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On the Chern number inequalities satisfied by all smooth complete intersection threefolds with ample canonical class

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We obtain all linear Chern number inequalities satisfied by any smooth complete intersection threefold with ample canonical bundle.

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1. Introduction

This small note is motivated by finding a new Chern number inequality for a smooth projective threefold X with ample canonical bundle. Let $c_i = c_i(T_X)$ be its Chern class for $i = 1, 2, 3$. Yau's famous inequality [4] in the three-dimensional case says that

$$8c_1c_2 \leq 3c_1^3,$$

with equality if and only if X is uniformized by the complex ball. As it contains no c_3 term, one may naturally wonder whether there exists a Chern number inequality involving c_3. This is possible because of the following result:

Theorem 1.1 (Chang–Lopez [2, Corollary 1.3]). *The region described by the Chern ratios* $(\frac{c_1^3}{c_1c_2}, \frac{c_3}{c_1c_2})$ *of smooth irreducible threefolds with ample canonical bundles is bounded.*

However, the result, as well as its proof, does not produce a new Chern number inequality, even for the subclass of smooth complete intersections. Before the discovery of a new method to handle the general case, it is valuable from a scientific standpoint to treat this subclass firstly by bare hands. This is what we are going to do here.

Our method is to determine the convex hull in $\mathbb{R}^2$ generated by Chern ratios $(\frac{c_1^3}{c_1c_2}, \frac{c_3}{c_1c_2})$ of all smooth complete intersection threefolds with ample canonical bundles. Let n be a natural number. A smooth complete intersection (SCI) threefold X in $\mathbb{P}^{n+3}$ is cut out by n hypersurfaces, and a nondegenerate one, i.e. not contained in a hyperplane, by n hypersurfaces of degrees $d_1+1, \ldots, d_n+1$ with $d_i \geq 1$ for $1 \leq i \leq n$. The Chern numbers of a smooth X is uniquely determined by the tuple $(d_1, \ldots, d_n)$. Therefore, we may use the notation

$$Q(n; d_1, \ldots, d_n) = \left(\frac{c_1^3}{c_1c_2}, \frac{c_3}{c_1c_2}\right) \in \mathbb{R}^2$$

for Chern ratios of X. Note that X has ample canonical bundle if and only if $\sum_{i=1}^n d_i \geq 5$. Put

$$Q = \left\{ Q(n; d_1, \ldots, d_n) \,|\, n \geq 1, d_i \geq 1, \sum_{i=1}^n d_i \geq 5 \right\} \subset \mathbb{R}^2.$$

Let P be the convex hull of Q. We obtain the following theorem.

Theorem 1.2. *P is a rational polyhedra with infinitely many faces. The corners of P are given by the following points*:

$$Q(1;5) = \left(\frac{1}{16}, \frac{43}{8}\right), \quad Q(2;2,3) = \left(\frac{1}{10}, \frac{19}{5}\right),$$

$$Q(3;2,3,3) = \left(\frac{1}{8}, \frac{13}{4}\right), \quad Q(3;2,2,2) = \left(\frac{1}{3}, \frac{23}{12}\right),$$

$$Q(n;1,\ldots,1) = \left(\frac{2(-4+n)^2}{12-5n+n^2}, \frac{-24+14n-3n^2+n^3}{3(-4+n)(12-5n+n^2)}\right), \quad n \geq 5.$$

Remark 1.1. In another work of Chang [1], she described a region R in the plane of Chern ratios such that any rational point in R can be realized by a SCI threefold with ample canonical bundle; Outside R there are infinitely many Chern ratios of smooth complete intersection threefolds but no accumulating points. These two results are related, but do not imply each other. See Fig. 1.

We proceed to deduce our main application of the above result. According to the values of their x-coordinates, we label the corner points of P as follows:

$$p_1 = Q(1;5), \quad p_2 = Q(2;2,3), \quad p_3 = Q(3;2,3,3),$$

$$p_4 = Q(5;1,1,1,1,1), \quad p_5 = Q(3;2,2,2), \quad p_n = Q(n;1,\ldots,1), \;\; n \geq 6.$$

The sequence of points $\{p_n\}$ converges to the point

$$p_\infty = \left(2, \frac{1}{3}\right).$$

The closure of P, denoted by $\bar{P}$, contains the points $p_n (n \geq 1), p_\infty$ as its corners.

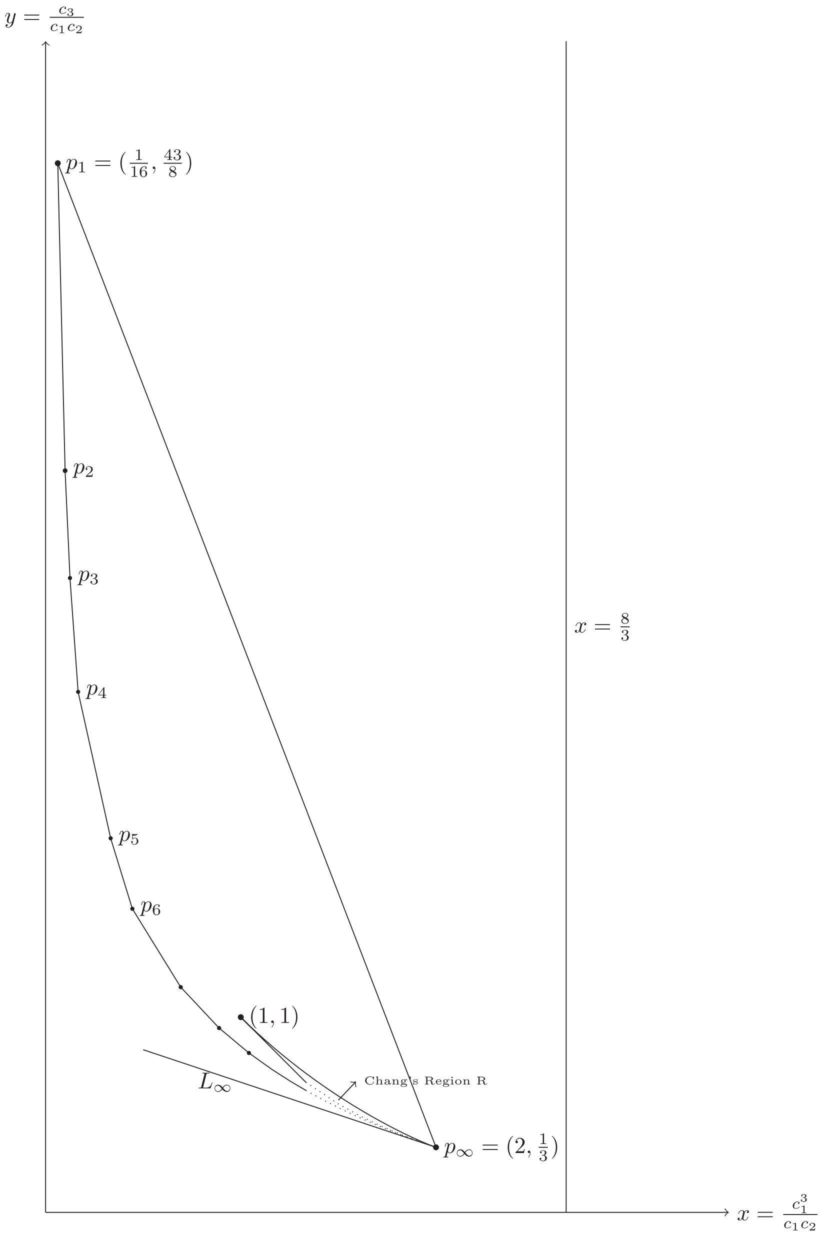

Fig. 1. Convex hull of Chern ratios.

For two distinct points $p, q \in \mathbb{R}^2$, denote the line through p, q by L_{pq}, and the line segment connecting p, q by pq. Denote the expressions of lines as follows:

$$L_{p_1 p_\infty} : y = k_0 x + b_0,$$
$$L_{p_m p_{m+1}} : y = k_m x + b_m, \quad m \geq 1.$$

The values of k_m, b_m are:

$$(k_0, b_0) = \left(-\frac{242}{93}, \frac{515}{93}\right), \quad (k_1, b_1) = (-42, 8), \quad (k_2, b_2) = (-22, 6),$$

$$(k_3, b_3) = (-14, 5), \quad (k_4, b_4) = \left(-\frac{9}{2}, \frac{41}{12}\right), \quad (k_5, b_5) = \left(-\frac{13}{5}, 3\right),$$

$$k_m = \frac{-28m + m^2 + 4m^3 - m^4}{(-4+m)(-3+m)(-20-5m+3m^2)}, \quad \forall\, m \geq 6,$$

$$b_m = \frac{-120 + 254m + 3m^2 - 50m^3 + 9m^4}{3(-4+m)(-3+m)(-20-5m+3m^2)}, \quad \forall\, m \geq 6.$$

The sequence of lines $L_{p_m p_{m+1}}$ converges to the line

$$L_\infty : y = k_\infty x + b_\infty,$$

where $k_\infty = -\frac{1}{3}, b_\infty = 1$.

Theorem 1.3. *Let C be the convex cone of linear inequalities satisfied by the Chern numbers of each SCI threefold with ample canonical bundle. That is,*

$$C = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \,|\, \lambda_1 c_1^3(X) + \lambda_2 c_1(X)c_2(X) + \lambda_3 c_3(X) \geq 0, \text{ for any SCI threefold } X \text{ with } K_X > 0\}.$$

Then C is a rational convex cone with edges

$$(-k_0, -b_0, 1), \quad (k_m, b_m, -1)(m \geq 1), \quad (k_\infty, b_\infty, -1),$$

where by an edge we mean a one-dimensional face.

Proof. Let $\check{C} \subset \mathbb{R}^3$ be the closure of the convex cone generated by the set

$$\{(c_1^3(X), c_1(X)c_2(X), c_3(X)) \in \mathbb{R}^3 \mid X \text{ is a SCI threefold with } K_X > 0\}.$$

Note that if X is a SCI threefold with $K_X > 0$, then $c_1(X)c_2(X) < 0$. Indeed, Yau's inequality gives us $8c_1(X)c_2(X) \leq 3c_1(X)^3$, and the ampleness of canonical bundle implies the inequality $c_1^3(X) < 0$.

Since $c_1(X)c_2(X) < 0$, it can be easily seen that

$$\check{C} = \{(\lambda x, \lambda, \lambda y) \,|\, \lambda \in \mathbb{R}_{\leq 0}, (x, y) \in \bar{P}\}.$$

By definition, C is the dual cone of $\check{C}$. By Theorem 1.2, the codimensional one faces of $\check{C}$ are exactly the hyperplanes in $\mathbb{R}^3$ determined by the vectors $(-k_0, -b_0, 1)$, $(k_m, b_m, -1)(m \geq 1)$, $(k_\infty, b_\infty, -1)$. From the duality of C and $\check{C}$ we get that $(-k_0, -b_0, 1)$, $(k_m, b_m, -1)(m \geq 1)$, $(k_\infty, b_\infty, -1)$ are exactly the edges (one-dimensional faces) of C. □

Corollary 1.1. *If X is a SCI threefold with $K_X > 0$, then its Chern numbers satisfy the inequality $86c_1^3 \leq c_3 < \frac{c_1^3}{6}$, with the equality $86c_1^3 = c_3$ holds if and only if X is isomorphic to a degree six hypersurface in $\mathbb{P}^4$.*

Proof. It can be checked that

$$\frac{744}{229}(-k_0, -b_0, 1) + \frac{515}{229}(k_1, b_1, -1) = (-86, 0, 1).$$

By Theorem 1.3, $(-86, 0, 1) \in C$, hence we have

$$c_3 - 86c_1^3 \geq 0,$$

with equality holds iff

$$\left(\frac{c_1^3}{c_1c_2}, \frac{c_3}{c_1c_2}\right) = L_{p_1p_\infty} \cap L_{p_1p_2} = Q(1; 5),$$

which is equivalent to that X is isomorphic to a degree six hypersurface in $\mathbb{P}^4$.

Similarly,

$$\frac{93}{422}(-k_0, -b_0, 1) + \frac{515}{422}(k_\infty, b_\infty, -1) = \left(\frac{1}{6}, 0, -1\right).$$

By Theorem 1.3, we have

$$\frac{1}{6}c_1^3 - c_3 \geq 0,$$

with equality holds iff

$$\left(\frac{c_1^3}{c_1c_2}, \frac{c_3}{c_1c_2}\right) = L_{p_1p_\infty} \cap L_\infty = p_\infty.$$

Since p_∞ is not in Q, the inequality $\frac{1}{6}c_1^3 - c_3 \geq 0$ is in fact strict. □

Remark 1.2. In [3], the authors prove that for a smooth projective threefold X admitting a smooth fibration of minimal surfaces of general type over a curve, it holds that

$$c_3(X) \geq \frac{1}{18}c_1^3(X).$$

According to Corollary 1.1, this inequality can never be satisfied for a SCI threefold with ample canonical bundle. As pointed out by Professor Kang Zuo, this is actually explained by the Lefschetz hyperplane theorem. Indeed, the hyperbolicity of the moduli space of minimal surfaces of general type means that the base curve of a smooth fibration is a hyperbolic curve and hence the fundamental group of the total space X is nontrivial, in contrast with the triviality of the fundamental group of a SCI implied by the Lefschetz hyperplane theorem.

2. Proof of Theorem 1.2

Suppose $n \in \mathbb{N}$ is a positive integer, and $d_1, \ldots, d_n \in \mathbb{R}_{\geq 0}$ are non-negative real numbers. Let $s_j = \sum_{i=1}^n d_i^j$, $j \geq 1$. We define

$$c_1(n; d_1, \ldots, d_n) := 4 - s_1,$$

$$c_2(n; d_1, \ldots, d_n) := \frac{s_1^2 + s_2}{2} - 3(s_1 - 2),$$

$$c_3(n; d_1, \ldots, d_n) := -\frac{s_1^3 + 3s_1s_2 + 2s_3}{6} + (s_1^2 + s_2) - 3s_1 + 4.$$

If $d_1 = \cdots = d_n = d$, we denote $c_i(n; d_1, \ldots, d_n)$ by $c_i(n; d)$, $i = 1, 2, 3$.

The following result is standard.

Lemma 2.1. *Let X be a SCI threefold in $\mathbb{P}^{n+3}$. If X is a complete intersection of hypersurfaces with degrees $d_1 + 1, \ldots, d_n + 1$, and $\forall\, 1 \leq i \leq n$, $d_i \geq 1$, then the Chern classes of X are: $c_i(X) = c_i(n; d_1, \ldots, d_n)H^i$, $i = 1, 2, 3$, where H is the hyperplane class of $\mathbb{P}^{n+3}$.*

We divide the proof of Theorem 1.2 into three steps, corresponding to three subsections.

Step 1: In Sec. 2.1, we firstly prove the x-coordinate of any point in Q is between the x-coordinates of p_1 and p_∞. Then we prove any point of Q is below the line $L_{p_1p_\infty}$.

Step 2: In Sec. 2.2, we prove that any point of Q is above the line $L_{p_mp_{m+1}}$, $\forall\, m \geq 6$.

Step 3: In Sec. 2.3, we prove that $\forall\, i = 1, \ldots, 5$, if a point $Q(n; d_1, \ldots, d_n) \in Q$ has x-coordinate less than or equal to the x-coordinate of p_6, then $Q(n; d_1, \ldots, d_n)$ lies above the line segment p_ip_{i+1}.

After the three steps above, it is obvious that we have finished the proof of Theorem 1.2.

Ideas of the proof in each step:

In steps 1 and 2, the idea of the proof is the following:

Given a line $L : y = kx + b$ in $\mathbb{R}^2$, to prove Q is below L is equivalent to verify $\forall\, n, d_i \in \mathbb{N}, \sum_{i=1}^n d_i \geq 5$,

$$c_3(n; d_1, \ldots, d_n) - kc_1^3(n; d_1, \ldots, d_n) - bc_1(n; d_1, \ldots, d_n)c_2(n; d_1, \ldots, d_n) \geq 0,$$

and by the following Lemma 2.2, it suffices to verify $\forall\, n \in \mathbb{N}, d \in \mathbb{R}, d \geq 1, nd \in \mathbb{N}, nd \geq 5$,

$$c_3(n; d) - kc_1^3(n; d) - bc_1(n; d)c_2(n; d) \geq 0.$$

Similarly, in order to prove Q is above a line $y = kx + b$, it suffices to verify $\forall\, n \in \mathbb{N}, d \in \mathbb{R}, d \geq 1, nd \in \mathbb{N}, nd \geq 5$,

$$kc_1^3(n; d) + bc_1(n; d)c_2(n; d) - c_3(n; d) \geq 0.$$

Lemma 2.2. *Let $\lambda, \mu, \nu \in \mathbb{R}$ be constants. $\forall\, m \in \mathbb{N}$, we have*

$$\inf\Bigg\{\lambda c_1^3(n; d_1, \ldots, d_n) + \mu c_1(n; d_1, \ldots, d_n)c_2(n; d_1, \ldots, d_n) + \nu c_3(n; d_1, \ldots, d_n) \mid n \in \mathbb{N}, d_i \in \mathbb{R}, \sum_{i=1}^n d_i = m, d_i \geq 1, \forall\, i = 1, \ldots, n\Bigg\}$$
$$\geq \inf\{\lambda c_1^3(n; d) + \mu c_1(n; d)c_2(n; d) + \nu c_3(n; d) \mid n \in \mathbb{N}, d \in \mathbb{R}, nd = m, d \geq 1\}$$

Proof. This lemma is a direct consequence of the following elementary proposition. □

Proposition 2.1. *Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be non-negative real numbers, $s_j = \sum_{i=1}^{n} d_i^j$, $j = 1, 2, 3$, and $\lambda, \mu \in \mathbb{R}$ be constants. For fixed n and s_1, there exists a natural number $k \leq n$, such that the function $\lambda s_2 + \mu s_3$ attains its minimal value when $d_1 = \cdots = d_k = 0$, and $d_{k+1} = \cdots = d_n = \frac{s_1}{n-k}$.*

In step 3, we firstly prove that there are only finite points of Q with x-coordinates less than or equal to the x-coordinate of p_6, then we verify case-by-case that if a point of Q has x-coordinate less than or equal to the x-coordinate of p_6, it lies above the union of line segments $\bigcup_{i=1}^{5} p_i p_{i+1}$.

2.1.

We first give an estimate of the x-coordinates of points in Q.

Lemma 2.3. *The x-coordinate of any point of Q is between the x-coordinates of p_1 and p_∞.*

Proof. Recall the x-coordinates of p_1 and p_∞ are $\frac{1}{16}$ and 2, respectively. For any point $Q(n; d_1, \ldots, d_n)$ in Q, by Lemma 2.1, its x-coordinate is

$$\frac{c_1(n; d_1, \ldots, d_n)^3}{c_1(n; d_1, \ldots, d_n)c_2(n; d_1, \ldots, d_n)} = \frac{2(4 - s_1)^2}{s_1^2 + s_2 - 6(s_1 - 2)},$$

where $s_j = \sum_{i=1}^{n} d_i$, $j = 1, 2$.

What we want to prove is equivalent to

$$\frac{1}{16} \leq \frac{2(4 - s_1)^2}{s_1^2 + s_2 - 6(s_1 - 2)} \leq 2.$$

Since $s_1 \geq 5$, the inequalities above can be verified easily. □

Recall the line $L_{p_1 p_\infty}$ has the expression $y = k_0 x + b_0$, where $k_0 = -\frac{242}{93}$, $b_0 = \frac{515}{93}$. According to the argument before Lemma 2.2, in order to prove Q is below $L_{p_1 p_\infty}$, it suffices to prove the following:

Lemma 2.4. $\forall\, n \in \mathbb{N}, d \in \mathbb{R}, nd \in \mathbb{N}, nd \geq 5, d \geq 1,$

$$f(n, d) := c_3(n; d) - k_0 c_1^3(n; d) - b_0 c_1(n; d) c_2(n; d) \geq 0.$$

Proof. Let $s_1 = nd$, we have

$$f(n, d) = \tilde{f}(s_1, d) = \frac{1}{93}(3500 - 2625 s_1 - 937 d s_1 - 31 d^2 s_1 + 422 s_1^2 + 211 d s_1^2).$$

Since $\tilde{f}(s_1, d)$ is a quadratic polynomial of d with negative leading term, and $1 \le d \le s_1$, we have $\tilde{f}(s_1, d) \ge \mathrm{Min}\{\tilde{f}(s_1, 1), \tilde{f}(s_1, s_1)\}$. By computations,

$$f(s_1, 1) = \frac{1}{93}(3500 - 3593s_1 + 633s_1^2),$$

$$f(s_1, s_1) = \frac{5}{93}(700 - 525s_1 - 103s_1^2 + 36s_1^3).$$

It is elementary to verify the above two polynomials of s_1 are non-negative when $s_1 \in \mathbb{N}$ and $s_1 \ge 5$. Hence $f(n, d) = \tilde{f}(s_1, d) \ge 0, \forall n \in \mathbb{N}, d \in \mathbb{R}, nd \in \mathbb{N}, nd \ge 5, d \ge 1$. □

2.2.

In this section, we prove that Q is above the line $L_{p_m p_{m+1}}$, $\forall m \ge 6$. Recall the line $L_{p_m p_{m+1}}$ has an expression $y = k_m x + b_m$, where

$$k_m = \frac{-28m + m^2 + 4m^3 - m^4}{(-4 + m)(-3 + m)(-20 - 5m + 3m^2)},$$

$$b_m = \frac{-120 + 254m + 3m^2 - 50m^3 + 9m^4}{3(-4 + m)(-3 + m)(-20 - 5m + 3m^2)}.$$

According to the argument before Lemma 2.2, to prove Q is above the line $L_{p_m p_{m+1}}$, we need to study the non-negativity of the function

$$f(m, n, d) := k_m c_1^3(n; d) + b_m c_1(n; d) c_2(n; d) - c_3(n; d).$$

We have the following lemma.

Lemma 2.5. $f(m, n, d) \ge 0$, *if one of the following conditions holds*:

(i) $m, n \in \mathbb{N}, d \in \mathbb{R}, d \ge 1, nd \in \mathbb{N}, nd \ge 5, m \ge 10$;
(ii) $m, n \in \mathbb{N}, d \in \mathbb{R}, d \ge 1, nd \in \mathbb{N}, nd \ge 11, m = 6, 7, 8, 9$.

Proof. Let $s_1 = nd$. In the new variable m, s_1, d, we denote the function $f(m, n, d)$ by $\tilde{f}(m, s_1, d)$, then by the expressions of k_m, b_m and $c_i(n; d)$,

$$\begin{aligned} \tilde{f}(m, s_1, d) &= f(m, n, d) \\ &= k_m(4 - s_1)^3 + b_m(4 - s_1)\left(\frac{s_1^2}{2} - 3s_1 + 6\right) + \frac{s_1^3}{6} - s_1^2 + 3s_1 - 4 \\ &\quad + \frac{s_1 d^2}{3} + \left(\frac{s_1^2}{2} - s_1 + \frac{b_m(4 - s_1)s_1}{2}\right) d. \end{aligned}$$

Suppose condition (i) holds, since $\tilde{f}(m, s_1, d)$ is a quadratic polynomial of d, we have two cases:

Case I:

$$\frac{-(s_1^2/2 - s_1 + b_m(4 - s_1)s_1/2)}{2s_1/3} \le 1.$$

In this case, $\tilde{f}(m, s_1, d) \geq \tilde{f}(m, s_1, 1) = f(m, s_1, 1)$, and $f(m, s_1, 1) \geq 0$ is equivalent to that the point $Q(s_1; 1, \ldots, 1)$ lies above the line $L_{p_m p_{m+1}}$, which can be easily verified under the condition $s_1 \geq 5$.

Case II:

$$\frac{-(s_1^2/2 - s_1 + b_m(4 - s_1)s_1/2)}{2s_1/3} \geq 1.$$

In this case, $s_1 \geq \frac{12b_m - 2}{3b_m - 3}$, and

$$\tilde{f}(m, s_1, d) \geq \tilde{f}\left(m, s_1, \frac{-(s_1^2/2 - s_1 + b_m(4 - s_1)s_1/2)}{2s_1/3}\right).$$

By computations, we get

$$g(m, s_1) := 12(-4 + m)^2(-3 + m)^2(-20 - 5m + 3m^2)^2 \cdot \tilde{f}\left(m, s_1, \frac{-(s_1^2/2 - s_1 + b_m(4 - s_1)s_1/2)}{2s_1/3}\right)$$

is a cubic polynomial of s_1 with polynomial coefficients of m. We only need to verify the positivity of $g(m, s_1)$.

Again, by computations, we get that if $m \in \mathbb{N}, m \geq 10$, then

$$g\left(m, \frac{12b_m - 2}{3b_m - 3}\right) > 0, \quad \frac{\partial g}{\partial s_1}\left(m, \frac{12b_m - 2}{3b_m - 3}\right) > 0,$$
$$\frac{\partial^2 g}{\partial s_1^2}\left(m, \frac{12b_m - 2}{3b_m - 3}\right) > 0, \quad \frac{\partial^3 g}{\partial s_1^3}\left(m, \frac{12b_m - 2}{3b_m - 3}\right) > 0.$$

Note $g(m, s_1)$ is a cubic polynomial of s_1, the positivity of $g(m, s_1)$ follows from the above computations, and we have verified the conclusion under condition (i).

Suppose condition (ii) holds, by computations, we have

$$\frac{12b_m - 2}{3b_m - 3} < 10, \quad \forall\, m = 6, 7, 8, 9.$$

This inequality and condition (ii) imply $s_1 > \frac{12b_m - 2}{3b_m - 3} + 1 > \frac{12b_m - 2}{3b_m - 3}$. We have

$$\tilde{f}(m, s_1, d) \geq \tilde{f}\left(m, s_1, \frac{-(s_1^2/2 - s_1 + b_m(4 - s_1)s_1/2)}{2s_1/3}\right).$$

Again let

$$g(m, s_1) := 12(-4 + m)^2(-3 + m)^2(-20 - 5m + 3m^2)^2 \cdot \tilde{f}\left(m, s_1, \frac{-(s_1^2/2 - s_1 + b_m(4 - s_1)s_1/2)}{2s_1/3}\right)$$

which is a cubic polynomial of s_1. We only need to show the positivity of $g(m, s_1)$. By direct computations,

$$g\left(m, \frac{12b_m - 2}{3b_m - 3} + 1\right) > 0, \quad \frac{\partial g}{\partial s_1}\left(m, \frac{12b_m - 2}{3b_m - 3} + 1\right) > 0,$$

$$\frac{\partial^2 g}{\partial s_1^2}\left(m, \frac{12b_m - 2}{3b_m - 3} + 1\right) > 0, \quad \frac{\partial^3 g}{\partial s_1^3}\left(m, \frac{12b_m - 2}{3b_m - 3} + 1\right) > 0.$$

Since $g(m, s_1)$ is a cubic polynomial of s_1, the positivity of $g(m, s_1)$ follows from the above computations. We have verified the conclusion under condition (ii). □

From Lemmas 2.2 and 2.5, we get that $Q(n; d_1, \ldots, d_n) \in \mathcal{Q}$ lies above the line $L_{p_m p_{m+1}}$, if one of the following conditions holds:

(i) $m \geq 10$;
(ii) $m = 6, 7, 8, 9, \sum_{i=1}^{n} d_i \geq 11$.

A case-by-case verification shows that if $5 \leq \sum_{i=1}^{n} d_i \leq 10$, then $Q(n; d_1, \ldots, d_n)$ lies above the line $L_{p_m p_{m+1}}$, $\forall\, m = 6, 7, 8, 9$. So we have verified $\mathcal{Q}$ lies above the line $L_{p_m p_{m+1}}$, $\forall\, m \geq 6$.

2.3.

In this section, we prove that $\forall\, i = 1, \ldots, 5$, if a point $Q(n; d_1, \ldots, d_n) \in \mathcal{Q}$ has x-coordinate less than or equal to the x-coordinate of p_6, then $Q(n; d_1, \ldots, d_n)$ lies above the line segment $p_i p_{i+1}$.

Note that the x-coordinate of p_6 is $\frac{4}{9}$. The following lemma tells us that there are only finite points in $\mathcal{Q}$ with x-coordinates less than or equal to the x-coordinate of p_6.

Lemma 2.6. *If* $s_1 = \sum_{i=1}^{n} d_i \geq 10$, *then*

$$\frac{c_1(n; d_1, \ldots, d_n)^3}{c_1(n; d_1, \ldots, d_n) c_2(n; d_1, \ldots, d_n)} = \frac{(4 - s_1)^2}{(s_1^2 + s_2)/2 - 3(s_1 - 2)} > \frac{4}{9}.$$

Proof. Since $s_1^2 \geq s_2$, we have

$$s_1 \geq 10 \Rightarrow s_1^2 - 12s_1 + 24 > 0 \Rightarrow 5s_1^2 - 60s_1 + 120 > 0$$

$$\Rightarrow 7s_1^2 - 60s_1 + 120 > 2s_1^2 \Rightarrow 7s_1^2 - 60s_1 + 120 > 2s_2$$

$$\Rightarrow \frac{(4 - s_1)^2}{(s_1^2 + s_2)/2 - 3(s_1 - 2)} > \frac{4}{9}.$$

□

A case-by-case verification shows that the finite points $\{Q(n, d_1, \ldots, d_n), 5 \leq \sum_{i=1}^{n} d_i \leq 9\}$ lie above the lines $\{L_{p_i p_{i+1}}, 1 \leq i \leq 5\}$. By these verifications and the above lemma, we have shown that, once the x-coordinate of a point $Q(n; d_1, \ldots, d_n)$ is less than or equal to that of p_6, it lies above the line $L_{p_i p_{i+1}}$, $\forall\, 1 \leq i \leq 5$. This completes the proof of Theorem 1.2.

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