $\mathcal{MF}^{\nabla}_{big,r}(M)$:*

(i) *for d even,*

$$\bigwedge^2 (H', F, \phi^r, \nabla) = \bigoplus_{i=1}^{r} \mathbf{E}^{\vee}(\mathrm{Sym}^2\tilde{\mathbb{V}}_i)_0\{-2^{d-1}\} \otimes \mathbf{E}^{\vee}(\chi) \oplus \textit{rest term};$$

(ii) *for d odd,*

$$\mathrm{Sym}^2(H', F, \phi^r, \nabla) = \bigoplus_{i=1}^{r} \mathbf{E}^{\vee}(\mathrm{Sym}^2\tilde{\mathbb{V}}_i)_0\{-2^{d-1}\} \otimes \mathbf{E}^{\vee}(\chi) \oplus \textit{rest term}.$$

Proof. We shall prove (i) only because (ii) can be similarly proved. By the discussion before the proposition, we obtain a decomposition in $\mathcal{MF}^{\nabla}_{[0,2]}(M)$:

$$\mathbf{E}^{\vee}(\bigwedge^2(\mathbb{H}') \otimes \chi^{-1} \otimes \mathbb{Z}_{p^r}) = \bigoplus_{i=1}^{r} \mathbf{E}^{\vee}(\mathrm{Sym}^2\tilde{\mathbb{V}}_i)\{-2^{d-1}\} \oplus \text{rest term}.$$

The claimed decomposition is obtained by considering the eigen-decomposition of both sides corresponding to the eigenvalue s_0: The argument is similar to that of Lemma 4.3. The right hand side is clear, and the question is the left hand side. It suffices to consider the eigen-component after inverting p. By Ch. II h) [10], one has a Γ_R-isomorphism

$$\mathbf{E}^{\vee}(\bigwedge^2(\mathbb{H}') \otimes \chi^{-1} \otimes \mathbb{Z}_{p^r}|_{\hat{U}}) \otimes_{\hat{R}} B(R) \cong (\bigwedge^2(\mathbb{H}') \otimes \chi^{-1} \otimes \mathbb{Z}_{p^r}|_{\hat{U}}) \otimes_{\mathbb{Z}_p} B(R).$$

It follows that

$$\begin{aligned}
[\mathbf{E}^{\vee}(\bigwedge^2(\mathbb{H}') \otimes \chi^{-1} \otimes \mathbb{Z}_{p^r})|_{\hat{U}}]_0[\frac{1}{p}] &\cong [\bigwedge^2(\mathbb{H}') \otimes_{\mathbb{Z}_p} \chi^{-1} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}|_{\hat{U}} \otimes_{\mathbb{Z}_{p^r}} B(R)]^{\Gamma_R} \\
&\cong [\bigwedge^2 \mathbb{H}' \otimes_{\mathbb{Z}_p} \chi^{-1} \otimes_{\mathbb{Z}_p} B(R)]^{\Gamma_R} \\
&\cong \mathbf{E}^{\vee}(\bigwedge^2 \mathbb{H}') \otimes \mathbf{E}^{\vee}(\chi^{-1})[\frac{1}{p}] \\
&\cong \bigwedge^2(\mathbf{E}^{\vee}(\mathbb{H}')) \otimes \mathbf{E}^{\vee}(\chi^{-1})[\frac{1}{p}].
\end{aligned}$$

This shows that the eigen-submodule of $\mathbf{E}^{\vee}(\bigwedge^2 \mathbb{H}' \otimes \chi^{-1} \otimes \mathbb{Z}_{p^r})$ to the eigenvalue s_0 is naturally isomorphic to $\bigwedge^2 H' \otimes \mathbf{E}^{\vee}(\chi^{-1})$. The claimed decomposition then follows. □

5.3. **Construction of the pair.** The aim of this subsection is to construct the pair $(\mathcal{P}_0, \tilde{F}_{rel})$ claimed in the introduction of the section. In the following, $\mathbf{E}_0$ denotes $\mathbf{E}^{\vee}(\mathrm{Sym}^2\tilde{\mathbb{V}}_1)_0\{-2^{d-1}\}$ and $\tilde{\mathbf{E}}_0 = \mathbf{E}_0 \otimes \mathbf{E}^{\vee}(\chi)$ for the factor in the decomposition in Proposition 5.6. Note that the square of $\mathbf{E}^{\vee}(\chi)$ is the trivial crystal. In particular, its filtration is trivial and its restriction to each $W(k)$-valued point is a unit crystal. Let $x_0 \in M_0(k)$ be a k-rational point and x a $W(k)$-valued point of M lifting x_0.

Proposition 5.7. *One has a natural isomorphism in the category* $\mathcal{MF}^{\nabla}_{big,r}(\hat{M}_{x_0})$*:*

$$\mathbf{E}_0|_{\hat{M}_{x_0}} \cong \hat{\xi}_{crys}[\mathrm{Sym}^2\mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1}\det(\mathcal{N}_i)].$$

Proof. Assume d to be even. By Corollary 4.13, one has a natural isomorphism in $\mathcal{MF}^{\nabla}_{big,r}(\hat{M}_{x_0})$:

$$H'|_{\hat{M}_{x_0}} \cong \hat{\xi}_{crys}(\bigotimes_{i=0}^{r-1}\mathcal{N}_i) \otimes M_{A_2}.$$

By a Schur functor calculation as in Lemma 5.5, one finds that, via the isomorphism, $\hat{\xi}_{crys}[\mathrm{Sym}^2\mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1}\det(\mathcal{N}_i)]$ is a direct factor of $\bigwedge^2(H'|_{\hat{M}_{x_0}}) \otimes \mathbf{E}^{\vee}(\chi^{-1})|_{\hat{M}_{x_0}}$. The point is to show that it is the direct factor $\mathbf{E}_0|_{\hat{M}_{x_0}}$. Note that $\hat{\xi}_{crys}[\mathrm{Sym}^2\mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1}\det(\mathcal{N}_i)]$ is the unique rank three direct factor with nontrivial filtration. So it suffices to show that the rank three direct factor $\mathbf{E}_0|_{\hat{M}_{x_0}}$ also has this property. To do so we show that the filtration of the filtered ϕ^r-module $(\mathbf{E}_0)_x \otimes \mathrm{Frac}W(k)$ is nontrivial. By Lemma 5.5, $\mathrm{Sym}^2(\tilde{\mathbb{V}}_1(-2^{d-2}))$ is a direct factor of the crystalline sheaf $\bigwedge^2(\mathbb{H}' \otimes \mathbb{Z}_{p^r})$. So by Lemma 5.2 (i), $(\mathrm{Sym}^2\tilde{\mathbb{V}}_1(-2^{d-2}))_{x^0}$ is a crystalline lattice for the group $\mathrm{Gal}_{\mathrm{Frac}W(k)}$, and by (iii), one has the equality (after taking the eigen-component to the eigenvalue s_0)

$$(\mathbf{E}_0)_x = \mathbf{E}^{\vee}(\mathrm{Sym}^2(\tilde{\mathbb{V}}_1(-2^{d-2}))_{x^0})_0.$$

Then by Lemma 5.2 (ii) we determine the filtration of $D_{crys}(\mathrm{Sym}^2(\tilde{\mathbb{V}}_1(-2^{d-2})_{x^0} \otimes \mathbb{Q}_p))_0$. Consider the $\mathrm{Gal}_{\mathrm{Frac}W(k)}$-representation $\mathrm{Sym}^2(\tilde{\mathbb{V}}_1(-2^{d-2})_{x^0} \otimes \mathbb{Q}_p)$. It is equal to $\mathrm{Sym}^2(\tilde{\mathbb{V}}_{1,x^0}(-2^{d-2}) \otimes \mathbb{Q}_p)$, and by Proposition 3.11 $\mathbb{V}_{1,x^0} \otimes \mathbb{Q}_p$ is crystalline for an open subgroup $\mathrm{Gal}_E \subset \mathrm{Gal}_{\mathrm{Frac}W(k)}$. As

$$\mathrm{Sym}^2(\tilde{\mathbb{V}}_{1,x^0}(-2^{d-2})) = \mathrm{Sym}^2(\mathbb{V}_{1,x^0}) \otimes_{\mathbb{Z}_{p^r}} \det(\mathbb{V}_{1,\sigma,x^0}) \otimes_{\mathbb{Z}_{p^r}} \cdots \otimes_{\mathbb{Z}_{p^r}} \det(\mathbb{V}_{1,\sigma^{r-1},x^0}),$$

and the functor D_{crys} commutes with a Schur functor for a crystalline representation, we have

$$D_{crys}(\mathrm{Sym}^2(\tilde{\mathbb{V}}_{1,x^0}(-2^{d-2}) \otimes \mathbb{Q}_p))_0 = \mathrm{Sym}^2(D_{crys}(\tilde{\mathbb{V}}_{1,x^0}(-2^{d-2}) \otimes \mathbb{Q}_p)_0),$$

which is naturally isomorphic to $[\mathrm{Sym}^2 M_0 \otimes \bigotimes_{i=1}^{r-1}\det(M_i)] \otimes \mathrm{Frac}(W(k_E))$. This shows that the filtration of $(\mathbf{E}_0)_x \otimes \mathrm{Frac}W(k_E)$ is nontrivial. Thus, so is the filtration on $(\mathbf{E}_0)_x \otimes \mathrm{Frac}W(k)$. □

Construction of $\mathcal{P}_0$. Consider the filtration on the factor $\mathbf{E}_0$. As the Hodge filtration on H is filtered free (see §2 [11]), the induced filtration on $\mathbf{E}_0$ by Proposition 5.6 is also filtered free.

Lemma 5.8. *The filtration F on $\mathbf{E}_0$ is nontrivial with form*

$$\mathbf{E}_0 = F^0\mathbf{E}_0 \supset F^1\mathbf{E}_0 \supset F^2\mathbf{E}_0,$$

and each grading is locally free of rank one.

Proof. As it is filtered free, it suffices to show this over a point x as above. Then it follows from Proposition 5.7 and the proofs of Propositions 3.11, 3.16. □

By the lemma we put $\mathcal{P}=\frac{\mathbf{E}_0}{F^1\mathbf{E}_0}=\frac{\tilde{\mathbf{E}}_0}{F^1\tilde{\mathbf{E}}_0}$, and $\mathcal{P}_0$ is the modulo p reduction of $\mathcal{P}$ which is defined over M_0. Next we consider the line bundle $\mathcal{P}^0$ over M^0 by taking a comparison with the variation of Hodge structures at infinity associated to the Mumford family. Let $(H^1_{dR}, F_{hod}, \nabla^{GM})$ be the automorphic vector bundle over M_K coming from the universal family of abelian varieties over M_K. One has a natural isomorphism

$$(H,F,\nabla)\otimes_{\mathcal{O}_{\mathfrak{p}}} F_{\mathfrak{p}} \cong (H^1_{dR}, F_{hod}, \nabla^{GM})\otimes_F F_{\mathfrak{p}}.$$

We intend to show a tensor decomposition of $(H',F,\nabla)\otimes_{\mathcal{O}_{\mathfrak{p},\tau}}\bar{\mathbb{Q}}_p$ of the form

$$(H',F,\nabla)\otimes_{\mathcal{O}_{\mathfrak{p},\tau}}\bar{\mathbb{Q}}_p=(H_1,F_1,\nabla_1)\otimes\cdots\otimes(H_d,F_d,\nabla_d),$$

where the first tensor factor on the right hand side is the unique one admitting the nontrivial filtration. For this we apply the theory of de Rham cycles as developed in §2.2 [15]. Let $\{s_{\alpha,B}\}\subset H^{\otimes}_{\mathbb{Q}}$ be a finite set of tensors defining the subgroup $G_{\mathbb{Q}}\subset \mathrm{GL}(H_{\mathbb{Q}})$. By Corollary 2.2.2 [15], it defines a set of de Rham cycles $\{s_{\alpha,dR}\}\subset (H^1_{dR})^{\otimes}$ defined over the reflex field $\tau(F)$, which are by definition ∇^{GM}-parallel and contained in Fil^0.

Lemma 5.9. *The set of de Rham cycles $\{s_{\alpha,dR}\}$ induces a direct-tensor decomposition*

$$(H^1_{dR},F_{hod},\nabla^{GM})\otimes_{F,\tau}\bar{\mathbb{Q}}=[\bigotimes_{i=1}^{d}(H^1_{dR,i},F_{hod,i},\nabla^{GM}_i)]^{\oplus 2^{\epsilon(D)}}$$

such that the factor $(H^1_{dR,1},F_{hod,1},\nabla^{GM}_1)$ is the unique one with nontrivial filtration.

Proof. Let $\pi:\tilde{M}_{an}:=X\times G(\mathbb{A}_f)/K\to M_K(\mathbb{C})$ be the natural projection of complex analytic spaces. The pull-back of $(H^1_{dR},\nabla^{GM})\otimes\mathbb{C}$ over $M_K(\mathbb{C})$ via π is trivialized, and by the de Rham isomorphism it is isomorphic to $(H_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathcal{O}_{\tilde{M}_{an}},1\otimes d)$. By a similar discussion on the direct-tensor decomposition of the $G(\mathbb{Q})$-representation $H_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C}$ as given in §2.1, the tensors $s_{\alpha,B}\otimes 1$s induce a tensor decomposition of $\pi^*((H^1_{dR},\nabla^{GM})\otimes\mathbb{C})$. It is $G(\mathbb{Q})$-equivariant by construction and hence descends to a decomposition on $(H^1_{dR},\nabla^{GM})\otimes\mathbb{C}$. This is the same tensor decomposition induced by the tensors $s_{\alpha,dR}$. Since they are defined over $\tau(F)$, the tensor decomposition already occurs over $\bar{\mathbb{Q}}$. We have also to check the property about the filtration in the tensor decomposition. Note that the Hodge filtration $\pi^*(H^1_{dR},F_{hod})\otimes\mathbb{C}$ over the point $[0\times id]$ is induced from $\mu_{h_0}:\mathbb{G}_m(\mathbb{C})\to G_{\mathbb{C}}\subset\mathrm{GL}(H_{\mathbb{C}})$. The assertion follows then from the definition of h_0 in §2. □

Composing with the embedding $\iota:\bar{\mathbb{Q}}\hookrightarrow\bar{\mathbb{Q}}_p$, we obtain the claimed tensor decomposition on $(H',F,\nabla)\otimes_{\mathcal{O}_{\mathfrak{p},\tau}}\bar{\mathbb{Q}}_p$. Taking the grading with respect to $F_{hod,i}$, one obtains the associated Higgs bundle (E_i,θ_i) with $(H^1_{dR,i},F_{hod,i},\nabla^{GM}_i)$. By the lemma, only θ_1 is nontrivial. In fact, it is a maximal Higgs field (see [29]), that is,

$$\theta_1: F^1_{hod,1}\xrightarrow{\cong}\frac{H^1_{dR,1}}{F^1_{hod,1}}\otimes\Omega_{M_K\otimes\bar{\mathbb{Q}}}.$$

Actually over each connected component of M_K, $\theta_1\otimes\mathbb{C}$ is a morphism of locally homogenous bundles of rank one. Then it must be an isomorphism, because it will otherwise be zero, and together with the zero Higgs fields on the other factors $E_i, i\geq 2$, this implies that the Kodaira-Spencer map of the universal family is trivial, which is absurd. As both $F^1_{hod,1}$ and $\frac{H^1_{dR,1}}{F^1_{hod,1}}$ are locally homogenous line bundles over each connected component of M_K, their isomorphism classes are determined by the corresponding representations of $K_{\mathbb{R}} \otimes \mathbb{C}$, where $K_{\mathbb{R}}$ is the stabilizer of $G(\mathbb{R})$ at $0 \in X$. In this way one easily shows that they are dual to each other. By putting $\mathcal{L} := F^1_{hod,1}$, one then has

$$\theta_1 : \mathcal{L} \cong \mathcal{L}^{-1} \otimes \Omega_{M_K \otimes \bar{\mathbb{Q}}}.$$

By abuse of notation, we use $\mathcal{L}$ again to denote the base change of $\mathcal{L}$ to

$$\bar{M}^0 := M^0 \otimes_{F_{\mathfrak{p}}} \bar{\mathbb{Q}}_p = (M_K \otimes_{F,\tau} \bar{\mathbb{Q}}) \otimes \bar{\mathbb{Q}}_p.$$

Lemma 5.10. *One has a natural isomorphism $\mathcal{P}^0 \cong \mathcal{L}^{-2}$ over $\bar{M}_0$.*

Proof. In fact we show that there is a natural isomorphism $(\mathbf{E}_0, F) \otimes \mathcal{O}_{\bar{M}^0} \cong \mathrm{Sym}^2(H_1, F_1)$. We raise the defining field of M^0 so that it contains the defining field of $H_i, 1 \leq i \leq d$ and $\mathbb{Q}_{p^d}$. By abuse of notation, we use the same notation to mean an above object after the base change. Let $U = \mathrm{Spec}R \subset M$ be a small affine subset. We have a natural isomorphism $\mathbb{H}' \otimes_{\mathbb{Z}_p} B(R) \cong H' \otimes_R B(R)$ respecting Γ_R-actions and filtrations (we forget the ϕ's in the isomorphism). As $\mathbb{Q}_{p^d} \subset B(R)$, we can write it as

$$(\tilde{\mathbb{V}}_1 \otimes \cdots \otimes \tilde{\mathbb{V}}_d) \otimes_{\mathbb{Q}_{p^d}} B(R) \cong (H_1 \otimes \cdots \otimes H_d) \otimes_{R[\frac{1}{p}]} B(R)$$

or

$$\bigotimes_{i=1}^{d} [\tilde{\mathbb{V}}_i \otimes_{\mathbb{Q}_{p^d}} B(R)] \cong \bigotimes_{i=1}^{d} [H_i \otimes_{R[\frac{1}{p}]} B(R)].$$

In the comparison the tensor factor with numbering is preserved, because it is also over a general $\bar{\mathbb{Q}}$-rational point of each connected component of U by a result of Blasius and Wintenberger (see [2]; see also §4 [24]), which asserts that in the p-adic comparison the tensors $s_{\alpha,et}$ and $s_{\alpha,dR}$ correspond. Then taking the second wedge (symmetric) power for n even (odd) of the above isomorphism, we find the isomorphism $\mathrm{Sym}^2\mathbb{V}_1 \otimes_{\mathbb{Q}_{p^d}} B(R) \cong \mathrm{Sym}^2 H_1 \otimes_{R[\frac{1}{p}]} B(R)$ which respects Γ_R-actions and filtrations. Taking Γ_R-invariants of both sides, we obtain the claimed isomorphism over $\mathrm{Spec}R[\frac{1}{p}]$. By the naturalness of the comparison, the local isomorphisms glue into a global one. □

By the main theorems of Langton [16], the line bundle $\mathcal{L}^{-1}$ extends over $M \otimes \bar{\mathbb{Z}}_p$ with the modulo p reduction $\mathcal{L}_0^{-1}$, and the isomorphism in the above lemma specializes to an isomorphism between $\mathcal{P}_0$ and $\mathcal{L}_0^{-2}$. So we have shown the first isomorphism in the following:

Proposition 5.11. *One has natural isomorphisms $\mathcal{P}_0 \cong \mathcal{L}_0^{-2} \cong \Omega^{-1}_{M_0}$ over $\bar{M}_0$.*

Proof. We have shown that over $\bar{M}^0$ the Higgs field θ_1 induces an isomorphism $\mathcal{L}^2 \cong \Omega_{M^0}$. For the same reason as above, this isomorphism specializes into an isomorphism $\mathcal{L}_0^2 \cong \Omega_{M_0}$. □

Construction of $\tilde{F}_{rel}$. For each small affine $U \in \mathcal{U}$, we choose a Frobenius lifting $F_U : \hat{U} \to \hat{U}$, where $\hat{U}$ is the p-adic completion of U. As $\mathbf{E}_0$ is an object in $\mathcal{MF}^{\nabla}_{big,r}(M)$, there is a map $\phi_{r,F_U} : F_U^{*r}\mathbf{E}_0|_{\hat{U}} \to \mathbf{E}_0|_{\hat{U}}$. By Proposition 5.6, ϕ_{r,F_U} is the restriction of the second wedge (resp. symmetric) power of the r-th iterated relative Frobenius morphism $\phi_{F_U} : F_U^* H'|_{\hat{U}} \to H'|_{\hat{U}}$ for d even (resp. odd) to the direct factor $\mathbf{E}_0|_{\hat{U}}$.

Lemma 5.12. *For each U, the image $\phi_{r,F_U}(F_U^{*r}\mathbf{E}_0|_{\hat{U}}) \subset \mathbf{E}_0|_{\hat{U}}$ is divisible by p^{r-1}, but not divisible by p^r.*

Proof. In the p-adic filtration

$$\mathbf{E}_0|_{\hat{U}} \supset p\mathbf{E}_0|_{\hat{U}} \supset \cdots \supset p^{i-1}\mathbf{E}_0|_{\hat{U}} \supset p^i\mathbf{E}_0|_{\hat{U}} \supset \cdots,$$

there is a unique i with the property

$$p^{i-1}\mathbf{E}_0|_{\hat{U}} \supset \phi_{r,F_U}(F_U^{*r}\mathbf{E}_0|_{\hat{U}}) \supsetneq p^i\mathbf{E}_0|_{\hat{U}}.$$

It suffices to show that $i = r$ or equivalently that the images of $\phi_{r,F_U}(F_U^{*r}\mathbf{E}_0|_{\hat{U}})$ in the successive gradings $\frac{p^{i-1}\mathbf{E}_0|_{\hat{U}}}{p^i\mathbf{E}_0|_{\hat{U}}}$ are zero for $1 \leq i < r$ and nonzero for $i = r$. Let $x_0 \in \hat{U}(k)$ and $\hat{U}_{x_0}$ be the completion of $\hat{U}$ at x_0. It is equivalent to show the above statement over each $\hat{U}_{x_0}$. This follows from the description of the relative Frobenius ϕ over the formal neighborhood $\hat{U}_{x_0}$ as described in §4.2 and the result for the closed point x_0. In detail it goes as follows: by Proposition 5.7, the filtered ϕ^r-module $\mathbf{E}_0|_x$ is isomorphic to $\mathrm{Sym}^2 M_0 \otimes (\bigotimes_{i=1}^{r-1} \det M_i) \otimes$unit crystal over $W(\bar{k})$. By Proposition 3.16, the Newton slope of the rank one ϕ^r-module $\det M_i, i \geq 1$, is either 1×1 or 1×2. In the former case, the Newton slopes of $\mathrm{Sym}^2 M_0$ are $\{1 \times 0, 1 \times 1, 1 \times 2\}$. These imply that $\phi_r(\mathbf{E}_0|_x)$ is always divisible by p^{r-1}. By Remark 3.18, the former case does occur for a certain x_0. So $\phi_r(\mathbf{E}_0|_x)$ is not divisible by p^r at such a closed point. □

As $\phi_{r,F_U}(F_U^{*r}F^1\mathbf{E}_0|_{\hat{U}}) \subset p^r\mathbf{E}_0|_{\hat{U}}$, the composite of the morphisms

$$F_U^{*r}F^1\mathbf{E}_0|_{\hat{U}} \hookrightarrow F_U^{*r}\mathbf{E}_0|_{\hat{U}} \xrightarrow{\frac{\phi_{r,F_U}}{p^{r-1}}} \mathbf{E}_0|_{\hat{U}} \overset{pr}{\twoheadrightarrow} \mathcal{P}|_{\hat{U}} \overset{\mathrm{mod}\ p}{\twoheadrightarrow} \mathcal{P}_0|_{U_0}$$

is zero. As a result we get the morphism

$$\frac{F_U^{*r}\mathbf{E}_0|_{\hat{U}}}{F_U^{*r}F^1\mathbf{E}_0|_{\hat{U}}} = F_U^{*r}\mathcal{P}|_{\hat{U}} \to \mathcal{P}_0|_{U_0},$$

which clearly factors further through $F_U^{*r}\mathcal{P}|_{\hat{U}} \overset{\mathrm{mod}\ p}{\twoheadrightarrow} F_{U_0}^{*r}\mathcal{P}_0|_{U_0}$. Thus we obtain a morphism $F_{U_0}^{*r}\mathcal{P}_0|_{U_0} \to \mathcal{P}_0|_{U_0}$ which is denoted by $[\frac{\phi_{r,F_U}}{p^{r-1}}]$.

Lemma 5.13. *The local morphisms $\{[\frac{\phi_{r,F_U}}{p^{r-1}}]\}_{U\in\mathcal{U}}$ glue into a global one, $\tilde{F}_{rel}: F_{M_0}^{*r}\mathcal{P}_0 \to \mathcal{P}_0$.*

Proof. It is equivalent to show the following statement: for two different Frobenius liftings F_U and F'_U of the absolute Frobenius F_{U_0}, and for a local section of $(F_{M_0}^{*r}\mathcal{P}_0)(U_0)$ of form $F_{U_0}^{*r}s_0$ with $s_0 \in \mathcal{P}_0(U_0)$, one has the equality $[\frac{\phi_{r,F_U}}{p^{r-1}}](F_{U_0}^{*r}s_0) = [\frac{\phi_{r,F'_U}}{p^{r-1}}](F_{U_0}^{*r}s_0)$. Let s be an element of $\mathbf{E}_0(\hat{U})$ lifting s_0. It suffices to show that $(\frac{\phi_{r,F'_U}}{p^{r-1}}F_U'^{*r} - \frac{\phi_{r,F_U}}{p^{r-1}}F_U^{*r})(s) \in p\mathbf{E}_0(\hat{U})$. Note that by replacing the Frobenius ϕ_r of $\mathbf{E}_0$ with $\frac{\phi_r}{p^{r-1}}$ one obtains another object in $\mathcal{MF}^{\nabla}_{big,r}(M)$, which is denoted by $\mathbf{E}'_0$. Let $x_0 \in \hat{U}(k)$ and $\hat{U}_{x_0}$ be the completion of U at x_0. Fix an isomorphism $\hat{U}_{x_0} \cong W(k)[[t]]$. Then F_U and F'_U restrict to two Frobenius liftings on $\hat{U}_{x_0}$. For any local section s' of $\mathbf{E}'_0(\hat{U}_{x_0})$, one has the Taylor formula (see §7 [11], Theorem

2.3 [10], page 16 [15]): write $\partial = \partial_t$ and $z = F'_U(t) - F_U(t)$. Then it holds that

$$\frac{\phi_{r,F'_U}}{p^{r-1}} F'^*_U(s') = \sum_{i=0}^{\infty} \frac{\phi_{r,F_U}}{p^{r-1}} F^*_U(\nabla^i_{\partial}(s')) \otimes \frac{z^i}{i!}.$$

Note that as z is divisible by p, $\frac{z^i}{i!}$ is divisible by p for all $i \geq 1$. So the difference $\frac{\phi_{r,F'_U}}{p^{r-1}} F'^*_U(s') - \frac{\phi_{r,F_U}}{p^{r-1}} F^*_U(s')$ belongs to $p\mathbf{E}'_0(\hat{U}_{x_0})$. The lemma follows. □

Let $\mathcal{S} \subset M_0(\bar{k})$ be the supersingular locus of $f_0 : X_0 \to M_0$.

Proposition 5.14. *The morphism $\tilde{F}_{rel}$ is nonzero and takes zero at $x_0 \in M_0(\bar{k})$ iff $x_0 \in \mathcal{S}$.*

Proof. The morphism $\tilde{F}_{rel}$ is nonzero because of Lemma 5.12. And when and only when it takes zero at x_0, the Newton slopes of the factors M_i in the proof of Lemma 5.12 take values in $\{2 \times 1\}$, which by the proof of Theorem 3.17 implies that $x_0 \in \mathcal{S}$. □

5.4. **A mass formula.** In this subsection we deduce a mass formula for the supersingular locus $\mathcal{S}$ from the pair $(\mathcal{P}_0, \tilde{F}_{rel})$. It is clear that we shall determine the multiplicity of the Frobenius degeneracy at a supersingular point. To that end we have the following result:

Proposition 5.15. *The vanishing order of $\tilde{F}_{rel}$ at each supersingular point is two.*

This is a local statement. Take an $x_0 \in \mathcal{S} \cap M_0(k)$. By discussions in §4.1, there is a Drinfel'd $\mathcal{O}_{\mathfrak{p}}$-divisible module B' such that Corollary 4.13 holds. It is also clear that B' is supersingular. In this case, it is a formal p-divisible group. By Proposition 5.7, the above statement can be deduced from the corresponding result for the universal filtered Dieudonné module associated to a versal deformation of a Drinfel'd $\mathcal{O}_{\mathfrak{p}}$-divisible module. To this end we shall apply the theory of display for a local expression of the Frobenius. Note that $\mathrm{Sym}^2\mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1} \det(\mathcal{N}_i)$ is contained as a direct factor in $\bigwedge^2(\bigotimes_{i=0}^{r} \mathcal{N}_i)$ (resp. $\mathrm{Sym}^2(\bigotimes_{i=0}^{r} \mathcal{N}_i)$) for r even (resp. odd). The induced Frobenius on the factor $\mathrm{Sym}^2\mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1} \det(\mathcal{N}_i)$ from the second wedge/symmetric power of ϕ^r_{ten} on $\bigotimes_{i=0}^{r} \mathcal{N}_i$ is denoted by ${\phi^r_{ten}}^{\otimes 2}$. We then have the following.

Proposition 5.16. *The vanishing order of $\phi_{\mathcal{N}_0} \mod p$ on $\frac{\mathcal{N}_0}{Fil^1\mathcal{N}_0}$ along the equal characteristic deformation at the point $[B']$ is one, and that of $\frac{{\phi^r_{ten}}^{\otimes 2}}{p^{r-1}} \mod p$ on $\frac{\mathrm{Sym}^2\mathcal{N}_0}{Fil^1\mathrm{Sym}^2\mathcal{N}_0} \otimes \bigotimes_{i=1}^{r-1} \det \mathcal{N}_i$ is two.*

Proof. Note that it suffices to write the display over the equal-characteristic deformation (see [22], [23], §2 [13]). For simplicity we shall take $r = 2$ in the following argument. The proof for a general r is completely the same. Let (N, F, V) be the covariant Dieudonné module of the Cartier dual of the Drinfel'd $\mathcal{O}_{\mathfrak{p}}$-divisible module B' over $\bar{k}$. So we have the eigen-decomposition $N = N_0 \oplus N_1$ with respect to the endomorphism $\mathcal{O}_{\mathfrak{p}} \cong \mathbb{Z}_{p^2}$. Choose a basis $\{X_i, Y_i\}$ for $N_i, i = 0, 1$. To write the display, we need to arrange the order of the basis elements into $\{Y_0, X_1, Y_1, X_0\}$ with the understanding that X_0 modulo p is the basis element of $\frac{VN}{pN}$ which is a one dimensional $\bar{k}$-vector space. Then the display under the chosen basis is given by the matrix:

$$\begin{pmatrix} A_{3\times 3} & B_{3\times 1} \\ C_{1\times 3} & D_{1\times 1} \end{pmatrix} = \begin{pmatrix} 0 & c_1 & d_1 & 0 \\ b_1 & 0 & 0 & a_1 \\ b_2 & 0 & 0 & a_2 \\ 0 & c_2 & d_2 & 0 \end{pmatrix}.$$

This is an invertible matrix; i.e., $\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \cdot \det \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}$ is a unit. Since both determinants are elements in $W(\bar{k})$, it implies that each determinant is a unit in $W(\bar{k})$. The universal equal-characteristic deformation ring of B' as a p-divisible group is $\bar{k}[[t_0, t_1, t_2]]$. Let $T_i \in W(\bar{k}[[t_0, t_1, t_2]])$ be the Teichmüller lifting of t_i for $0 \leq i \leq 2$. Then by Norman [22] and Norman-Oort [23] the display over the universal equal-characteristic deformation is given by $\begin{pmatrix} A + TC & B + TD \\ C & D \end{pmatrix}$, where $T = \begin{pmatrix} T_0 & T_1 & T_2 \end{pmatrix}^t$, and the Frobenius on the universal display is given by

$$M_1 := \begin{pmatrix} A + TC & p(B + TD) \\ C & pD \end{pmatrix}.$$

We need to determine the one dimensional sublocus of $\mathrm{Spf}(\bar{k}[[t_0, t_1, t_2]])$ where B' deforms as a Drinfel'd module. Take $s \in \mathbb{Z}_{p^2}$ to be a primitive element. Then the endomorphism of N given by s has the matrix form (using the same basis):

$$M_2 := \begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \xi^\sigma & 0 & 0 \\ 0 & 0 & \xi^\sigma & 0 \\ 0 & 0 & 0 & \xi \end{pmatrix}.$$

The universal display of the Drinfel'd module has the property that the endomorphism matrix commutes with the Frobenius. That is, one has $M_1 M_2^\sigma = M_2 M_1$. Now by an easy computation one finds that the one dimensional deformation as the Drinfel'd module is given by $t_1 = t_2 = 0$. Write $t = t_0$. Thus the two-iterated Frobenius $\phi^2_{\mathcal{N}_{B'}}$ on $\mathcal{N}_{B'}$ along the equal-characteristic deformation is displayed by

$$\phi^2_{\mathcal{N}_{B'}}\{Y_0, X_1, Y_1, X_0\} = \{Y_0, X_1, Y_1, X_0\}\Phi,$$

where $\Phi = M_1 M_1^\sigma$ is equal to

$$\begin{pmatrix} \Phi_{11} & 0 & 0 & \Phi_{14} \\ 0 & \Phi_{22} & \Phi_{23} & 0 \\ 0 & \Phi_{32} & \Phi_{33} & 0 \\ \Phi_{41} & 0 & 0 & \Phi_{44} \end{pmatrix}.$$

The nontrivial entries are given by

$$\begin{aligned} \Phi_{11} &= (b_1^\sigma c_1 + b_2^\sigma d_1) + (b_1^\sigma c_2 + b_2^\sigma d_2)t, & \Phi_{14} &= (pa_1^\sigma c_1 + pa_2^\sigma d_1) + (pa_1^\sigma c_2 + pa_2^\sigma d_2)t, \\ \Phi_{22} &= (b_1 c_1^\sigma + pa_1 c_2^\sigma) + b_1 c_2^\sigma t^\sigma, & \Phi_{23} &= (b_1 d_1^\sigma + pa_1 d_2^\sigma) + b_1 d_2^\sigma t^\sigma, \\ \Phi_{32} &= (b_2 c_1^\sigma + pa_2 c_2^\sigma) + b_2 c_2^\sigma t^\sigma, & \Phi_{33} &= (b_2 d_1^\sigma + pa_2 d_2^\sigma) + b_2 d_2^\sigma t^\sigma, \\ \Phi_{41} &= b_1^\sigma c_2 + b_2^\sigma d_2, & \Phi_{44} &= pa_1^\sigma c_2 + pa_2^\sigma d_2. \end{aligned}$$

Consider first the element Φ_{11}: its modulo p reduction is equal to the iterated Hasse-Witt map on $\frac{\mathcal{N}_0}{Fil^1 \mathcal{N}_0}$. As we require that B' lies in the supersingular locus which is a finite set, it follows that

$$b_1^\sigma c_1 + b_2^\sigma d_1 = 0 \mod p, \qquad b_1^\sigma c_2 + b_2^\sigma d_2 \neq 0 \mod p.$$

This shows the first assertion in the statement. So we can write that $b_1^\sigma c_1 + b_2^\sigma d_1 = pv_1, b_1^\sigma c_2 + b_2^\sigma d_2 = u_1$, where u_1 is a unit. Consider the induced Frobenius on $\mathrm{Sym}^2\mathcal{N}_0 \otimes \bigwedge^2 \mathcal{N}_1$. We shall compute the coefficient before the element $Y_0^2 \otimes X_1 \wedge Y_1$, which is the basis element of $\frac{\mathrm{Sym}^2\mathcal{N}_0}{Fil^1\mathrm{Sym}^2\mathcal{N}_0} \otimes \det \mathcal{N}_1$, under the map $\frac{\phi_{ten}^{2\ \otimes 2}}{p} \mod p$. Using the above matrix expression of $\phi^2_{\mathcal{N}_{B'}}$ one computes that the local expression is given by

$$[-u_1^2 \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \det \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}^\sigma]t^2 + v_2 t^p + v_3 t^{p^2}.$$

By the previous discussion we know that the coefficient before t^2 is a unit. As p is assumed to be odd, it follows that the multiplicity is equal to two. □

Now the proof of Proposition 5.15 is clear:

Proof. By the construction of $\tilde{F}_{rel}$, its vanishing order at x_0 is equal to that of $\frac{\phi_r}{p^{r-1}} \mod p$ on $\frac{\mathbf{E}_0}{F^1\mathbf{E}_0}$ along $\hat{M}_{0,x_0}$. Note that the closed formal subscheme $\hat{M}_{0,x_0} \subset \hat{M}_{x_0}$ represents the equal-characteristic deformation direction. By Proposition 5.7, the restriction of $\mathbf{E}_0$ to $\hat{M}_{x_0}$ is naturally isomorphic to $\hat{\xi}_{crys}[\mathrm{Sym}^2\mathcal{N}_0 \otimes \bigotimes_{i=1}^{r-1} \det(\mathcal{N}_i)]$. Thus the result follows from Proposition 5.16. □

Corollary 5.17. *Let $\mathcal{S}$ be the supersingular locus of $f_0 : X_0 \to M_0$. Then in the Chow ring of $\bar{M}_0$ one has the cycle formula*

$$2\mathcal{S} = (1 - p^r)c_1(M_0),$$

where $r = [F_{\mathfrak{p}} : \mathbb{Q}_p]$. Consequently one has the mass formula

$$|\mathcal{S}| = (p^r - 1)(g - 1),$$

where g is the genus of M_0.

Proof. By Propositions 5.14 and 5.15, it follows that

$$2\mathcal{S} = (p^r - 1)c_1(\mathcal{P}_0).$$

By Proposition 5.11, one further has

$$c_1(\mathcal{P}_0) = -2c_1(\mathcal{L}_0) = -c_1(M_0).$$

By taking the degree of the cycle formula, one obtains the mass formula as claimed. □

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