

ON THE COHOMOLOGY GROUPS OF LOCAL SYSTEMS OVER HILBERT MODULAR VARIETIES VIA HIGGS BUNDLES

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Abstract. Let X be a Hilbert modular variety and \mathbb{V} a non-trivial local system over X with infinite monodromy. In this paper we study Saito's mixed Hodge structure (MHS) on the cohomology group $H^k(X, \mathbb{V})$ using the method of Higgs bundles. Among other results we prove the Eichler-Shimura isomorphism, give a dimension formula for the Hodge numbers and show that the mixed Hodge structure is split over \mathbb{R} . These results are analogous to [24] in the cocompact case and complement the results in [12] for constant coefficients.

1. Introduction. Consider the Lie group $G = \mathrm{SL}(2, \mathbb{R})^n \times U$, where U is connected and compact. In their classical work [24] Matsushima and Shimura study the cohomology groups $H^*(X, \mathbb{V})$ with values in a local system \mathbb{V} attached to a linear representation of G on a compact quotient X of a product of upper half planes by a discrete subgroup $\Gamma \subset G$, see Section 1 in [24]. The main result of [24] is a dimension formula for the Hodge numbers of the pure Hodge structure on $H^*(X, \mathbb{V})$.

The arguments and results of the present paper grew out of an attempt to find a generalization of the results in [24]. The use of the maximum principle in the proof of the vanishing result of Theorem 3.1 in [24] presents an obvious difficulty for a direct generalization to the non-compact case. The case of Hilbert modular surfaces was already studied in the thesis [37] of the second named author. The final approach we have taken is a mixture of the one of Zucker in [41] for general locally symmetric varieties and the original theory of harmonic forms in [24] for discrete quotients of products of upper half planes. The technique of Higgs bundles made it possible to combine both methods effectively. The vanishing theorem of Mok on locally homogenous vector bundles [26] in the case of Hilbert modular varieties is indispensable to obtain our results for *all* non-trivial local systems with infinite monodromy groups, i.e., also non-regular ones. We are aware of the fact that some of our results can be also explained in an automorphic setting. Harris and Zucker [15, 16, 17] have developed a general framework using automorphic forms which is related to our work through the BGG-complex. However our approach is

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purely Hodge theoretic and can be applied to other Shimura varieties and to more general non-locally-homogenous situations. For example, we have started to apply the same machinery to orthogonal and unitary Shimura varieties (see [6, 25, 27]).

Now let X be a Hilbert modular variety and \mathbb{V} an irreducible local system over X . Then the cohomology group $H^k(X, \mathbb{V})$ carries a natural real MHS, which is the principal object of study in the present paper. The case of a constant local system has been treated in the book [12] by Freitag. The relevant result is Theorem 7.9, Ch. III in *ibid.* where a formula for the Hodge numbers of the real mixed Hodge structures $H^k(X, \mathbb{R}), 0 \leq k \leq 2n$ is provided. Using with Proposition 7.4 of the current paper and Propositions 7.2 and 7.7 in *ibid.*, one can even show that the real MHS on the cohomology group is actually split. Note that when \mathbb{V} has finite monodromy, it becomes trivial after a finite étale base change. Thus we shall assume throughout the paper that \mathbb{V} is a non-trivial local system with infinite monodromy. Any such irreducible \mathbb{V} is induced by an irreducible linear representation of G and is of the form \mathbb{V}_m for a certain n -tuple $m = (m_1, \dots, m_n) \in \mathbb{N}_0^n$. For such an m , one puts $|m| = \sum_{i=1}^n m_i$, and for any subset $I \subset \{1, \dots, n\}$, $|m_I| = \sum_{i \in I} m_i$. Each $\mathbb{V} = \mathbb{V}_m$ underlies a natural real polarized variation of Hodge structure (\mathbb{R} -PVHS) $\mathbb{V}_{\mathbb{R}}$ of weight $|m|$. After the work of Deligne, Saito and Zucker [8, 29, 30, 31, 32, 40], the cohomology group $H^k(X, \mathbb{V}_{\mathbb{R}})$ carries a natural real MHS with weights $\geq |m| + k$ (Saito's MHS). This MHS is defined over \mathbb{Q} if all m_i are equal. See Section 2 for details. Our main results are summarized as follows:

THEOREM 1.1. *Let X be a Hilbert modular variety of dimension $n \geq 2$ and \mathbb{V}_m be the irreducible non-trivial local system determined by $m = (m_1, \dots, m_n) \in \mathbb{N}_0^n$. Then:*

- (i) $H^k(X, \mathbb{V}_m) = 0$ for $0 \leq k \leq n-1$ and $k = 2n$.
- (ii) If $m_1 = \dots = m_n$, then for $n+1 \leq k \leq 2n-1$

$$\begin{aligned} h_k^{|m|+n, |m|+n} &:= \dim_{\mathbb{C}} F^{|m|+n} W_{2(|m|+n)} H^k(X, \mathbb{V}_m) \\ &= \dim_{\mathbb{C}} H^k(X, \mathbb{V}_m) = \binom{n-1}{k-n} h, \end{aligned}$$

where h is the number of cusps.

- (iii) If not all m_i are equal, then $H^k(X, \mathbb{V}_m) = 0$ for $n+1 \leq k \leq 2n-1$.
- (iv) One has the formula

$$\dim H^n(X, \mathbb{V}_m) = \delta(m)h + \sum_{I \subset \{1, \dots, n\}} h(I, m).$$

where $\delta(m) = 1$ if $m_1 = \dots = m_n$ is satisfied and otherwise zero, and $h(I, m)$ is the dimension of cusp forms on \mathbb{H}^n with respect to Γ_I (see Sections 5 and 7 for

more details). Moreover, for $P + Q = |m| + n$,

$$h_n^{P,Q} := \dim_{\mathbb{C}} Gr_F^P Gr_{\bar{F}}^Q Gr_{|m|+n}^W H^n(X, \mathbb{V}_m) = \sum_{\substack{I \subset \{1, \dots, n\}, \\ |m_I| + |I| = P}} h(I, m),$$

and $h_n^{|m|+n, |m|+n} = \delta(m)h$. Otherwise $h_n^{P,Q} = 0$.

For the convenience of the reader we give a short account on a conjecture of Harris-Zucker [17]. Recall that the Zucker conjecture, which has been solved by Looijenga and Saper-Stern via different methods, asserts the following statement.

THEOREM 1.2. (Looijenga [22] and Saper-Stern [34]) *Let M be a smooth arithmetic quotient of a Hermitian symmetric domain with M^* the Baily-Borel compactification. Let \mathbb{W} be a locally homogenous VHS over M . Let g (resp. h) be a group invariant metric on M (resp. \mathbb{W}), and let $H_{(2)}^k(M, \mathbb{W})$ be the L^2 -cohomology group of degree k with coefficients in \mathbb{W} with respect to the above metrics. Let $IH^k(M^*, \mathbb{W})$ be the k -th (middle perversity) intersection cohomology. Then one has a natural isomorphism*

$$r_k : H_{(2)}^k(M, \mathbb{W}) \cong IH^k(M^*, \mathbb{W}).$$

The case of Hilbert modular varieties had been known to Zucker [42] before the conjecture was proven in the general case. If \mathbb{W} is assumed to be an \mathbb{R} -VHS, then the L^2 -cohomology $H_{(2)}^k(M, \mathbb{W})$ carries a real Hodge structure by the theory of L^2 -harmonic forms and the intersection cohomology $IH^k(M^*, \mathbb{W})$ a real Hodge structure by Saito's theory on mixed Hodge modules. A priori, these two Hodge structures do not coincide under the isomorphism in the above theorem. The following conjecture remains open in general:

CONJECTURE 1.3. (Harris-Zucker [17, Conjecture 5.3]) *Assume \mathbb{W} to be an \mathbb{R} -VHS. Then the isomorphism in Theorem 1.2 is an isomorphism of real Hodge structures.*

Zucker showed some instances for the above conjecture, including the case of Hilbert modular surface with constant coefficients [43]. As a byproduct of our study of Saito's MHS on $H^k(X, \mathbb{V}_{\mathbb{R}})$, we obtain the following case of the conjecture:

THEOREM 1.4. *Let X be a Hilbert modular variety and X^* its Baily-Borel compactification. Let $\mathbb{V}_{\mathbb{R}}$ be an irreducible \mathbb{R} -VHS with infinite monodromy. Then the natural isomorphism $r_k : H_{(2)}^k(X^*, \mathbb{V}_{\mathbb{R}}) \cong IH^k(X^*, \mathbb{V}_{\mathbb{R}})$ is an isomorphism of real Hodge structures.*

Henceforth, we identify these two Hodge structures under the natural isomorphism in Theorem 1.2. Our next result is to show that Saito's MHS is split over

\mathbb{R} . Let $j : X \hookrightarrow X^*$ be the natural inclusion. Then it induces an injective morphism of MHS $IH^k(X^*, \mathbb{V}_{\mathbb{R}}) \cong H_{(2)}^k(X^*, \mathbb{V}_{\mathbb{R}}) \rightarrow H^k(X, \mathbb{V}_{\mathbb{R}})$ (see Proposition 7.1). We denote again by $IH^k(X^*, \mathbb{V}_{\mathbb{R}})$ the image of the embedding in the following. The theory of Eisenstein cohomology (see [13, 36]) provides a decomposition $H^k(X, \mathbb{V}_{\mathbb{R}}) = H_!^k(X, \mathbb{V}_{\mathbb{R}}) \oplus H_{\text{Eis}}^k(X, \mathbb{V}_{\mathbb{R}})$, where $H_!^k(X, \mathbb{V}_{\mathbb{R}})$ is the image of $H_c^k(X, \mathbb{V}_{\mathbb{R}})$, the cohomology of $\mathbb{V}_{\mathbb{R}}$ with compact supports, in $H^k(X, \mathbb{V}_{\mathbb{R}})$.

THEOREM 1.5. *For $n \leq k \leq 2n - 1$ let $(H^k(X, \mathbb{V}_{\mathbb{R}}), W_., F_.)$ be Saito's MHS. Then:*

(i) *For $n + 1 \leq k \leq 2n - 1$, one has $H^k(X, \mathbb{V}_{\mathbb{R}}) = H_{\text{Eis}}^k(X, \mathbb{V}_{\mathbb{R}})$ and the MHS on $H^k(X, \mathbb{V}_{\mathbb{R}})$ is pure Hodge-Tate of type $(|m| + n, |m| + n)$.*

(ii) *$IH^n(X^*, \mathbb{V}_{\mathbb{R}}) = H_!^n(X, \mathbb{V}_{\mathbb{R}})$ and $H^n(X, \mathbb{V}_{\mathbb{R}}) = IH^n(X^*, \mathbb{V}_{\mathbb{R}}) \oplus H_{\text{Eis}}^n(X, \mathbb{V}_{\mathbb{R}})$ is a natural splitting of MHS over \mathbb{R} into two pieces with weights $|m| + n$ and $2(|m| + n)$.*

Due to the splitting of the MHS $(H^k(X, \mathbb{V}_{\mathbb{R}}), W_., F_.)$ over \mathbb{R} , one has a bigrading $H^k(X, \mathbb{V}_m) = \bigoplus_{P+Q=k} H_k^{P,Q}$ with $H_k^{P,Q} = F^P \cap \bar{F}^Q \cap W_{P+Q, \mathbb{C}}$ satisfying

$$W_{l, \mathbb{C}} = \bigoplus_{P+Q \leq l} H_k^{P,Q}, \quad F^P = \bigoplus_{r \geq P} H_k^{r,s}.$$

In addition, we can give an algebraic description of the (non-zero) Hodge (P, Q) -components $H_k^{P,Q}$. This result can be regarded as a generalized *Eichler-Shimura isomorphism* for Hilbert modular varieties:

THEOREM 1.6. *Let \bar{X} be a smooth toroidal compactification of X (see [2]) with boundary divisor $S = \bar{X} - X$. For $1 \leq i \leq n$ let \mathcal{L}_i be the good extension (see [28]) to \bar{X} of the locally homogenous line bundle over X determined by the automorphy factor $c_i z_i + d_i$ (using the notation of Section 2). Then one has the following natural isomorphisms:*

(i) *For $n + 1 \leq k \leq 2n - 1$,*

$$H^k(X, \mathbb{V}_m) = H_k^{|m|+n, |m|+n} \cong H^{k-n} \left(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2} \right).$$

(ii) $H_n^{|m|+n, 0} \cong H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$,

$$H_n^{|m|+n, |m|+n} \cong H^0 \left(S, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2} |_S \right),$$

We changed the two displayed equations in items (i) and (ii) from inline equations to displayed ones to avoid the bad line breaks. Please check.

and for $0 \leq P \leq |m| + n - 1$, $P + Q = |m| + n$,

$$H_n^{P,Q} \cong \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |m_I| + |I| = P}} H^{k-|I|} \left(\tilde{X}, \bigotimes_{i \in I} \mathcal{L}_i^{m_i+2} \otimes \bigotimes_{i \in I^c} \mathcal{L}_i^{-m_i} \right).$$

In the above results, Theorem 1.1(i) and the first half of Theorem 1.5(i) for regular local systems are special cases of Li-Schwermer [21] (see also Saper [33]). Wildeshaus has recently informed us that the main result in [3] also implies Theorem 1.1(ii). Combined with Lemma 7.3, one is able to show then the second half of Theorem 1.5(i) for regular local systems.

The paper is organized as follows: Section 2 contains the basic set-up. In Section 3 we compute the logarithmic Higgs cohomology and present an algebraic description of the gradings of the Hodge filtration on cohomology. Section 5 studies the pure Hodge structure on the L^2 -cohomology which relies on the theory of L^2 -harmonic forms. This provides an L^2 -generalization of the results in [24]. Section 6 introduces Eisenstein cohomology. This has been extensively investigated in [12, 14] for constant coefficients and in [15, 16, 17, 21, 42] for non-constant coefficients. We complement these results by computing the Hodge type of Eisenstein cohomology. The last section combines all results and contains the proofs of the above theorems.

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2. Preliminaries and Saito's mixed Hodge structure. Let $F \subset \mathbb{R}$ be a totally real number field of degree $n \geq 2$ over \mathbb{Q} , with the set of real embeddings $\text{Hom}_{\mathbb{Q}}(F, \mathbb{R}) = \{\sigma_1 = \text{id}, \dots, \sigma_n\}$. Let \mathcal{O}_F be the integer ring of F and \mathcal{O}_F^* be the unit group of \mathcal{O}_F . For an element $a \in F$ one puts $a^{(i)} = \sigma_i(a)$, the i -th Galois conjugate of a . Let $G = R_{F|\mathbb{Q}} \text{SL}_2$ be the \mathbb{Q} -algebraic group obtained by Weil restriction. The set of real points $G(\mathbb{R})$ of G is identified with $G_1 \times \dots \times G_n$, where each G_i is a copy of $\text{SL}(2, \mathbb{R})$. The subset $G(\mathbb{Q}) \subset G(\mathbb{R})$ is then given by

$$\left\{ \left(\left(\begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix} \right) \in G_1 \times \dots \times G_n \mid a, b, c, d \in F \right\}.$$

Now let $\mathbb{H}^n = \mathbb{H}_1 \times \dots \times \mathbb{H}_n$ be the product of n copies of the upper half plane with coordinates

$$z = (z_1 = x_1 + iy_1, \dots, z_l = x_l + iy_l, \dots, z_n = x_n + iy_n).$$

The group $G(\mathbb{R})$ acts on \mathbb{H}^n by a product of linear fractional transformation. Namely, for $g = (g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}) \in G(\mathbb{R})$ and $z \in \mathbb{H}^n$, the action is given by $g \cdot z = (g_1 \cdot z_1, \dots, g_n \cdot z_n)$, where $g_i \cdot z_i = (a_i z_i + b_i)(c_i z_i + d_i)^{-1}$. The action is transitive and the isotropy subgroup of $G(\mathbb{R})$ at the base point $z_0 = (i, \dots, i)$ is $K = \mathrm{SO}(2)^n$, a maximally compact subgroup. K acts on G by right multiplication and one identifies the set G/K of left cosets naturally with \mathbb{H}^n . Let $\Gamma \subset G(\mathbb{Q})$ be a torsion-free subgroup which is commensurable with $G(\mathbb{Z}) \subset G(\mathbb{Q})$. It is called a *Hilbert modular group* in this paper and will be fixed throughout. By the theorem of Baily-Borel, the quotient space $X_\Gamma := \Gamma \backslash G(\mathbb{R})/K$ is naturally a smooth quasi-projective variety, which is called the *Hilbert modular variety* for Γ . As Γ is fixed, we shall denote X_Γ simply by X . The topological space X is non-compact and admits several natural compactifications. The Baily-Borel compactification X^* of X is obtained by adding a number h of *cusps* as a set (see [12]) and has the structure of a projective variety by Baily-Borel. It is however singular, and admits a natural family of resolutions of singularities, the so-called smooth toroidal compactifications (see [2] for general locally symmetric varieties and [9] more details for Hilbert modular varieties). Let \bar{X} be such a smooth compactification, and let $S = \bar{X} - X$ be the divisor at infinity, which has simple normal crossings. In addition, we also use the Borel-Serre compactification X^\sharp in Section 6. It is a smooth compact manifold with boundary, which contains X as the interior open subset. The boundary $\partial X^\sharp = X^\sharp - X$ has h components in total, with each component an $(S^1)^n$ -bundle over $(S^1)^{n-1}$.

For $A \subset \mathbb{C}$ a \mathbb{Q} -subalgebra, we define an A -local system over X as a locally constant sheaf of free A -modules of finite rank with respect to the analytic topology on X . Let 0 be the point of X given by the Γ -equivalence class of $z_0 \in \mathbb{H}^n$. Then it is well-known that an A -local system over X corresponds to a representation $\pi_1(X, 0) \rightarrow \mathrm{GL}(A)$. In this paper A is either \mathbb{Q} , \mathbb{R} or \mathbb{C} . Since $\pi_1(X, 0)$ is naturally identified with Γ , by the super-rigidity theorem of Margulis (see [23]), equivalence classes of complex local systems with infinite monodromy groups over X are in one-to-one correspondence with equivalence classes of finite dimensional complex representations of $G(\mathbb{R})$. Let \mathbb{V} be the complex local system corresponding to the irreducible representation $\rho : G(\mathbb{R}) \rightarrow \mathrm{GL}(V)$. By Schur's lemma, there exists an n -tuple $m = (m_1, \dots, m_n)$ of non-negative integers and n copies V_i of \mathbb{C}^2 for $i = 1, \dots, n$ such that $\rho = \rho_{m_1} \otimes \dots \otimes \rho_{m_n}$, where for each i , $\rho_{m_i} : G_i \rightarrow \mathrm{GL}(S^{m_i} V_i)$ is isomorphic to the m_i -th symmetric power of the standard complex representation of $\mathrm{SL}(2, \mathbb{R})$. A local system \mathbb{V} is called *regular* if ρ is a regular representation, i.e., its highest weight is contained in the interior of the Weyl chamber. It is clear that $\mathbb{V} = \mathbb{V}_m$ is regular if and only if each m_i in the above is positive. To summarize, for each $m \in \mathbb{N}_0^n$, there is a unique complex local system \mathbb{V}_m over X up to isomorphism, and any complex local system over X is a finite direct sum of such. For $(0, \dots, m_i, \dots, 0)$ we denote the corresponding local system by \mathbb{V}_{i, m_i} . So $\mathbb{V}_m = \mathbb{V}_{1, m_1} \otimes \dots \otimes \mathbb{V}_{n, m_n}$. The complex local system

$\mathbb{V} = \mathbb{V}_m$ is the complexification of a natural real local system $\mathbb{V}_{\mathbb{R}}$. Moreover, $\mathbb{V}_{\mathbb{R}}$ is naturally an \mathbb{R} -PVHS, which is a special case considered by Zucker (see [41]). This can be seen as follows. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis of \mathbb{C}^2 on which $\mathrm{SL}(2, \mathbb{R})$ acts by matrix multiplication. Then $\mathbb{R}^2 = \mathbb{R}e_1 + \mathbb{R}e_2 \subset \mathbb{C}^2$ is an invariant \mathbb{R} -structure. Define a symplectic form ω on \mathbb{R}^2 such that $\{e_1, e_2\}$ is the symplectic basis for ω . Let $\mathbb{C}^{1,0} = \mathbb{C}\{e_1 - ie_2\}$ and $\mathbb{C}^{0,1} = \mathbb{C}\{e_1 + ie_2\}$. Then the decomposition $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^{1,0} \oplus \mathbb{C}^{0,1}$ defines a polarized weight one Hodge structure on \mathbb{R}^2 , and this decomposition has the special property that it is also the eigen-decomposition of the induced action of $\mathrm{SO}(2)$ by restriction. For each $1 \leq i \leq n$, the i -th factor of $G(\mathbb{R})$ acts on V_i via the standard representation and trivially on any other factor. Applying the foregoing construction on \mathbb{C}^2 to V_i , one obtains a polarized weight one Hodge structure on the fiber of the constant bundle $\mathbb{H}^n \times V_i$ at $z_0 \in \mathbb{H}^n$. By using the homogeneity property of \mathbb{H}^n , one defines a \mathbb{R} -PVHS on the constant bundle $\mathbb{H}^n \times V_i$, and it descends to a \mathbb{R} -PVHS on $\mathbb{V}_{i,1}$ over X (see Section 4 in [41]). Taking the m_i -th symmetric power, one obtains a \mathbb{R} -PVHS of weight m_i on \mathbb{V}_{i,m_i} , and further by taking tensor products one obtains a \mathbb{R} -PVHS of weight $|m| = \sum_{i=1}^n m_i$ on \mathbb{V}_m as claimed. It is clear that $\mathbb{V}_{\mathbb{R}}$ is in fact defined over $F \subset \mathbb{R}$, and is even defined over \mathbb{Q} if (and only if) $m_1 = \cdots = m_n$ holds.

Now we consider an even more general setting. Let M be a quasi-projective manifold of dimension d and $(\mathbb{W}_{\mathbb{R}}, \nabla, F^\cdot)$ a \mathbb{R} -PVHS over M of weight n . Let \bar{M} be a smooth, projective compactification of M such that $S = \bar{M} - M$ is a simple normal crossing divisor. For simplicity of exposition, we assume that the local monodromy around each irreducible component of S is unipotent. Put $\mathbb{W}_{an} = \mathbb{W}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{O}_{M_{an}}$, where $\mathcal{O}_{M_{an}}$ is the sheaf of germs of holomorphic functions on M . Deligne's canonical extension (see Section 4, Ch. II in [7]) gives a unique extended vector bundle $\bar{\mathbb{W}}_{an}$ of \mathbb{W}_{an} over \bar{M} , together with a flat logarithmic connection $\bar{\nabla} : \bar{\mathbb{W}}_{an} \rightarrow \bar{\mathbb{W}}_{an} \otimes \Omega_{\bar{M}_{an}}^1(\log S)$. Using this we obtain the logarithmic de Rham complex $\Omega_{\log}^*(\bar{\mathbb{W}}_{an}, \bar{\nabla})$. By Deligne in loc. cit., the complex computes the cohomology groups $H^*(M, \mathbb{W}_{\mathbb{R}})$. Schmid's nilpotent orbit theorem implies that the Hodge filtration F^\cdot extends to a filtration \bar{F}^\cdot of holomorphic subbundles (not merely subsheaves) of $\bar{\mathbb{W}}_{an}$ as well (see Section 4 in [35]). By GAGA, the extended holomorphic objects over \bar{M} are in fact algebraic. One defines a Hodge filtration on the logarithmic de Rham complex by $F^r \Omega_{\log}^*(\bar{\mathbb{W}}_{an}, \bar{\nabla}) = \Omega_{\bar{M}}^*(\log S) \otimes \bar{F}^{r-*}$, which is a subcomplex by Griffiths transversality. After Saito (see [29]–[32]), there is a naturally defined weight filtration W on the logarithmic de Rham complex.

For convenience, we shall briefly recall the construction of Saito's MHS on $H^*(M, \mathbb{W}_{\mathbb{R}})$ (all the details can be found in [1]).

Extend $(\mathbb{W}_{an}, \nabla)$ to a vector bundle $\bar{\mathbb{W}}_{an}^I$ with a logarithmic connection such that the eigenvalues of its residues lie in a half open interval I of length one. Note that $\bar{\mathbb{W}}_{an} = \bar{\mathbb{W}}_{an}^{[0,1)}$ and define $\mathcal{M} := \bigcup_I \bar{\mathbb{W}}_{an}^I \subseteq j_* \mathbb{W}_{an}$. Then \mathcal{M} is a D_M -module

which corresponds to the perverse sheaf $\mathbb{R}j_*\mathbb{W}_{\mathbb{R}}[n] \otimes \mathbb{C}$. Filter this by

$$F_p\mathcal{M} = \sum_i F_i D_M F_{p-i} \bar{\mathbb{W}}_{an}^{(-1,0]}$$

Then according to [30]:

THEOREM 2.1. *There exist compatible filtrations W on $\mathbb{R}j_*\mathbb{W}_{\mathbb{R}}[n]$ and \mathcal{M} such that $(\mathbb{R}j_*\mathbb{W}_{\mathbb{R}}[n], W, \mathcal{M}, F)$ becomes a mixed Hodge module.*

Then one deduces Saito's MHS on $H^*(M, \mathbb{W}_{\mathbb{R}})$ by elaborating on a remark in [30]. There is a natural isomorphism in the derived category of sheaves of abelian groups:

$$\alpha : \mathbb{R}j_*\mathbb{W}_{\mathbb{R}} \otimes \mathbb{C} \cong \Omega_{\log}^*(\bar{\mathbb{W}}_{an}, \bar{\mathbb{V}}).$$

Using the isomorphism, the filtration on the left is transported into the Hodge filtration on the logarithmic de Rham complex described as above. There is also the induced weight filtration W on the complex. One constructs W by taking the filtration induced by $(\Omega_M^* \otimes W_{*+n}\mathcal{M})[n]$ under the inclusion

$$\Omega_M^*(\log S) \otimes \bar{\mathbb{W}}_{an} \subset \Omega_M^* \otimes \mathcal{M}[n].$$

We recall at this point that Deligne introduced a device, called a *cohomological mixed Hodge complex* [8, Section 8.1], for producing mixed Hodge structures. This consists of a bifiltered complex $(A_{\mathbb{C}}, W_{\mathbb{C}}, F^{\cdot})$ of sheaves of \mathbb{C} -vector spaces, a filtered complex of (A, W) of sheaves over \mathbb{R} , and a filtered quasi-isomorphism $(A, W) \otimes \mathbb{C} \cong (A_{\mathbb{C}}, W_{\mathbb{C}})$. The crucial axioms are that

- this datum should induce a pure weight $i + k$ real Hodge structure on $H^i(Gr_k^W A)$;
- the filtration induced by F^{\cdot} on Gr_k^W is strict, i.e., the map $H^i(F^p Gr_k^W A) \rightarrow H^i(Gr_k^W A)$ is injective.

PROPOSITION 2.2. *With these filtrations*

$$(\mathbb{R}j_*\mathbb{W}_{\mathbb{R}}, W; \Omega_M^*(\log S)^{\bullet} \otimes \bar{\mathbb{W}}_{an}, W, F^{\cdot}; \alpha)$$

becomes a cohomological mixed Hodge complex.

To verify the above axioms, one observes that $Gr_k^W \mathbb{R}j_*\mathbb{W}_{\mathbb{R}}$ decomposes into a direct sum of intersection cohomology complexes associated to pure variations of Hodge structure of the correct weight. An appeal to theorem of [5] or [19] shows that $H^i(Gr_k^W \mathbb{R}j_*\mathbb{W}_{\mathbb{R}})$ carries a pure Hodge structure of weight $k + i$. Strictness follows from similar considerations.

From [8, Section 8.1], we obtain Saito's MHS on $H^k(M, \mathbb{W}_{\mathbb{R}})$:

COROLLARY 2.3. *Notation as above. Then the cohomology group $H^k(M, \mathbb{W}_{\mathbb{R}})$ admits a real mixed Hodge structure of weight $n + k$. Namely, the Hodge filtration is induced by F^\cdot under the isomorphism*

$$H^k(M, \mathbb{W}_{\mathbb{R}} \otimes \mathbb{C}) \cong \mathbb{H}^k(\Omega_{\log}^*(\bar{\mathbb{W}}_{an}, \bar{\mathbb{V}})),$$

and the weight filtration is determined by

$$W_{l+k}H^k(M, \mathbb{W}_{\mathbb{R}} \otimes \mathbb{C}) := \mathbb{H}^k(X, W_l(\Omega_{\log}^*(\bar{\mathbb{W}}_{an}, \bar{\mathbb{V}}))).$$

There is an extra left delimiter or a missing right one. Please check.

Finally, we remark that in the simplest case $\mathbb{W}_{\mathbb{R}} = \mathbb{R}$, W above coincides with the filtration W defined by Deligne [8, Section 3.1]. In particular, Saito's MHS coincides with Deligne's in this case.

It is this MHS that we intend to understand properly in the case of Hilbert modular varieties.

After taking the graded object associated to the pair $(\bar{\mathbb{W}}_{an}, \bar{\mathbb{V}})$ with respect to the filtration \bar{F}^\cdot , one obtains the logarithmic Higgs bundle $(E = \bigoplus_{p+q=n} E^{p,q}, \theta = \bigoplus_{p+q=n} \theta^{p,q})$. Precisely, $E^{p,q} = \bar{F}^p / \bar{F}^{p+1}$ and $\theta^{p,q} = Gr_{\bar{F}}^p \bar{\mathbb{V}} : E^{p,q} \rightarrow E^{p-1,q+1} \otimes \Omega_{\bar{M}}^1(\log S)$ which is $\mathcal{O}_{\bar{M}}$ -linear. The Higgs field is integrable, i.e., $\theta \wedge \theta = 0$, and so one can form the *logarithmic Higgs complex* $\Omega_{\log}^*(E, \theta)$, which is the logarithmic version of the Higgs complex over a compact case. The complex appears on page 24 of [39] under the name holomorphic Dolbeault complex. More specifically, the p -th term of the complex $\Omega_{\log}^p(E, \theta) = E \otimes \Omega_{\bar{M}}^p(\log S)$ and the differential $\Omega_{\log}^p(E, \theta) \rightarrow \Omega_{\log}^{p+1}(E, \theta)$ is the composite

$$\begin{aligned} E \otimes \Omega_{\bar{M}}^p(\log S) &\xrightarrow{\theta \otimes \text{id}} [E \otimes \Omega_{\bar{M}}^1(\log S)] \otimes \Omega_{\bar{M}}^p(\log S) \\ &= E \otimes [\Omega_{\bar{M}}^1(\log S) \otimes \Omega_{\bar{M}}^p(\log S)] \xrightarrow{\text{id} \otimes \text{pr}} E \otimes \Omega_{\bar{M}}^{p+1}(\log S). \end{aligned}$$

Note that the sheaves and the differentials in this complex are algebraic. The hypercohomology of this complex is called *logarithmic Higgs cohomology*. One observes that the logarithmic Higgs complex is a direct sum $\bigoplus_{P=0}^{d+n} \Omega_P^*(E, \theta)$ of sub-complexes, where $\Omega_P^l(E, \theta) = E^{P-l, n-P+l} \otimes \Omega_{\bar{M}}^l(\log S)$. We define $\mathcal{C}^{P,l}(E, \theta)$ to be the l -th cohomology sheaf of the subcomplex $\Omega_P^*(E, \theta)$. The relation of logarithmic Higgs cohomology with Saito's MHS is given by the following

PROPOSITION 2.4. *Let $\mathbb{W} = \mathbb{W}_{\mathbb{R}} \otimes \mathbb{C}$ and F^\cdot be the Hodge filtration on $H^k(M, \mathbb{W})$ as above. For $0 \leq k \leq d$, one has a natural isomorphism*

$$Gr_{F^\cdot}^P H^k(M, \mathbb{W}) \cong \mathbb{H}^k(\bar{M}, \Omega_P^*(E, \theta)).$$

Proof. Deligne Scholie 8.1.9(v) [8] asserts that the Hodge filtration on a cohomological mixed Hodge complex degenerates at the E_1 -term. So the statement is just a direct consequence of Proposition 2.2. \square

When \mathbb{W} is constant, the Higgs field is trivial. Therefore the logarithmic Higgs cohomology is just the usual sheaf cohomology. In another important case the above result can also be improved:

PROPOSITION 2.5. *Assume that M is a smooth arithmetic quotient of a Hermitian symmetric domain and \bar{M} a smooth toroidal compactification. Moreover, assume that $\mathbb{W}_{\mathbb{R}}$ is locally homogenous. Then one has a natural isomorphism*

$$Gr_F^P H^k(M, \mathbb{W}_{\mathbb{R}}) \cong \bigoplus_{l=0}^d H^{k-l}(\bar{M}, \mathcal{C}^{P,l}(E, \theta)).$$

The proof is based on the one of Proposition 5.19 in [41] for compact M . The new ingredient is the relation between the Mumford extension of Higgs complex over M and the logarithmic Higgs complex over \bar{M} .

LEMMA 2.6. *Let $\Omega^*(E^0, \theta^0)$ be the Higgs complex of $\mathbb{W}_{\mathbb{R}}$ over M , and $\Omega_{\log}^*(E, \theta)$ the logarithmic Higgs complex of $\mathbb{W}_{\mathbb{R}}$ over \bar{M} . Then the cohomology sheaf of the logarithmic Higgs complex is the good extension of the corresponding cohomology sheaf in the sense of Mumford [28].*

Proof. The Higgs complex is just the restriction of the logarithmic Higgs complex to M . We consider the terms in the log complex. Each term is a tensor product of a wedge power of $\Omega_{\bar{M}}^1(\log S)$ and the Deligne-Schmid extension of $E^0 = E|_M$. Proposition 3.4 in [28] shows that $\Omega_{\bar{M}}^1(\log S)$ is the good extension of Ω_M^1 . The estimates of the Hodge metric, which is the invariant metric on E^0 up to scalar, provided by Theorem 5.21 in [4], shows that the sections of E are at most of logarithmic growth around S with respect to the Hodge metric. By the characterization of the good extension in [28, Proposition 1.3], one concludes that E is the good extension of E^0 . We consider also the bundle homomorphism: we may choose a local basis of group invariant sections for E^0 and conclude that θ^0 is a constant morphism in this basis, since, by homogeneity, θ^0 is determined by its value at one point. In the extended group invariant basis θ is also constant on \bar{M} . Therefore, it follows that the l -th cohomology sheaf $\mathcal{C}^{P,l}(E, \theta)$ of $\Omega_P^*(E, \theta)$ is the good extension of l -th cohomology of $\Omega_P^*(E^0, \theta^0)$ to \bar{M} , which is a direct summand of $\Omega_P^l(E, \theta)$. Moreover, the inclusion $\bigoplus_l \mathcal{C}^{P,l}(E, \theta)[-l] \hookrightarrow \Omega_P^*(E, \theta)$ is a quasi-isomorphism. \square

3. Logarithmic Higgs cohomology over a Hilbert modular variety. Let $\mathbb{V} = \mathbb{V}_m$ (resp. \mathbb{V}_{i,m_i}) be an irreducible complex local system over the Hilbert modular variety X . After a possible finite étale base change of X , the local monodromies of \mathbb{V} at infinity can be made unipotent, so we assume this from now on. This assumption can be removed once a result is unaffected by finite étale base change. Let (E_m, θ_m) (resp. $(E_{i,m_i}, \theta_{i,m_i})$) be the resulting logarithmic Higgs bundle over \bar{X} by the construction in Section 2. It is clear that the construction is

compatible with direct sums and tensor products, and hence

$$(E_m, \theta_m) = (E_{1,m_1}, \theta_{1,m_1}) \otimes \cdots \otimes (E_{n,m_n}, \theta_{n,m_n}).$$

In this section the logarithmic Higgs cohomology of (E_m, θ_m) will be determined.

We start with a basic property of the set $\{(E_{i,1}, \theta_{i,1}), 1 \leq i \leq n\}$ of logarithmic Higgs bundles.

Definition 3.1. For $1 \leq i \leq n$ define \mathcal{L}_i to be the good extension of the locally homogenous line bundle over X corresponding to the automorphy factor $c_i + d_i z_i$.

The \mathcal{L}_i 's are locally free, either by the proof of Main Theorem 3.1 [28], or by noticing that \mathcal{L}_i is the extended Hodge filtration of $\bar{\mathbb{V}}_{i,1,an}$ by the nilpotent orbit theorem.

PROPOSITION 3.2. *For each i , $E_{i,1} = E_{i,1}^{1,0} \oplus E_{i,1}^{0,1}$ with $E_{i,1}^{1,0} = \mathcal{L}_i$ and $E_{i,1}^{0,1} = \mathcal{L}_i^{-1}$. There exists a natural isomorphism $\Omega_{\bar{X}}^1(\log S) = \bigoplus_{i=1}^n \mathcal{L}_i^2$ such that $\theta_{i,1}^{1,0} : \mathcal{L}_i \rightarrow \mathcal{L}_i^{-1} \otimes \Omega_{\bar{X}}^1(\log S)$ is the composition of the tautological maps:*

$$\mathcal{L}_i \xrightarrow{\cong} \mathcal{L}_i^{-1} \otimes \mathcal{L}_i^2 \hookrightarrow \mathcal{L}_i^{-1} \otimes \Omega_{\bar{X}}^1(\log S).$$

Proof. This follows from the fact that the period map for a Hilbert modular variety is an embedding together with uniqueness of the Mumford extension. \square

PROPOSITION 3.3. *For any subset $I \subset \{1, \dots, n\}$ define I^c to be the complement of I , $|m_I| = \sum_{i \in I} m_i$ and $\mathcal{C}_I = \bigotimes_{i \in I} \mathcal{L}_i^{m_i+2} \otimes \bigotimes_{i \in I^c} \mathcal{L}_i^{-m_i}$. Then one has the formula*

$$\mathcal{C}^{P,l}(E_m, \theta_m) = \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |m_I| + |I| = P, |I| = l}} \mathcal{C}_I.$$

Proof. By Lemma 2.6, each $\mathcal{C}^{P,l}(E_m, \theta_m)$ is the good extension of the corresponding cohomology sheaf for the Higgs complex, the restriction of $\Omega_{\mathcal{P}}^*(E_m, \theta_m)$ to X . By the homogeneity of the Higgs complex, it suffices to carry out the computation at 0, or equivalently the computation of the pull-back Higgs complex over \mathbb{H}^n at z_0 . For simplicity, we shall not change the previous notation when we carry out the calculation over \mathbb{H}^n . First we observe that over \mathbb{H}^n the Higgs complex decomposes

$$\Omega_{\mathcal{P}}^*(E_m, \theta_m) = \bigoplus_{P_1 + \dots + P_n = P} \Omega_{P_1}^*(E_{1,m_1}, \theta_{1,m_1}) \otimes \cdots \otimes \Omega_{P_n}^*(E_{n,m_n}, \theta_{n,m_n}).$$

Now, by the homogeneity of the Higgs complex, the condition in Appendix 8 in [11] is satisfied. In fact, it suffices to check this condition at the point z_0 . Therefore,

it follows from Appendix 8 in [11] that each cohomology sheaf of the previous complex decomposes into the following form:

$$\mathcal{C}^{P,l}(E_m, \theta_m) = \bigoplus_{\substack{P_1+\dots+P_n=P, \\ l_1+\dots+l_n=l}} \mathcal{C}^{P_1,l_1}(E_{1,m_1}, \theta_{1,m_1}) \otimes \dots \otimes \mathcal{C}^{P_n,l_n}(E_{n,m_n}, \theta_{n,m_n}).$$

Now we calculate the single factor $\mathcal{C}^{P_1,l_1}(E_{1,m_1}, \theta_{1,m_1})$: From the Higgs complex

$$\begin{array}{ccccccc} \mathcal{L}_1^{m_1} & \oplus & \dots & \oplus & \mathcal{L}_1^{-m_1+2} & \oplus & \mathcal{L}_1^{-m_1} \\ \downarrow \cong & & & & \downarrow \cong & & \downarrow \\ \mathcal{L}_1^{m_1} \otimes \Omega_{\mathbb{H}} & \oplus & \mathcal{L}_1^{m_1-2} \otimes \Omega_{\mathbb{H}} & \oplus & \dots & \oplus & \mathcal{L}_1^{-m_1} \otimes \Omega_{\mathbb{H}} & \oplus & 0, \end{array}$$

and the isomorphism $\Omega_{\mathbb{H}} \cong \mathcal{L}_1^2$, we see that

$$\mathcal{C}^{P_1,l_1}(E_{1,m_1}, \theta_{1,m_1}) = \begin{cases} \mathcal{L}_1^{-m_1}, & \text{if } P_1 = 0, l_1 = 0; \\ \mathcal{L}_1^{m_1+2}, & \text{if } P_1 = m_1 + 1, l_1 = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The remaining step is elementary. \square

Since C_I is a tensor product of powers of \mathcal{L}_i 's, it is locally free. The above proposition and Proposition 2.5 imply the following

COROLLARY 3.4. *Let $0 \leq k \leq 2n$ and F^\cdot be the Hodge filtration on $H^k(X, \mathbb{V}_m)$. For each $0 \leq P \leq |m| + k$, one has a natural isomorphism*

$$\text{Gr}_F^P H^k(X, \mathbb{V}_m) \cong \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |m_I| + |I| = P}} H^{k-|I|}(\bar{X}, \mathcal{C}_I).$$

4. A vanishing theorem of Mok. In this section we deduce a certain vanishing result of global sections of \mathcal{C}_I from a vanishing theorem of N. Mok, which contribute to a priori information about the sheaf cohomologies in Corollary 3.4. We first recall the following

Definition 4.1. Let M be a complex manifold and (F, h) a hermitian holomorphic vector bundle over M . Let Θ be the curvature tensor of (F, h) . (F, h) is said to be semi-positive (in the sense of Griffiths) at the point $x \in M$, if for any non-zero tangent vector $v \in T_{X,x}$ and any non-zero vector $e \in F_x$, $\Theta(e, \bar{e}, v, \bar{v}) \geq 0$. It is said to be properly semi-positive if furthermore for certain non-zero vectors e_0 and v_0 and has $\Theta(e_0, \bar{e}_0, v_0, \bar{v}_0) = 0$.

The significance of the notion *properly semi-positive* in the case of locally hermitian symmetric domains lies in the following theorem due to N. Mok, as a direct

consequence of his metric rigidity theorem on irreducible locally homogenous bundles:

THEOREM 4.2. (Mok) *Let Ω be a hermitian symmetric domain and Γ a torsion-free discrete subgroup of $G = \text{Aut}^0(\Omega)$ such that the quotient $M = \Gamma \backslash \Omega$ is irreducible (i.e., not a product of two complex manifolds of positive dimensions) and has finite volume with respect to the canonical metric. Let (F, h) be a non-trivial irreducible locally homogenous vector bundle over M with an invariant hermitian metric h . If (F, h) is properly semi-positive at one point (and hence for all points), then $H^0(X, F) = 0$.*

Proof. See the beginning of Section 2.3, Chapter 10 in [26], which is a direct application of Theorem 1, Section 2.2 in loc. cit. \square

The following proposition gives a sufficient condition for a semi-positive locally homogenous bundle being properly semi-positive.

PROPOSITION 4.3. *Let M be as above, and (F, h) an irreducible semi-positive locally homogenous bundle over M . If there is a Higgs bundle (E, θ) corresponding to a locally homogenous VHS \mathbb{W} over M such that $F \subset E^{p,q}$ and there exist non-zero vectors $e \in F_0$ and $v \in T_{X,0}$ such that $\theta_v(e) = 0$ (where $\theta_v(e)$ is given by $\theta(e)(v)$), then (F, h) is properly semi-positive.*

Proof. Because F is an irreducible component of $E^{p,q}$, the second fundamental form vanishes (see Chapter I in [20]). So we can use Griffiths curvature formula (see Lemma 7.18 in [35] for example) for $E^{p,q}$ to calculate the curvature of F :

$$\Theta(e, \bar{e}') = (\theta(e), \theta(e')) - (\theta^\dagger(e), \theta^\dagger(e')) \quad \forall e, e' \in E^{p,q},$$

where θ^\dagger is the $\text{Hom}(E^{p,q}, E^{p+1,q-1})$ -valued $(0, 1)$ -form determined by requiring θ_v^\dagger be the adjoint of θ_v relative to the Hodge metric. By assumption, $\theta_v(e) = 0$. So

$$\Theta(e, \bar{e}, v, \bar{v}) = (\theta_v(e), \theta_v(e)) - (\theta_v^\dagger(e), \theta_v^\dagger(e)) = -(\theta_v^\dagger(e), \theta_v^\dagger(e)) \leq 0.$$

Since F is semi-positive, $\Theta(e, \bar{e}, v, \bar{v}) \geq 0$. Therefore $\Theta(e, \bar{e}, v, \bar{v}) = 0$. \square

Note that the line bundle \mathcal{C}_I is of the form $\bigotimes_{i=1}^n \mathcal{L}_i^{s_i}$ for an n -tuple of integers (s_1, \dots, s_n) .

PROPOSITION 4.4. *For an n -tuple $(s_1, \dots, s_n) \in \mathbb{N}_0^n \setminus \{(0, \dots, 0)\}$ one has the vanishing*

$$H^0\left(X, \bigotimes_{i=1}^n \mathcal{L}_i^{s_i}|_X\right) = 0$$

if one of the entries s_i is zero.

Proof. Take $m = (s_1, \dots, s_n)$. By the discussion at the beginning of Section 3, it is clear that $\bigotimes_{i=1}^n \mathcal{L}_i^{s_i}|_X$ is the first Hodge bundle, namely the $E^{|m|,0}$ part of the corresponding Higgs bundle to \mathbb{V}_m . It follows from the Griffiths curvature formula as in the proof of Proposition 4.3 that $E^{|m|,0}$ is semi-positive. Assume $s_1 = 0$ without loss of generality. Let e_i be a nonvanishing local section of \mathcal{L}_i at z_0 . Via the natural projection map, X and \mathbb{H}^n are analytically local isomorphic to each other. A generator of the tangent subspace $T_{\mathbb{H}^n, i} \subset T_{\mathbb{H}^n, z_0}$ gives rise to a tangent vector v of $T_{X,0}$. By Proposition 3.2, the Higgs field $\theta_{i,1}$ along the direction v acts trivially on e_i for $2 \leq i \leq n$. For $\theta_m = \bigotimes_{i=2}^n \theta_{i,1}^{s_i}$, it follows that for the local section $e = \bigotimes_{i=2}^n e_i^{s_i}$ of $\bigotimes_{i=1}^n \mathcal{L}_i^{s_i}$, $\theta_{m,v}(e) = 0$. By Proposition 4.3 and Theorem 4.2, the proposition follows. \square

We remark that the condition on the irreducibility of Γ in the above result is crucial: for a product of modular curves the analogous statement does not hold.

5. L^2 -cohomology of local systems over Hilbert modular variety and the pure Hodge structure. Let g (resp. h) be the group invariant metric on X (resp. \mathbb{V}_m) (unique up to constant), and let $H_{(2)}^k(X^*, \mathbb{V}_m)$ be the L^2 -cohomology group of degree k with coefficients in \mathbb{V}_m with respect to the invariant metrics. Our principal aim in this section is to study the pure Hodge structure on $H_{(2)}^k(X^*, \mathbb{V}_m)$ given by theory of L^2 -harmonic forms.

Let us remind the reader that the local system \mathbb{V}_m in our case has been assumed to be non-trivial from the beginning, i.e., $m \neq 0$. Let D be the C^∞ -Gauß-Manin connection and its $(1,0)$ -part and $(0,1)$ -part decomposition reads

$$D = d' + d''.$$

Let $\bar{\partial}$ be the holomorphic structure of E_m and ∂ the $(1,0)$ -part of the metric connection of the Hodge metric. We define

$$D' = \partial + \bar{\theta}, \quad D'' = \bar{\partial} + \theta.$$

Our notation follows Section 4 in [39] which differs unfortunately from Zucker [41]. On page 256 loc. cit. the operators $D', D'', d'_\rho, d''_\rho$ are defined, which are $\partial, \bar{\partial}, \theta, \bar{\theta}$ respectively. The operators $\mathcal{D}', \mathcal{D}''$ defined on page 265 loc. cit. are our D', D'' respectively. For any of the above operators \mathcal{D} , we denote its L^2 -adjoint by \mathcal{D}^* and Laplacian operator by $\square_{\mathcal{D}}$.

For a subset $I \subset \{1, \dots, n\}$, we denote by \mathcal{C}_I^0 the restriction of \mathcal{C}_I to X , which is a locally homogenous line bundle over X , and by $H_{(2)}^*(X, \mathcal{C}_I^0)$ the L^2 -Dolbeault cohomology of \mathcal{C}_I^0 with respect to the *invariant* metric g of X and the group invariant metric h on \mathcal{C}_I^0 . That is, it is the cohomology of the complex $(A_{(2)}^{0,*}(X, \mathcal{C}_I^0), \bar{\partial})$, where $A_{(2)}^{0,i}(X, \mathcal{C}_I^0)$ is the space of L^2 -integrable smooth $(0, i)$ -forms ϕ over X with

coefficients in \mathcal{C}_I^0 such that $\bar{\partial}\phi$ is L^2 . Let $\mathfrak{h}_{\bar{\partial},(2)}^i(X, \mathcal{C}_I^0) \subset A_{(2)}^{0,i}(X, \mathcal{C}_I^0)$ be the subspace of L^2 - $\bar{\partial}$ -harmonic forms. By the standard L^2 -harmonic theory, one has a natural injection $\mathfrak{h}_{\bar{\partial},(2)}^i(X, \mathcal{C}_I^0) \rightarrow H_{(2)}^i(X, \mathcal{C}_I^0)$, and it is an isomorphism **iff** the range of $\bar{\partial}$ is closed (in the Hilbert space of weakly differentiable L^2 -forms). But we know from the finite dimensionality of $H_{(2)}^*(X^*, \mathbb{V}_m)$ that the range of D is closed, and hence the range of \square_D is closed. As shown in page 609 [43] $\square_D = 2\square_{D'}$ holds in the strict operator sense, it follows that $\square_{D'}$ has closed range and thus the range of D' is closed as well. Now that θ acts on \mathcal{C}_I^0 by zero, the range of $\bar{\partial}$ is closed. In summary, one has a natural isomorphism

$$H_{(2)}^i(X, \mathcal{C}_I^0) \cong \mathfrak{h}_{\bar{\partial},(2)}^i(X, \mathcal{C}_I^0),$$

and particularly the finite dimensionality of $H_{(2)}^i(X, \mathcal{C}_I^0)$.

THEOREM 5.1. *Let $I \subset \{1, \dots, n\}$ be a subset. Then for $i \neq n - |I|$, it holds that*

$$\dim H_{(2)}^i(X, \mathcal{C}_I^0) = 0.$$

In order to prove it, we need to employ the full machinery of L^2 -harmonic theory. Let $A_{(2)}^n(X, \mathbb{V}_m)$ (resp. $A_{(2)}^{p,q}(X, \mathbb{V}_m)$) be the space of smooth L^2 - n -forms (resp. type (p, q) -forms) on X with coefficients in \mathbb{V}_m . Denote the subspace of L^2 -harmonic forms by

$$\mathfrak{h}_{(2)}^n(X, \mathbb{V}_m) = \{\alpha \in A_{(2)}^n(X, \mathbb{V}_m) \mid \square_D \alpha = 0\},$$

similarly the subspace of L^2 -harmonic (p, q) -forms by $\mathfrak{h}_{(2)}^{p,q}(X, \mathbb{V}_m)$. By Zucker [41], one has natural isomorphisms

$$H_{(2)}^n(X^*, \mathbb{V}_m) \cong \mathfrak{h}_{(2)}^n(X, \mathbb{V}_m) = \bigoplus_{p+q=n} \mathfrak{h}_{(2)}^{p,q}(X, \mathbb{V}_m).$$

By abuse of notation, in the following we denote again by (E_m, θ_m) the restriction of (E_m, θ_m) to X , which is a Higgs bundle over X . We consider the cohomology sheaf of the Higgs complex of (E_m, θ_m) over X at the i -th place:

$$E_m \otimes \Omega_X^{i-1} \xrightarrow{\theta_m^{i-1}} E_m \otimes \Omega_X^i \xrightarrow{\theta_m^i} E_m \otimes \Omega_X^{i+1},$$

and denote by $\theta_m^{*,i}$ the adjoint of θ_m^i with respect to the Hodge metric on E_m and the invariant metric on X . We have the following

LEMMA 5.2. *The i -th cohomology sheaf of the Higgs complex can be characterized by those sections of $E_m \otimes \Omega_X^i$ satisfying the equations $\theta_m^i = \theta_m^{*,i-1} = 0$.*

Should we change "iff" to "if"?

Proof. Let $\mathcal{C} = \frac{\text{Ker}\theta_m^i}{\text{Im}\theta_m^{i-1}}$ be the cohomology sheaf. By the homogeneity, one has the holomorphic and metric decomposition of Hermitian vector bundles

$$\text{Ker}\theta_m^i = \mathcal{C} \oplus \text{Im}\theta_m^{i-1}.$$

So $\mathcal{C} = \text{Ker}\theta_m^i \cap (\text{Im}\theta_m^{i-1})^\perp$. On the other hand, one has clearly that $\text{Ker}\theta_m^{*,i-1} = (\text{Im}\theta_m^{i-1})^\perp$. The lemma follows. \square

LEMMA 5.3. *For each I , one has the inclusion $\mathfrak{h}_{\bar{\partial},(2)}^i(X, \mathcal{C}_I^0) \subset \mathfrak{h}_{(2)}^{|I|,i}(X, \mathbb{V}_m)$.*

Proof. This follows from (5.22) and (5.14) in [41]. We need to explain the notation. By Proposition 3.3, \mathcal{C}_I^0 is a direct summand of the $|I|$ -th cohomology sheaf of the Higgs subcomplex $\Omega_{|m_I|+|I|}^*(E_m, \theta_m)$. By (5.22) and (5.14) in [41], it follows that

$$\mathfrak{h}_{\bar{\partial},(2)}^i(X, \mathcal{C}_I^0) \subset \mathfrak{h}_{(2)}^{|I|,i}(X, \mathbb{V}_m). \quad \square$$

Let $I = (i_1, \dots, i_p), J = (j_1, \dots, j_q)$ be two multi-indices with $1 \leq i_1 < \dots < i_p \leq n$ and $1 \leq j_1 < \dots < j_q \leq n$. Put $dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$ and $\overline{dz}_J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$. Let $\mathfrak{h}_{(2)}(I, J; \mathbb{V}_m)$ be the subspace of $\mathfrak{h}_{(2)}^{p,q}(X, \mathbb{V}_m)$ consisting of those elements whose pull-back to \mathbb{H}^n are of form $f_{I,J} dz_I \wedge \overline{dz}_J$. The following lemma is the L^2 -analogue of Proposition 1.2 in [24] and its proof holds verbatim for L^2 -harmonic forms.

LEMMA 5.4. *One has an orthogonal decomposition into a direct sum of subspaces of L^2 -harmonic forms for each (p, q) :*

$$\mathfrak{h}_{(2)}^{p,q}(X, \mathbb{V}_m) = \bigoplus_{\substack{I, J \subset \{1, \dots, n\}, \\ |I|=p, |J|=q}} \mathfrak{h}_{(2)}(I, J; \mathbb{V}_m).$$

The next proposition gives the L^2 -analogue of Proposition 4.1-4.3 in [24] in the case of non-trivial local systems.

PROPOSITION 5.5. *Let I, J be as above. Then $\mathfrak{h}_{(2)}(I, J; \mathbb{V}_m) = 0$ unless $I \cup J$ (as a set) is equal to $\{1, \dots, n\}$ and $I \cap J = \emptyset$.*

Proof. Let $\alpha = \alpha_{I,J} \in \mathfrak{h}_{(2)}(I, J; \mathbb{V}_m)$. We prove the statement by induction on $q = |J|$. When $q = 0$, $\alpha \in A^{p,0}(X, \mathbb{V}_m) = A^{0,0}(X, E_m \otimes \Omega_X^p)$. Because α is D -harmonic and equivalently D'' -harmonic, one has $\bar{\partial}(\alpha) = 0$, $\theta(\alpha) = 0$ and $\theta^*(\alpha) = 0$ (see Corollary 3.20 in [41]). This implies that α is a global holomorphic section of $E_m \otimes \Omega_X^p$ and by Lemma 5.2 it is even a global section of the cohomology sheaf of the Higgs complex at the $p = |I|$ -th place. Thus it must be a global section of $\mathcal{C}_I^0 = \bigotimes_{i \in I} \mathcal{L}_i^{m_i+2} \otimes \bigotimes_{i \in I^c} \mathcal{L}_i^{-m_i}|_X$ by consideration of $\{I, J; \mathbb{V}_m\}$ -type.

Case 1: $I \neq \{1, \dots, n\}$ and $m_i = 0$ for all $i \in I^c$. In this case, \mathcal{C}_I^0 is of the type in Proposition 4.4. Since $\alpha \in \mathfrak{h}_{\bar{\partial}, (2)}^0(X, \mathcal{C}_I^0) \subset H^0(X, \mathcal{C}_I^0)$, $\alpha = 0$ by Proposition 4.4.

Case 2: $m_i \neq 0$ for certain $i \in I^c$. In this case, we recall the decomposition of the differential operator D over the space $A_{(2)}^p(X, \mathbb{V}_m)$

$$D = D' + D'' = d' + d''; \quad D' = \partial + \bar{\theta}, \quad D'' = \bar{\partial} + \theta, \quad d' = \partial + \theta, \quad d'' = \bar{\partial} + \bar{\theta}.$$

$\square_D(\alpha) = 0$ implies that $d''(\alpha) = 0$. Since $\bar{\partial}\alpha = 0$, it follows that $\bar{\theta}(\alpha) = 0$. Now we take a smooth open neighborhood U of $0 \in X$ with the local coordinates $\{z_1, \dots, z_n\}$ and e be a holomorphic basis of the line bundle \mathcal{C}_I^0 on U . Write $\alpha = f(z_1, \dots, z_n)e$ over U . Assuming the following claim, one has

$$\bar{\theta}(\alpha) = \bar{\theta}(f(z)e) = \bar{f}(z)\bar{\theta}(e) = 0,$$

and then $f = 0$. So $\alpha = 0$. The following claim is a direct consequence of Proposition 3.2.

Claim 5.6. One has $\bar{\theta}(e) \neq 0$, where e is as above and

$$\bar{\theta} : A_{(2)}^{(p,0)}(E^{|m_I|, |m| - |m_I|}) \longrightarrow A_{(2)}^{(p,1)}(E^{|m_I|+1, |m| - |m_I| - 1}).$$

Proof. Let $\theta_i = \theta_{\partial_{z_i}}$ be the Higgs field along the tangent direction ∂_{z_i} . Let e_i (resp. e_i^*) be a local basis of \mathcal{L}_i (resp. \mathcal{L}_i^{-1}) over U such that $\theta_i(e_i) = e_i^*$. By Proposition 3.2, one has $\theta_i(e_j) = 0$ for $j \neq i$, and of course $\theta_i(e_i^*) = 0$ for any j . Put $e_I^{m_I} = \bigotimes_{i \in I} e_i^{\otimes m_I}$ and $e_{I^c}^{*m_{I^c}} = \bigotimes_{i \in I^c} e_i^{*m_i}$. Then one has $e = dz_I \otimes e_I^{m_I} \otimes e_{I^c}^{*m_{I^c}}$ up to an invertible holomorphic function, which does not affect the proof. Recall the local formula of $\bar{\theta}$ (see Section 1 in [39]):

$$\bar{\theta} = \sum_i \bar{\theta}_i d\bar{z}_i,$$

where $\bar{\theta}_i$ is the adjoint of the matrix θ_i with respect to the Hodge metric. By the product rule for the Higgs field with respect to tensor products (see Section 1 in [39]), one has for $i \in I^c$,

$$\begin{aligned} \bar{\theta}_i(dz_I \otimes e_I^{m_I} \otimes e_{I^c}^{*m_{I^c}}) &= dz_I \otimes \bar{\theta}_i(e_I^{m_I}) \otimes e_{I^c}^{*m_{I^c}} + dz_I \otimes e_I^{m_I} \otimes \bar{\theta}_i(e_{I^c}^{*m_{I^c}}) \\ &= dz_I \otimes e_I^{m_I} \otimes \bar{\theta}_i(e_{I^c}^{*m_{I^c}}) \\ &= \frac{m_i}{4y_i^2} \cdot dz_I \otimes e_I^{m_I} \otimes e_{I^c \setminus \{i\}}^{*m_{I^c \setminus \{i\}}} \otimes (e_i^{*m_i-1} \otimes e_i). \end{aligned}$$

Similarly one gets $\bar{\theta}_i(dz_I \otimes e_I^{m_I} \otimes e_{I^c}^{*m_{I^c}}) = 0$ for $i \in I$. Thus one has the formula

$$\bar{\theta}(e) = \sum_{i \in I^c, m_i \neq 0} \frac{m_i}{4y_i^2} \left(dz_I \wedge d\bar{z}_i \otimes e_I^{m_I} \otimes e_{I^c \setminus \{i\}}^{*m_{I^c \setminus \{i\}}} \otimes e_i^{*m_i-1} \otimes e_i \right).$$

By the assumption of the case, the above expression is non-zero. The claim is proved. \square

In summary, the space $\mathfrak{h}_{(2)}(I, \emptyset; \mathbb{V}_m)$ is zero unless I is the whole set. This proves the $q = 0$ case. Now we assume $q > 0$. There are two possibilities, namely the case $I \cap J \neq \emptyset$ and the case $I \cap J = \emptyset$. Consider the former case. Let Λ be the adjoint of the Lefschetz operator $L = \wedge \omega$ on the space of differential forms, where ω is the Kähler form of the metric g on X . By the standard L^2 -harmonic theory, $\Lambda(\alpha)$ is again an L^2 -harmonic form. As $I \cap J \neq \emptyset$, $\Lambda(\alpha) = 0$ if and only if $\alpha = 0$. Write $\Lambda(\alpha) = \sum_{I', J'} \beta_{I', J'}$. Then $|J'| = q - 1$ and $I' \cup J'$ is not equal to $\{1, \dots, n\}$. So one proves by induction that each term $\beta_{I', J'}$ of $\Lambda(\alpha)$ is zero and hence $\alpha = 0$. Consider the latter case. Let \mathbb{H}_J^n be the complex manifold whose underlying riemannian structure is the same as that of \mathbb{H}^n but the complex structure differs from the usual one by that at the j -th factor for $j \in J$, one takes the complex conjugate complex structure of \mathbb{H} (see Section 4 in [24]). One puts $X_J = \Gamma \backslash \mathbb{H}_J^n$. As observed in Section 4 [24], such an operation identifies the space of L^2 -harmonic forms $\mathfrak{h}_{(2)}^n(X, \mathbb{V}_m)$ with $\mathfrak{h}_{(2)}^n(X_J, \mathbb{V}_m)$, but maps the subspace of type $\{I, J; \mathbb{V}_m\}$ on X to the subspace of type $\{I \cup J, \emptyset; \mathbb{V}_m\}$ on X_J . This allows us to reduce the proof to the case $q = 0$ on X_J . However the above arguments, particularly the truth of Proposition 4.4, holds for X_J as well. Therefore the second case also follows. \square

Now we can proceed to the proof of Theorem 5.1.

Proof. Let $\alpha \in \mathfrak{h}_{(2), \bar{\partial}}^i(X, \mathcal{C}_I)$ be the L^2 -harmonic representative of a non-zero cohomology class of $H_{(2)}^i(X, \mathcal{C}_I^0)$. By Lemma 5.3, $\alpha \in \mathfrak{h}_{(2)}^{p, q}(X, \mathbb{V}_m)$ where we rewrite $p = |I|, q = i$. By Lemma 5.4, one writes $\alpha = \sum_{I, J, |I|=p, |J|=q} \alpha_{I, J}$ into sum of L^2 -harmonic forms with $\alpha_{I, J}$ type $\{I, J; \mathbb{V}_m\}$. By Proposition 5.5, $\alpha_{I, J} = 0$ unless J is the complement of I . This implies that if $q \neq n - p$, namely $i \neq n - |I|$, $\alpha = 0$ and hence the theorem. \square

In the course of the above proof, we have introduced for a subset $J \subset \{1, \dots, n\}$ a complex manifold $X_{\bar{J}}$. It is again a discrete quotient of \mathbb{H}^n : Putting Γ_J to be $g_J \Gamma g_J^{-1}$ where $g_J = (g_1, \dots, g_n) \in \mathrm{GL}_2(\mathbb{R})^{\times n}$ with

$$g_i = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & i \in J; \\ \mathrm{id}, & i \notin J, \end{cases}$$

one sees easily that $X_{\bar{J}} = \Gamma_J \backslash \mathbb{H}^n$. By Margulis's arithmeticity theorem (see [23]), $\Gamma_J \subset G(\mathbb{R})$ is again an arithmetic lattice. One defines the locally homogenous line bundle \mathcal{L}_i over $X_{\bar{J}}$ as well as its good extension to a smooth toroidal compactification $\bar{X}_{\bar{J}}$ of $X_{\bar{J}}$. As a byproduct of the previous proof, we obtain the following result:

PROPOSITION 5.7.

$$\dim H_{(2)}^{n-|I|}(X, \mathcal{C}_I^0) = \dim H_{(2)}^0 \left(X_{\bar{I}^c}, \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2} |_{X_{\bar{I}^c}} \right).$$

Proof. Using the trick of taking the conjugate complex structures on \mathbb{H}^n at the factors I^c (see the proof of Proposition 5.5), one obtains an identification of the space of harmonic forms of type $\{I, I^c; \mathbb{V}_m\}$ on X with that of type $\{(1, \dots, n), \emptyset; \mathbb{V}_m\}$ on $X_{\bar{I}^c}$. \square

The space of L^2 -sections admits a natural analytical description using a smooth toroidal compactification.

PROPOSITION 5.8. *For each $J \subset \{1, \dots, n\}$, one has a natural isomorphism*

$$H_{(2)}^0(X_{\bar{J}}, \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2} |_{X_{\bar{J}}}) \cong H^0 \left(\bar{X}_{\bar{J}}, \mathcal{O}_{\bar{X}_{\bar{J}}}(-S_{\bar{J}}) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2} \right),$$

where $S_{\bar{J}} = \bar{X}_{\bar{J}} - X_{\bar{J}}$ is the boundary divisor.

Proof. Obviously it suffices to show the statement for $J = \emptyset$, i.e., for $X_{\bar{J}} = X$. Let $\iota : X \rightarrow \bar{X}$ be the inclusion. One has the relation

$$H_{(2)}^0(X, \mathcal{C}_{\{1, \dots, n\}}^0) \subset A_{(2)}^{0,0}(X, \mathcal{C}_{\{1, \dots, n\}}^0) \subset A^0(\bar{X}, \iota_* \mathcal{C}_{\{1, \dots, n\}}^0).$$

We define $\Omega_{(2)}^0(\mathcal{C}_{\{1, \dots, n\}}^0)$ to be the subsheaf of $\iota_* \mathcal{C}_{\{1, \dots, n\}}^0$ consisting of germs of L^2 -holomorphic sections. It is clear that one has a natural isomorphism

$$H_{(2)}^0(X, \mathcal{C}_{\{1, \dots, n\}}^0) \cong H^0(\bar{X}, \Omega_{(2)}^0(\mathcal{C}_{\{1, \dots, n\}}^0)).$$

Recall that $\mathcal{C}_{\{1, \dots, n\}}^0 = \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2} |_X$. In the following we show that

$$\Omega_{(2)}^0 \left(\bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2} |_X \right) = \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}.$$

The question is local at infinity, and we shall only show the case where $m = (1, 0, \dots, 0)$, since the proof in general case is completely analogous. It suffices also to consider a small neighborhood U of a maximal singular point $P \in S$. Let $\{u_1, \dots, u_n\}$ be a set of local coordinates of U such that $S \cap U$ is defined by $\prod_{i=1}^n u_i = 0$. We also put $\omega_{pc} = \sum_{i=1}^n \frac{\sqrt{-1}}{\pi} \frac{du_i \wedge d\bar{u}_i}{|u_i|^2 |\log |u_i||^2}$, the Poincaré metric on

$U - S \cong (\Delta^*)^n$, and $\omega = c_1 \sum_{i=1}^n \frac{dz_i \wedge d\bar{z}_i}{y_i^2}$ (in the following the c_i are certain constants), the invariant metric. Their volume forms are computed respectively by

$$\text{Vol}_{\omega_{pc}} = \frac{c_2 \bigwedge_{i=1}^n (du_i \wedge d\bar{u}_i)}{\prod_{i=1}^n |u_i|^2 |\log |u_i||^2}, \quad \text{Vol}_{\omega} = \frac{c_3 \bigwedge_{i=1}^n (dz_i \wedge d\bar{z}_i)}{\prod_{i=1}^n y_i^2}.$$

By the theory of toroidal resolutions of a cusp singularity (see [2, 10]), one has the following formula for the change of local coordinates:

$$2\pi\sqrt{-1} \cdot z_i = \sum_{j=1}^n a_{i,j} \log u_j,$$

where $a_{i,j} > 0$ for all i, j . Comparing the real parts of the above equality, one obtains

$$y_i = \sum_{j=1}^n -a'_{i,j} \log |u_j|$$

with $a'_{i,j} = \frac{a_{i,j}}{2\pi}$. The estimate of Hodge norms in [4, Theorem 5.21] is taken over the following type of region

$$D_\epsilon = \left\{ (u_1, \dots, u_n) \in (\Delta^*)^n \mid \frac{\log |u_1|}{\log |u_2|} > \epsilon, \dots, \frac{\log |u_{n-1}|}{\log |u_n|} > \epsilon, -\log |u_n| > \epsilon \right\}$$

for some $\epsilon > 0$. For an element $\sigma \in S_n$, the permutation group over n elements, we put D_σ to be the region obtained by permutating the indices of $\{u_i\}$ in the definition of D_ϵ . So $D_{\text{id}} = D_\epsilon$, and note that $\{D_\sigma\}_{\sigma \in S_n}$ is an open covering of a small neighborhood of P for suitable chosen ϵ . By shrinking U if necessary, it covers U . It is clear that the square-integrability over U is equivalent to that over each D_σ . Now let v_i be a local trivializing section of \mathcal{L}_i over U , and write $f(u_1, \dots, u_n) v_1^3 \otimes v_2^2 \otimes \dots \otimes v_n^2$ for an element in $\iota_*(\mathcal{L}_1^3 \otimes \mathcal{L}_2^2 \otimes \dots \otimes \mathcal{L}_n^2)|_X(U)$. By [4, Theorem 5.21], $\|v_i\|^2 \sim |\log |u_i||$ over D_{id} . The condition that $f v_1^3 \otimes v_2^2 \otimes \dots \otimes v_n^2$ being L^2 over D_{id} means that

$$\int_{D_{\text{id}}} |f| \cdot \|v_1^3 \otimes v_2^2 \otimes \dots \otimes v_n^2\| \text{Vol}_{\omega} < \infty.$$

Since over D_{id} ,

$$\frac{\text{Vol}_{\omega}}{\text{Vol}_{\omega_{pc}}} \sim \frac{\prod_{i=1}^n |\log |u_i||^2}{\prod_{i=1}^n y_i^2} = \frac{\prod_{i=1}^n |\log |u_i||^2}{\prod_{i=1}^n (\sum_{j=1}^n -a'_{i,j} \log |u_j|)^2} \sim \frac{\prod_{i=1}^n |\log |u_i||^2}{|\log |u_1||^{2n}}$$

holds, it follows that

$$\begin{aligned}
\int_{D_{\text{id}}} |f| \cdot \|v_1^3 \otimes v_2^2 \otimes \cdots \otimes v_n^2\| \text{Vol}_\omega &\sim \int_{D_{\text{id}}} |f| \cdot |\log |u_1||^{2n+1} \text{Vol}_\omega \\
&\sim \int_{D_{\text{id}}} |f| \cdot |\log |u_1||^3 \cdot \prod_{i=1}^n |\log |u_i||^2 \text{Vol}_{\omega_{pc}} \\
&\sim \int_{D_{\text{id}}} \frac{|f| \cdot |\log |u_1||^3}{\prod_{i=1}^n |u_i|^2} \bigwedge_{i=1}^n (du_i \wedge d\bar{u}_i).
\end{aligned}$$

It is clear now that over D_{id} , the above section is L^2 if and only if $f = u_1 \cdot f'$ for certain holomorphic f' . Running the above arguments for the other regions D_σ , one knows that the section is L^2 over all D_σ if and only if $f = (u_1, \dots, u_n) \cdot f''$ for certain holomorphic f'' . This shows the equality

$$\Omega_{(2)}^0 \left(\bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2} \Big|_X \right) = \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}$$

over U . By the previous discussion, the above equality actually holds over \bar{X} , which shows the proposition. \square

COROLLARY 5.9. *For $k \neq n$, $H_{(2)}^k(X^*, \mathbb{V}_m) = 0$.*

Proof. By Zucker (5.14), (5.22) in [41] and the finite dimensionality of $H_{(2)}^*(X, \mathcal{C}_I^0)$, one has the equality

$$\begin{aligned}
\dim IH^k(X^*, \mathbb{V}_m) &= \dim H_{(2)}^k(X^*, \mathbb{V}_m) \\
&= \sum_{P=0}^{k+|m|} \sum_{l=0}^n \sum_{\substack{I \subset \{1, \dots, n\}, \\ |I|=l, |m_I|+|I|=P}} \dim H_{(2)}^{k-l}(X, \mathcal{C}_I^0).
\end{aligned}$$

In the above formula, it is clear that for $0 \leq k \leq n-1$ fixed and for all l , $k-l < n-l = n-|I|$ holds. By Theorem 5.1, it follows that for $0 \leq k \leq n-1$, each direct summand in the right hand side of the formula is zero, and hence $IH^k(X^*, \mathbb{V}_m) = 0$. By Poincaré duality for intersection cohomology, $IH^k(X^*, \mathbb{V}_m)$ vanishes for $n+1 \leq k \leq 2n$ as well. Hence the result holds for the L^2 -cohomology groups. \square

COROLLARY 5.10. *The Hodge decomposition of $H_{(2)}^n(X^*, \mathbb{V}_m)$ reads*

$$H_{(2)}^n(X^*, \mathbb{V}_m) = \bigoplus_{P+Q=n+|m|} H_{(2)}^{P,Q},$$

where

$$H_{(2)}^{P,Q} \cong \bigoplus_{l=0}^n \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |I|=l, |m_I|+|I|=P}} H_{(2)}^{n-l}(X, \mathcal{C}_I^0).$$

In particular,

$$H_{(2)}^{n+|m|,0} \cong H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2}), \quad H_{(2)}^{0,n+|m|} \cong H^n(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i}).$$

Proof. Continuing the arguments in the above proof, one sees that for each P ,

$$H_{(2)}^{P,Q} \cong \bigoplus_{l=0}^n \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |I|=l, |m_I|+|I|=P}} H_{(2)}^{n-l}(X, \mathcal{C}_I^0).$$

It is clear that, for $P = 0$ (resp. $P = n + |m|$), the above expression consists of the unique term $H_{(2)}^n(X, \mathcal{C}_\emptyset^0)$ (resp. $H_{(2)}^0(X, \mathcal{C}_{\{1, \dots, n\}}^0)$). By Proposition 5.8, $H_{(2)}^{n+|m|,0} \cong H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2})$. By Serre duality, one has a natural isomorphism $H_{(2)}^{0,n+|m|} \cong H^n(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i})$. \square

6. Eisenstein cohomology of a Hilbert modular group. Let X^\sharp be the Borel-Serre compactification of X with the boundary ∂X^\sharp . Recall that X is homotopy equivalent to X^\sharp and hence one has the natural restriction map $r : H^*(X, \mathbb{V}_{\mathbb{R}}) \rightarrow H^*(\partial X^\sharp, \mathbb{V}_{\mathbb{R}})$. The theory of Eisenstein series (see [13, 36]) provides the following space decomposition

$$H^*(X, \mathbb{V}_{\mathbb{R}}) = H_!^*(X, \mathbb{V}_{\mathbb{R}}) \oplus H_{\text{Eis}}^*(X, \mathbb{V}_{\mathbb{R}}),$$

where $H_!^*(X, \mathbb{V}_{\mathbb{R}})$ is the image of the cohomology with compact support, and $H_{\text{Eis}}^*(X, \mathbb{V}_{\mathbb{R}})$ maps isomorphically to the image of r . Its elements can be represented using Eisenstein series. In this section, we study the Eisenstein cohomology $H_{\text{Eis}}^*(X, \mathbb{V}_{\mathbb{R}})$. Before doing anything, we first recall the following result, which is a special case of the main theorems in [21]:

THEOREM 6.1. (Li-Schwermer [21]) *If \mathbb{V}_m is regular, then $H^i(X, \mathbb{V}_m) = 0$ for $0 \leq i \leq n-1$, and $H^i(X, \mathbb{V}_m) = H_{\text{Eis}}^i(X, \mathbb{V}_m) \stackrel{r}{\cong} H^i(\partial X^\sharp, \mathbb{V}_m)$ for $i \geq n+1$.*

The following lemma is known by (6.13-18) in [42]:

LEMMA 6.2. *$H^i(\partial X^\sharp, \mathbb{V}_m) = 0$ unless $m_1 = \dots = m_n$. As a consequence, $H_{\text{Eis}}^i(X, \mathbb{V}_m) = 0$ if the relation $m_1 = \dots = m_n$ is not satisfied.*

The main result of this section is the following

THEOREM 6.3. *Assume $m_1 = \dots = m_n$ and $l \geq n$. Then the restriction map $r : H^l(X, \mathbb{V}_{\mathbb{R}}) \rightarrow H^l(\partial X^{\sharp}, \mathbb{V}_{\mathbb{R}})$ is surjective and $\dim H_{\text{Eis}}^l(X, \mathbb{V}_m) = \binom{n-1}{l-n} h$. Moreover,*

$$H_{\text{Eis}}^l(X, \mathbb{V}_m) \subset F^{|m|+n} H^l(X, \mathbb{V}_m)$$

holds, where $F \cdot$ is the Hodge filtration on $H^l(X, \mathbb{V}_m)$.

After the paper was posted, Wildeshaus informed us that the main result in [3] also shows the statement about the Hodge type of the Eisenstein cohomology in the above theorem. The current argument in the proof is based on the treatment of the Eisenstein cohomology for constant coefficients in [12] (see Sections 3 and 4, Ch. III in loc. cit.). In the following we assume $m_1 = \dots = m_n$. It is clear that the proof of the theorem can be reduced to the statement for the standard cusp ∞ , which is the Γ -equivalence class of (∞, \dots, ∞) . From now on we pretend that ∂X^{\sharp} has only one component. Let $\Gamma_{\infty} \subset \Gamma$ be the stabilizer of ∞ and put $X_{\infty} = \Gamma_{\infty} \backslash \mathbb{H}^n$.

Proof. We divide the whole proof into several steps.

Step 1. A basis of $H^l(X_{\infty}, \mathbb{V}_m)$ for $n \leq l \leq 2n - 1$. Let $X_{\infty}(1)$ be the quotient of the following set

$$\left\{ (z_1, \dots, z_n) \in \mathbb{H}^n \mid \prod_{i=1}^n y_i = 1 \right\}$$

by Γ_{∞} , which is naturally identified with ∂X^{\sharp} . The group Γ_{∞} is of the form

$$1 \longrightarrow U \longrightarrow \Gamma_{\infty} \longrightarrow M \longrightarrow 1,$$

where

$$U = \left\{ \begin{pmatrix} 1 & u^{(1)} \\ 0 & 1 \end{pmatrix} \times \dots \times \begin{pmatrix} 1 & u^{(n)} \\ 0 & 1 \end{pmatrix} \in \Gamma_{\infty} \mid u \in \mathcal{O}_F \right\}$$

$$M = \left\{ \begin{pmatrix} t^{(1)} & 0 \\ 0 & (t^{-1})^{(1)} \end{pmatrix} \times \dots \times \begin{pmatrix} t^{(1)} & 0 \\ 0 & (t^{-1})^{(n)} \end{pmatrix} \in \Gamma_{\infty} \mid t \in \mathcal{O}_F^* \right\}$$

are free abelian groups of rank n and $n - 1$ respectively. The $X_{\infty}(1)$ has two distinguished submanifolds: One is the quotient of the set $\{(iy_1, \dots, iy_n) \in \mathbb{H}^n \mid \prod_{i=1}^n y_i = 1\}$ by M , which is isomorphic to $(S^1)^{n-1}$ with ‘‘coordinates’’ $\{\log y_1, \dots, \log y_{n-1}\}$ and denoted temporarily by Y , and the quotient of $\{(x_1 + i, \dots, x_n + i) \in \mathbb{H}^n \mid x_1, \dots, x_n \in \mathbb{R}\}$ by U , which is isomorphic to $(S^1)^n$ with ‘‘coordinates’’ $\{x_1, \dots, x_n\}$ and denoted temporarily by Z . In fact, $X_{\infty}(1)$ is naturally a fiber bundle with Y (resp. Z) a section (resp. fiber) of it (see Section 2 in [13]).

*There is an extra right
delimiter or a missing left one.
Please check.*

Claim 6.4. For $n \leq l \leq 2n - 1$, the following set of vector valued differential forms over \mathbb{H}^n is Γ_∞ -invariant and defines a basis of $H^l(X_\infty(1), \mathbb{V}_m)$:

$$\left\{ \omega'_a = \frac{dy_a}{y_a} \wedge dx_1 \wedge \cdots \wedge dx_n \otimes \bigotimes_{i=1}^n (e_{i1} + x_i e_{i2})^{m_1} \mid a \subset \{1, \dots, n-1\}, |a| = l - n \right\},$$

where for $a = (i_1, \dots, i_{l-n})$, $i_1 < \cdots < i_{l-n}$, $\frac{dy_a}{y_a} = \frac{dy_{i_1}}{y_{i_1}} \wedge \cdots \wedge \frac{dy_{i_{l-n}}}{y_{i_{l-n}}}$, and $\{e_{i1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_{i2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ is the standard basis of V_i at $z_0 \in \mathbb{H}^n$ (see Section 2).

Proof. Let $\pi : X_\infty(1) \rightarrow Y$ be the fibration. By Section 2 in [13], the Leray spectral sequence of π for $H^l(X_\infty(1), \mathbb{V}_m)$ degenerates at E_2 . By the theorem of van Est (see Section 2 in [13]), each grading $H^*(Y, R^{l-*} \pi_* \mathbb{V}_m)$ is computed by its corresponding Lie algebra cohomology. By the computations on the Lie algebra cohomology in Section 6 of [42] (see particularly (6.18) and Lemma (6.13)), one knows that

$$H^l(X_\infty(1), \mathbb{V}_m) = H^{l-n}(Y, \mathbb{C}) \otimes H^n(Z, \mathbb{V}_m).$$

Now it is straightforward to check that $\{\frac{dy_a}{y_a} \mid a \subset \{1, \dots, n-1\}, |a| = l - n\}$ provides a basis for $H^{l-n}(Y, \mathbb{C})$ and the element $\bigwedge_{i=1}^n dx_i \otimes \bigotimes_{i=1}^n (e_{i1} + x_i e_{i2})^{m_1}$ is a basis for the one dimensional space $H^n(Z, \mathbb{V}_m)$. \square

Note that the inclusion $X_\infty(1) \subset X_\infty$ is a homotopy equivalence. We claim the following

Claim 6.5. The following set of Γ_∞ -invariant vector valued differential forms over \mathbb{H}^n

$$\left\{ \omega_a = \frac{dz_a \wedge d\bar{z}_a}{y_a} \wedge dz_b \otimes \bigotimes_{i=1}^n (e_{i1} + z_i e_{i2})^{m_1} \mid a \subset \{1, \dots, n-1\}, |a| = l - n, b = a^c \right\}$$

defines a basis of $H^l(X_\infty, \mathbb{V}_m)$.

Proof. By Remark 3.1, Ch. III in [12], ω_a is cohomologous to

$$\frac{dy_a}{y_a} \wedge dx_1 \wedge \cdots \wedge dx_n \otimes \bigotimes_{i=1}^n (e_{i1} + z_i e_{i2})^{m_1}$$

up to scalar. By Claim 6.4 and the homotopy equivalence, it remains to show that

$$\frac{dy_a}{y_a} \wedge dx_1 \wedge \cdots \wedge dx_n \otimes \bigotimes_{i=1}^n (e_{i1} + z_i e_{i2})^{m_1}$$

is cohomologous to ω'_a up to a scalar. Note that the difference of the above two forms is a linear combination of forms of the following type:

$$\frac{dy_a}{y_a} \wedge dx_1 \wedge \cdots \wedge dx_n \otimes \bigotimes_{i=1}^n (y_i e_{i2})^{t_i} \otimes (e_{i1} + x_i e_{i2})^{m_1 - t_i},$$

which is exact once one of the t_i is positive. In fact, assume $t_1 \geq 1$ for example, the exterior differential of the following form

$$y_1 \cdot \frac{dy_a}{y_a} \wedge \bigwedge_{i=2}^n dx_i \otimes (y_1 e_{12})^{t_1 - 1} \otimes (e_{11} + x_1 e_{12})^{m_1 - t_1 + 1} \\ \otimes \bigotimes_{i=2}^n (y_i e_{i2})^{t_i} \otimes (e_{i1} + x_i e_{i2})^{m_1 - t_i}$$

is up to a scalar equal to

$$\frac{dy_a}{y_a} \wedge dx_1 \wedge \cdots \wedge dx_n \otimes \bigotimes_{i=1}^n (y_i e_{i2})^{t_i} \otimes (e_{i1} + x_i e_{i2})^{m_1 - t_i}.$$

This shows the claim. \square

Step 2. Convergence of Eisenstein series. For each ω_a , we consider the following formal differential form $E(\omega_a)$ on X obtained by symmetrization (see Section 3, Ch. III in [12]):

$$E(\omega_a) = \sum_{M \in \Gamma/\Gamma_\infty} \omega_a|_M,$$

where $M = \begin{pmatrix} a^{(1)} & b^{(1)} \\ c^{(1)} & d^{(1)} \end{pmatrix} \times \cdots \times \begin{pmatrix} a^{(n)} & b^{(n)} \\ c^{(n)} & d^{(n)} \end{pmatrix}$ runs through a set of representatives of Γ/Γ_∞ , and $\omega_a|_M = M^* \omega_a$ by considering M as transformation on X_∞ . If the above series converges, then $E(\omega_a)$ defines a genuine vector valued differential form on X . The following simple transformation formulas

$$dz_i|_M = (c^{(i)} z_i + d^{(i)})^{-2} dz_i, \quad d\bar{z}_i|_M = (c^{(i)} \bar{z}_i + d^{(i)})^{-2} d\bar{z}_i, \\ y_i|_M = |c^{(i)} z_i + d^{(i)}|^{-2} y_i, \quad (e_{i1} + z_i e_{i2})|_M = (c^{(i)} z_i + d^{(i)})^{-1} (e_{i1} + z_i e_{i2})$$

show that the series $E(\omega_a)$ obeys the relation $E(\omega_a) = E_{\alpha,\beta}(z) \cdot \omega_a$, where

$$E_{\alpha,\beta}(z) = \sum_{M \in \Gamma/\Gamma_\infty} \prod_{i=1}^n (c^{(i)} z_i + d^{(i)})^{-\alpha_i} (c^{(i)} \bar{z}_i + d^{(i)})^{-\beta_i}$$

with

$$\alpha_i = \begin{cases} m_1 + 1 & \text{if } i \in a \\ m_1 + 2 & \text{if } i \in a^c \end{cases}, \quad \beta_i = \begin{cases} 1 & \text{if } i \in a \\ 0 & \text{if } i \in a^c. \end{cases}$$

This is the type of Eisenstein series considered in [12]. Note that for the constant coefficient they consider the border case $r = 1$, which requires the technique of Hecke summation to show the convergence of the series. In the current case, Lemma 5.7, Ch. I in [12] shows that $E_{\alpha,\beta}(z)$ is absolutely convergent. We show next that $E(\omega_a)$ is closed. For that, one considers the Fourier expansion of $E_{\alpha,\beta}(z)$ at ∞ , and as argued in Proposition 3.3, Ch. III in [12], the key point is to study the constant Fourier coefficient. The formula $a_0(y, s)$ in Page 170, [12] for $s = 0$, $r = \frac{m_1+2}{2}$ and α_i, β_i as above shows that the constant Fourier coefficient of $E_{\alpha,\beta}(z)$ is of the form $A + \frac{B}{\prod_{i=1}^n y_i^{2r-1}}$ with $A = 1, B = 0$ (see Theorem 4.9, Ch. III in [12]). One concludes from the proof of Proposition 3.3, Ch. III in [12] that $E(\omega_a)$ is closed (hence a cohomology class in $H^l(X, \mathbb{V}_m)$), and it induces the same cohomology class as ω_a in $H^l(X_\infty, \mathbb{V}_m)$. The same proposition also shows that the restriction of $E(\omega_a)$ to other cusps is zero. By the theory of Eisenstein cohomology, $\{E(\omega_a)\}_{a \subset \{1, \dots, n\}, |a|=l-n}$ forms a basis of $H_{\text{Eis}}^l(X, \mathbb{V}_m)$.

Step 3. Hodge type of Eisenstein cohomology classes. By the expression of ω_a in Claim 6.5, one knows that ω_a extends naturally to an element in $A_{\bar{X}}^{n,0}(\log S) \wedge \overline{A_{\bar{X}}^{|a|,0}(*S)} \otimes E_m^{|m|,0}$. Now by the expression of the Eisenstein series $E_{\alpha,\beta}(z)$ in [12, Theorem 4.9] for $A = 1, B = 0$, one sees that $E(\omega_a)$ lies again in $A_{\bar{X}}^{n,0}(\log S) \wedge \overline{A_{\bar{X}}^{|a|,0}(*S)} \otimes E_m^{|m|,0}$. The expression of $E(\omega_a)$ shows that it is of logarithmic singularity at infinity S . Therefore $E(\omega_a)$ represents a cohomology class in $F^{|m|+n}H^l(X, \mathbb{V}_m)$. \square

7. The mixed Hodge structure on the cohomology groups. Let $j : X \rightarrow X^*$ be the natural inclusion, and $H_{(2)}^n(X^*, \mathbb{V}_m)$ the L^2 -cohomology group. Recall that by the L^2 -harmonic theory, one has the following natural isomorphism:

$$H_{(2)}^n(X^*, \mathbb{V}_m) \cong \mathfrak{h}_{(2)}^n(X, \mathbb{V}_m) = \bigoplus_{p+q=n} \mathfrak{h}_{(2)}^{p,q}(X, \mathbb{V}_m).$$

Therefore, one has a natural map $H_{(2)}^n(X^*, \mathbb{V}_m) \rightarrow H^n(X, \mathbb{V}_m)$ by taking a cohomology class to its L^2 -harmonic representative. We assert the following

PROPOSITION 7.1. *The natural map $H_{(2)}^n(X^*, \mathbb{V}_m) \rightarrow H^n(X, \mathbb{V}_m)$ is injective.*

Proof. The proposition boils down to show the following statement: Assume we have $\omega \in \mathfrak{h}_{(2)}^n(X, \mathbb{V}_m)$ and $\alpha \in A_X^{n-1}(\mathbb{V}_m)$ satisfying $D(\alpha) = \omega$, then $\omega = 0$. In order to prove this we write $\omega = \sum_{p+q=n} \omega_{p,q}$ and further

$\omega_{p,q} = \sum_{I \subset \{1, \dots, n\}, |I|=p} \omega_{I, \bar{I}^c}$, where $\omega_{I, \bar{I}^c} = f_{I, \bar{J}} dz_I \wedge \overline{dz_{\bar{I}^c}}$ (see Lemma 5.4 and Proposition 5.5). It is enough to show $\omega_{I, \bar{I}^c} = 0$ for all possible I . Let $X_{\bar{I}^c}$ be the complex manifold considered in the proof of Proposition 5.5 for $J = I^c$ and $\bar{X}_{\bar{I}^c}$ a smooth toroidal compactification of $X_{\bar{I}^c}$. Let $\iota : X \rightarrow \bar{X}$ be the natural inclusion. By Deligne [7], the inclusion

$$\left(\left[\bigoplus_{p+q=\cdot} \mathcal{A}_{\bar{X}}^{p,0}(\log S) \wedge \overline{\mathcal{A}_{\bar{X}}^{q,0}(*S)} \right] \otimes \bar{\mathbb{V}}_m, D \right) \hookrightarrow (\iota_* \mathcal{A}_X(\mathbb{V}_m), D)$$

is a quasi-isomorphism. Furthermore, by E_1 -degeneration of the Hodge filtration, one has also the quasi-isomorphism

$$\begin{aligned} & \left(\left[\bigoplus_{p+q=\cdot} \mathcal{A}_{\bar{X}_{\bar{I}^c}}^{p,0}(\log S) \wedge \overline{\mathcal{A}_{\bar{X}_{\bar{I}^c}}^{q,0}(*S)} \right] \otimes \bar{\mathbb{V}}_m, D \right) \\ & \cong \left(\left[\bigoplus_{p+q=\cdot} \mathcal{A}_{\bar{X}_{\bar{I}^c}}^{p,0}(\log S) \wedge \overline{\mathcal{A}_{\bar{X}_{\bar{I}^c}}^{q,0}(*S)} \right] \otimes \bar{\mathbb{V}}_m, D''_{\bar{X}_{\bar{I}^c}} \right). \end{aligned}$$

It is not difficult to check that $\omega \in \left[\bigoplus_{p+q=n} \mathcal{A}_{\bar{X}_{\bar{I}^c}}^{p,0}(\log S) \wedge \overline{\mathcal{A}_{\bar{X}_{\bar{I}^c}}^{q,0}(*S)} \right] \otimes \bar{\mathbb{V}}_m$. So by the quasi-isomorphisms, we find actually $\alpha' \in \left[\bigoplus_{p+q=n-1} \mathcal{A}_{\bar{X}_{\bar{I}^c}}^{p,0}(\log S) \wedge \overline{\mathcal{A}_{\bar{X}_{\bar{I}^c}}^{q,0}(*S)} \right] \otimes \bar{\mathbb{V}}_m$ such that $D''\alpha' = \omega$. Note for fixed I , ω_{I, \bar{I}^c} is holomorphic over $\bar{X}_{\bar{I}^c}$. One has then

$$\begin{aligned} \langle \omega_{I, \bar{J}}, \omega_{I, \bar{J}} \rangle &= \langle (D''\alpha')_{I, \bar{J}}, \omega_{I, \bar{J}} \rangle \\ &= \langle (\bar{\partial}_{\bar{X}}\alpha')_{I, \bar{J}}, \omega_{I, \bar{J}} \rangle + \langle (\theta_{\bar{X}}\alpha')_{I, \bar{J}}, \omega_{I, \bar{J}} \rangle \\ &= \langle \bar{\partial}_{\bar{X}}\alpha', \omega_{I, \bar{J}} \rangle + \langle \theta_{\bar{X}}\alpha', \omega_{I, \bar{J}} \rangle \\ &= \langle \theta_{\bar{X}}\alpha', \omega_{I, \bar{J}} \rangle \\ &= \langle \alpha', \theta_{\bar{X}}^* \omega_{I, \bar{J}} \rangle \\ &= \langle \alpha', 0 \rangle \\ &= 0, \end{aligned}$$

and therefore we get $\omega_{I, \bar{J}} = 0$. So $\omega = 0$, and the proof is completed. \square

This proposition allows us to obtain an important byproduct of our study of the MHS on the cohomology groups. Namely, we are able to show the truth of Conjecture 1.3 in the case of Hilbert modular varieties with coefficients.

THEOREM 7.2. *The natural isomorphism $r_k : H_{(2)}^k(X^*, \mathbb{V}_m) \cong IH^k(X^*, \mathbb{V}_m)$, $0 \leq k \leq n$ is an isomorphism of Hodge structures.*

Proof. Since by Corollary 5.9 the above statement is trivial for $k \neq n$, it suffices to consider the case $k = n$. Theorem 5.4 and Remark 5.5(i) [17] asserts

that the natural map $H_{(2)}^n(X^*, \mathbb{V}_m) \rightarrow H^n(X, \mathbb{V}_m)$, which is just the composite of the isomorphism r_k (of real vector spaces) with the natural morphism of mixed Hodge structures $IH^k(X^*, \mathbb{V}_m) \rightarrow H^n(X, \mathbb{V}_m)$, is actually a morphism of mixed Hodge structures and its image is identified with the lowest weight of the MHS of $H^n(X, \mathbb{V}_m)$. Now Proposition 7.1 implies further:

(i) The morphism $IH^k(X^*, \mathbb{V}_m) \rightarrow H^n(X, \mathbb{V}_m)$ is injective and therefore an isomorphism of Hodge structures $IH^k(X^*, \mathbb{V}_m) \cong W_{n+|m|}H^n(X, \mathbb{V}_m)$.

(ii) An isomorphism of Hodge structures $H_{(2)}^n(X^*, \mathbb{V}_m) \cong W_{n+|m|}H^n(X, \mathbb{V}_m)$.

Altogether, since both are identified with the same Hodge structure, the L^2 -cohomology and the intersection cohomology are isomorphic as Hodge structures. \square

LEMMA 7.3. *Let $(H_{\mathbb{R}}, W_{\cdot}, F^{\cdot})$ be a MHS with weights $\geq m+k$ and the following properties:*

$$H_{\mathbb{C}} = F^0 = \cdots = F^{m+n} \supsetneq F^{m+n+1} = 0, \quad 0 = W_{m+k} \subset \cdots \subset W_{2(m+k)} = H_{\mathbb{R}}$$

for certain $\frac{k+1}{2} \leq n \leq k$. Then the weight filtration must be of the form

$$0 = W_{m+k} = \cdots = W_{2(m+n)-1} \subsetneq W_{2(m+n)} = \cdots = W_{2(m+k)} = H_{\mathbb{R}}.$$

Proof. By the assumption on the Hodge filtration and the Hodge symmetry, it is easy to see that each graded piece of the weight filtration can have at most one Hodge type. This implies that the first possible weight with non-zero dimension is $W_{2(m+n)}$. But then $W_{2(m+n)}$ must be the whole space. This is because for any $i \geq 2(m+n)+1$, the unique Hodge component $(\frac{i}{2}, \frac{i}{2})$ of Gr_i^W (assume i even), which is a quotient of $F^{\frac{i}{2}} \cap W_{i, \mathbb{C}} = 0$, is zero. This implies the result. \square

PROPOSITION 7.4. *Let $(H_{\mathbb{R}}, W_{\cdot}, F^{\cdot})$ be a MHS with weights $\geq m+n$ with*

$$H_{\mathbb{C}} = F^0 \supset \cdots \supset F^{m+n} \supsetneq F^{m+n+1} = 0, \quad 0 \subset W_{m+n} \subset \cdots \subset W_{2(m+n)} = H_{\mathbb{R}}.$$

Let $H_{\mathbb{R}} = H_{1, \mathbb{R}} \oplus H_{2, \mathbb{R}}$ be a vector space decomposition. Assume that $H_{2, \mathbb{R}} \subset F^{m+n}$ and $H_{1, \mathbb{R}} \subset W_{m+n}$. Then the weight filtration is of the form $0 \subset H_{1, \mathbb{R}} = W_{m+n} = \cdots = W_{2(m+n)-1} \subsetneq W_{2(m+n)} = H_{\mathbb{R}}$, and the MHS $(H_{\mathbb{R}}, W_{\cdot}, F^{\cdot})$ is split over \mathbb{R} .

Proof. Consider the quotient MHS on $(\frac{H_{\mathbb{R}}}{W_{m+n}}, \tilde{W}_{\cdot}, \tilde{F}^{\cdot})$, where \sim means the quotient filtration. By the assumption on $H_{1, \mathbb{R}}$ and $H_{2, \mathbb{R}}$, one sees that the above quotient MHS is of the form in Lemma 7.3. Thus one obtains the assertion about the weight filtration W_{\cdot} except the equality $W_{m+n} = H_{1, \mathbb{R}}$.

Set $H^{p,q} := F^p \cap \bar{F}^q \cap W_{p+q, \mathbb{C}}$. As W_{m+n} is of the lowest weight in the weight filtration, it has a pure Hodge structure of weight $m+n$ induced by F^{\cdot} and its Hodge (p, q) -component on W_{m+n} is given by $F^p \cap \bar{F}^q \cap W_{m+n, \mathbb{C}}$. So $W_{m+n, \mathbb{C}} = \bigoplus_{p+q=m+n} H^{p,q}$ and $H^{p,q} \cap W_{m+n, \mathbb{C}} = 0$ for $p+q \neq m+n$. Because for $m+n <$

$p + q < 2(m + n)$ $W_{p+q} = W_{m+n}$, one has $H^{p,q} = H^{p,q} \cap W_{m+n, \mathbb{C}} = 0$. Now consider the weight $2(m + n)$ pure Hodge structure on $Gr_{2(m+n)}^W$. By Hodge symmetry and the indexing of the Hodge filtration, $H^{p, 2(m+n)-p}$ is zero unless $p = m + n$. In that case, $H^{m+n, m+n} = F^{m+n} \cap \bar{F}^{m+n}$. Note that $H_{2, \mathbb{R}} \subset H^{m+n, m+n}$ by the assumption. This implies the following relation:

$$W_{2(m+n), \mathbb{C}} = H_{\mathbb{C}} = H_{1, \mathbb{C}} \oplus H_{2, \mathbb{C}} \subset W_{m+n, \mathbb{C}} \oplus H^{m+n, m+n} \subset W_{2(m+n), \mathbb{C}}.$$

Therefore $H_{2, \mathbb{R}} = W_{m+n}$ and $H_{2, \mathbb{C}} = H^{m+n, m+n}$ hold, which also shows the relation $W_{l, \mathbb{C}} = \bigoplus_{p+q \leq l} H^{p,q}$ for each $m + n \leq l \leq 2(m + n)$. To show that $(H_{\mathbb{R}}, W, F)$ is split over \mathbb{R} , it remains to show that $F^p = \bigoplus_{r \geq p} H^{r,s}$ holds for each p (see Section 2 in [4]). Because $H = W_{m+n, \mathbb{C}} \oplus H_{2, \mathbb{C}}$ as shown above, and $H_{2, \mathbb{C}} \subset F^{m+n}$ by assumption, it follows that $F^p = F^p(W_{m+n, \mathbb{C}}) \oplus H_{2, \mathbb{C}}$ for each p . Now that $F^p(W_{m+n, \mathbb{C}}) = \bigoplus_{r \geq p} H^{r, m+n-r}$, one obtains then $F^p = \bigoplus_{r \geq p} H^{r,s}$ for each p . This proves the result. \square

Now we proceed to deduce our main results of the paper from the above results, together with the established information in previous sections. Let us return to the decomposition (see Section 6):

$$H^k(X, \mathbb{V}_{\mathbb{R}}) = H_{\dagger}^k(X, \mathbb{V}_{\mathbb{R}}) \oplus H_{\text{Eis}}^k(X, \mathbb{V}_{\mathbb{R}}).$$

By Proposition 7.1 and Theorem 7.2, we denote again by $IH^k(X^*, \mathbb{V}_m)$ the image of it in $H^k(X, \mathbb{V}_{\mathbb{R}})$. The cohomology classes in $H_{\dagger}^k(X, \mathbb{V}_{\mathbb{R}})$ are representable by differential forms with compact support, which are square integrable with respect to any complete Kähler metric on X . Therefore $H_{\dagger}^k(X, \mathbb{V}_{\mathbb{R}}) \subset H_{(2)}^k(X^*, \mathbb{V}_m) = IH^k(X^*, \mathbb{V}_m)$. The following result improves Theorem 6.1:

THEOREM 7.5. *For $k \neq n$, one has $H^k(X, \mathbb{V}_m) = H_{\text{Eis}}^k(X, \mathbb{V}_m)$. Furthermore, for $0 \leq k \leq n - 1$ and $k = 2n$, it holds $H^k(X, \mathbb{V}_m) = 0$, and for $n + 1 \leq k \leq 2n - 1$, $H^k(X, \mathbb{V}_m) = H_{\text{Eis}}^k(X, \mathbb{V}_m) \stackrel{r}{\cong} H^k(\partial X^{\sharp}, \mathbb{V}_m)$.*

Proof. By the above discussion, one knows that

$$\dim H_{\text{Eis}}^k(X, \mathbb{V}_m) \leq \dim H^k(X, \mathbb{V}_m) \leq \dim H_{(2)}^k(X^*, \mathbb{V}_m) + \dim H_{\text{Eis}}^k(X, \mathbb{V}_m).$$

By Corollary 5.9, it follows that $H^k(X, \mathbb{V}_m) = H_{\text{Eis}}^k(X, \mathbb{V}_m)$ for $k \neq n$. The remaining part of the theorem follows from Lemma 6.2 and Theorem 6.1. Also one notices that $H_{\text{Eis}}^{2n}(X, \mathbb{V}_m) = 0$, since ∂X^{\sharp} is of real dimension $2n - 1$. \square

THEOREM 7.6. *Let \mathbb{V}_m be the irreducible non-trivial local system as above and the cohomology group $H^k(X, \mathbb{V}_m)$ is equipped with Saito's MHS. Then:*

(i) *For $n + 1 \leq k \leq 2n - 1$, one has $H^k(X, \mathbb{V}_m) = H_{\text{Eis}}^k(X, \mathbb{V}_m)$ and the MHS on $H^k(X, \mathbb{V}_m)$ is pure and of pure type $(|m| + n, |m| + n)$.*

(ii) $IH^n(X^*, \mathbb{V}_m) = H_1^n(X, \mathbb{V}_m)$ and $H^n(X, \mathbb{V}_m) = IH^n(X^*, \mathbb{V}_m) \oplus H_{\text{Eis}}^n(X, \mathbb{V}_m)$ is the splitting of the weight filtration over \mathbb{R} .

(iii) For $n \leq k \leq 2n - 1$ the weight filtration is of the following form:

$$0 \subset W_{|m|+k} = \cdots = W_{2(|m|+n)-1} \subset W_{2(|m|+n)} = \cdots = W_{2(|m|+k)} = H^k(X, \mathbb{V}_m),$$

where $W_{|m|+k} = IH^k(X^*, \mathbb{V}_m)$ and $Gr_{2(|m|+n)}^W \cong H_{\text{Eis}}^k(X, \mathbb{V}_m)$. The Hodge filtration is of the following form:

$$H^k(X, \mathbb{V}_m) = F^0 \supset \cdots \supset F^{|m|+n} \supset 0,$$

in which $H_{\text{Eis}}^k(X, \mathbb{V}_m) \subset F^{|m|+n}$ holds.

Proof. For $n+1 \leq k \leq 2n-1$, (i) and (iii) follows from Theorems 7.5, 6.3 and Lemma 7.3. For $k=n$, one applies Proposition 7.4 for $H_{\mathbb{R}} = H^n(X, \mathbb{V}_m)$, $H_{1,\mathbb{R}} = H_1^n(X, \mathbb{V}_m)$ and $H_{2,\mathbb{R}} = H_{\text{Eis}}^n(X, \mathbb{V}_m)$. The condition for H_2 follows from Theorem 6.3. The relation $H_{1,\mathbb{R}} \subset IH^n(X^*, \mathbb{V}_m) \subset W_{|m|+n}$ follows from the above discussion. Then Proposition 7.4 implies that

$$H_{1,\mathbb{R}} = IH^n(X^*, \mathbb{V}_m) = W_{|m|+n},$$

and the splitting of the MHS over \mathbb{R} . □

For the MHS $(H^k(X, \mathbb{V}_m), W, F^\cdot)$, put

$$h_k^{P,Q} := \dim Gr_F^P Gr_{\bar{F}}^Q Gr_{P+Q}^W H^k(X, \mathbb{V}_m), \quad H_k^{P,Q} := F^P \cap \bar{F}^Q \cap W_{P+Q, \mathbb{C}}.$$

By Theorem 7.6, $\dim H_k^{P,Q} = h_k^{P,Q}$.

THEOREM 7.7. *Notation as above. The following statements hold:*

(i) $H^k(X, \mathbb{V}_m) = 0$ for $0 \leq k \leq n-1$ and $k = 2n$.

(ii) If $m_1 = \cdots = m_n$, then for $n+1 \leq k \leq 2n-1$

$$\begin{aligned} h_k^{[m|+n, |m|+n]} &:= \dim_{\mathbb{C}} F^{|m|+n} W_{2(|m|+n)} H^k(X, \mathbb{V}_m) \\ &= \dim_{\mathbb{C}} H^k(X, \mathbb{V}_m) = \binom{n-1}{k-n} h, \end{aligned}$$

where h is the number of cusps.

(iii) If not all m_i are equal, then $H^k(X, \mathbb{V}_m) = 0$ for $n+1 \leq k \leq 2n-1$.

Proof. They follow directly from Theorems 7.5, 7.6, 6.3 and Lemma 6.2. □

THEOREM 7.8. *One has the following natural isomorphisms:*

(i) For $n+1 \leq k \leq 2n-1$, $H_k^{[m|+n, |m|+n]} \cong H^{k-n}(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$.

(ii) $H_n^{|m|+n,0} \cong H^0(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$, $H_n^{|m|+n,|m|+n} \cong H^0(S, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2}|_S)$, and for $0 \leq P \leq |m| + n - 1$, $P + Q = |m| + n$,

$$H_n^{P,Q} \cong \bigoplus_{\substack{I \subset \{1, \dots, n\}, \\ |m_I| + |I| = P}} H^{k-|I|} \left(\bar{X}, \bigotimes_{i \in I} \mathcal{L}_i^{m_i+2} \otimes \bigotimes_{i \in I^c} \mathcal{L}_i^{-m_i} \right).$$

Proof. (i) follows directly from Theorem 7.6(i) and Corollary 3.4. By Theorem 7.6, one has for $0 \leq P \leq |m| + n - 1$ and $P + Q = |m| + n$,

$$H_n^{P,Q} = Gr_F^P H^n(X, \mathbb{V}_m).$$

The isomorphisms for these $H_n^{P,Q}$ follow from Corollary 3.4. By Theorems 7.6 and 7.7(ii), one has

$$Gr_F^{|m|+n} H^n(X, \mathbb{V}_m) = F^{|m|+n} = H_n^{|m|+n,0} \oplus H_n^{|m|+n,|m|+n},$$

and $H_n^{|m|+n,0} = H_{(2)}^{|m|+n,0}(X^*, \mathbb{V}_m)$. By Corollary 5.10,

$$H_n^{|m|+n,0} \cong H^0 \left(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2} \right),$$

and by Corollary 3.4, $Gr_F^{|m|+n} H^n(X, \mathbb{V}_m) \cong H^0(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$ (the previous two isomorphisms are in fact equalities). Finally, one notes that $H^1(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2})$ is dual to $H^{n-1}(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i})$ by the Serre duality. Since it is zero by the next lemma, the long exact sequence of sheaf cohomologies of the following short exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2} \longrightarrow \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2} \longrightarrow \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2}|_S \longrightarrow 0$$

yields the isomorphism $H_n^{|m|+n,|m|+n} \cong H^0(S, \bigotimes_{i=1}^n \mathcal{L}_i^{m_i+2}|_S)$. \square

LEMMA 7.9. For $l < n$, $H^l(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i}) = 0$.

Proof. By Propositions 2.5 and 3.3, $\dim H^l(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i}) \leq \dim H^l(X, \mathbb{V}_m)$, which is zero for $0 \leq l \leq n - 1$ by Theorem 7.5. Thus $H^l(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i}) = 0$ for $l < n$. \square

We end this paper with some discussions on $\dim H^0(\bar{X}_{\bar{j}}, \mathcal{O}_{\bar{X}_{\bar{j}}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2})$, as well as the dimension of the middle degree cohomology.

PROPOSITION 7.10. Let c_i be the i -th chern class of $\bar{X}_{\bar{j}}$, c'_i be the i -th chern class of $T_{\bar{X}_{\bar{j}}}(-\log S_{\bar{j}})$, the dual vector bundle of $\Omega_{\bar{X}_{\bar{j}}}^1(\log S_{\bar{j}})$, and $P(c_1, \dots, c_n)$

We changed the opposite equation from inline equation to displayed one to avoid the bad line break. Please check.

be the degree n polynomial computing $\chi(\bar{X}_{\bar{J}}, \mathcal{O}_{\bar{X}_{\bar{J}}})$ in the Hirzebruch-Riemann-Roch formula. The following formula holds:

$$\begin{aligned} & \dim H^0 \left(\bar{X}_{\bar{J}}, \mathcal{O}_{\bar{X}_{\bar{J}}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2} \right) \\ &= (-1)^n \left[\prod_{i=1}^n (m_i + 1) - 1 \right] P(c'_1, \dots, c'_n) + (-1)^n P(c_1, \dots, c_n). \end{aligned}$$

Proof. Consider first the $J = \emptyset$ case. The above lemma implies that

$$\begin{aligned} & \dim H^0 \left(\bar{X}, \mathcal{O}_{\bar{X}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2} \right) \\ &= \dim H^n \left(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i} \right) \\ &= (-1)^n \chi \left(\bar{X}, \bigotimes_{i=1}^n \mathcal{L}_i^{-m_i} \right) \\ &= (-1)^n \left[\prod_{i=1}^n (m_i + 1) - 1 \right] \prod_{i=1}^n \mathcal{L}_i + (-1)^n \chi(\bar{X}, \mathcal{O}_{\bar{X}}). \end{aligned}$$

By Hirzebruch proportionality in the non-compact case (see Theorem 3.2 in [28]) and Proposition 3.2, it follows that $\prod_{i=1}^n \mathcal{L}_i = P(c'_1, \dots, c'_n)$. For a general J , one notices that the corresponding statement in Lemma 7.9 holds also for $X_{\bar{J}}$. The above argument works verbatim for the general case. \square

However one can not conclude that $P(c_1, \dots, c_n) = P(c'_1, \dots, c'_n)$ in general. In fact, by Proposition 5.8, the space $H^0(\bar{X}_J, \mathcal{O}_{\bar{X}_J}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2})$ is exactly the space of cusp forms on \mathbb{H}^n with respect to the discrete subgroup Γ_J . The dimension formula of cusp forms by means of Selberg's trace formula in such a case is the main result of the work [38] by Shimizu for regular m (see [18] for the irregular case). By his formula in the regular case (Theorem 11 [38]) the dimension is not proportional to $\prod_{i=1}^n (m_i + 1)$ because of an error term coming from the cusps. The same formula also shows that the dimension of cusp forms with respect to Γ_J is generally different from that with respect to Γ . These subtleties do not arise in the case of compact quotients studied by Matsushima-Shimura [24]. Now for each subset $J \subset \{1, \dots, n\}$, we put

$$h(J, m) = \dim H^0 \left(\bar{X}_{\bar{J}}, \mathcal{O}_{\bar{X}_{\bar{J}}}(-S) \otimes \bigotimes_{j=1}^n \mathcal{L}_j^{m_j+2} \right).$$

It is independent of the choice of a smooth toroidal compactification of $X_{\bar{j}}$ and $h(J, m) = h(J^c, m)$ by Hodge isometry.

PROPOSITION 7.11. *Let \mathbb{V}_m be an irreducible non-trivial local system over X . Put $\delta(m) = 1$ when $m_1 = \cdots = m_n$ is satisfied and otherwise zero. Then it holds that*

$$\dim H^n(X, \mathbb{V}_m) = \delta(m)h + \sum_{I \subset \{1, \dots, n\}} h(I, m).$$

Moreover, for $P + Q = |m| + n$,

$$h_n^{P, Q} = \sum_{\substack{I \subset \{1, \dots, n\}, \\ |m_I| + |I| = P}} h(I, m),$$

and $h_n^{|m|+n, |m|+n} = \delta(m)h$. Otherwise $h_n^{P, Q} = 0$.

Proof. It follows from Corollary 5.10, Propositions 5.7, 5.8, Theorems 7.6(ii), 6.3 and Lemma 6.2. \square

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