On the monodromy of the moduli space of Calabi–Yau threefolds coming from eight planes in $\mathbb{P}^3$

**Ralf Gerkmann · Mao Sheng · Duco van Straten ·
Kang Zuo**

Abstract It is a fundamental problem in geometry to decide which moduli spaces of polarized algebraic varieties are embedded by their period maps as Zariski open subsets of locally Hermitian symmetric domains. In the present work we prove that the moduli space of Calabi–Yau threefolds coming from eight planes in $\mathbb{P}^3$ does *not* have this property. We show furthermore that the monodromy group of a good family is Zariski dense in the corresponding symplectic group. Moreover, we study a natural sublocus which we call hyperelliptic locus, over which the variation of Hodge structures is naturally isomorphic to wedge product of a variation of Hodge structures of weight one. It turns out the hyperelliptic locus does not extend to a Shimura subvariety

R. Gerkmann (✉)
Mathematisches Institut, Universität München, 80333 Munich, Germany
e-mail: ralf.gerkmann@math.lmu.de

M. Sheng
Department of Mathematics, East China Normal University, Shanghai 200062,
People's Republic of China
e-mail: sheng@uni-mainz.de; msheng@math.ecnu.edu.cn

D. van Straten · K. Zuo
Institut für Mathematik, Universität Mainz, 55099 Mainz, Germany
e-mail: straten@mathematik.uni-mainz.de

K. Zuo
e-mail: zuok@uni-mainz.de

of type III (Siegel space) within the moduli space. Besides general Hodge theory, representation theory and computational commutative algebra, one of the proofs depends on a new result on the tensor product decomposition of complex polarized variations of Hodge structures.

1 Introduction

A fundamental result of E. Viehweg [31] states that for any polarized algebraic variety the coarse moduli space $\mathfrak{M}$ exists as a quasi-projective variety. It is of great interest to characterize those cases in which $\mathfrak{M}$ is a locally Hermitian symmetric variety via a period map. It has been shown in [19,32,33] that Arakelov-type equalities lead to sufficient conditions for this to happen. In this paper we describe the techniques of *characteristic varieties* that leads to necessary conditions that can be checked by a straightforward calculation in concrete examples. This leads to a computational tool that we apply to the moduli space of double octics ramified over an arrangement of eight planes in $\mathbb{P}^3$.

As over a coarse moduli space $\mathfrak{M}$ there usually does not exist a family, we use the following weaker notion. We say that a proper smooth map $f : \mathcal{X} \to S$ over a smooth connected base S is a *good family* for $\mathfrak{M}$, if the moduli map $S \to \mathfrak{M}$ of f is *dominant and generically finite*. The local system $\mathbb{V} := (R^n f_* \mathbb{Q}_\mathcal{X})_{pr}$ of primitive cohomomogies has the structure of a weight n polarized variation of $\mathbb{Q}$-Hodge structures, in short $\mathbb{Q}$-PVHS, over S. Recall that this means, among other things, that there is an Hodge filtration $\mathcal{F}^\bullet$ on the vector bundle $\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_S$ with flat connection ∇ for which *Griffiths transversality* $\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega_S^1$ holds. The associated graded object $(E, \theta) = (gr_F \mathcal{V}, gr_F \nabla) = (\sum_{p+q=n} E^{p,q}, \oplus_{p+q=n} \theta^{p,q})$ is called the *associated Higgs bundle*, where $E^{p,q} := \mathcal{F}^p/\mathcal{F}^{p-1}$ are the Hodge bundles and θ is induced by ∇. In general, we call a PVHS $\mathbb{V}$ over S of *Calabi–Yau type* (CY-type) if rank $E^{n,0} = 1$ and the morphism of vector bundles

$$\mathcal{T}_S \longrightarrow \mathrm{Hom}(E^{n,0}, E^{n-1,1})$$

induced by $\theta^{n,0} : E^{n,0} \to E^{n-1,1} \otimes \Omega_S^1$ is an isomorphism at the generic point. It follows from the Bogomolov-Tian-Todorov theorem on the unobstructedness of the infinitesimal deformations of a Calabi–Yau (CY) manifold X, that for a good family the map $\theta^{n,0}$ is naturally identified with the Kodaira-Spencer isomorphism, so in such a situation we obtain a PVHS of CY-type.

In this paper we study a particular interesting family of CY threefolds. An arrangement $\mathfrak{A}$ of eight planes in general position in $\mathbb{P}^3$ determines a double cover X, which is a CY variety with singularities along 28 lines. A resolution $\widetilde{X}$ of such a double octic has dim $H^3(\widetilde{X}) = 20$ and carries a weight 3 polarized Hodge structure with Hodge numbers $(1, 9, 9, 1)$. If we vary the arrangement $\mathfrak{A}$ in a good family, we obtain an irreducible weight three $\mathbb{Q}$-PVHS $\mathbb{V}$ of CY-type over a smooth 9 dimensional base S. We will show several theorems about $\mathbb{V}$.

Theorem 1.1 $\mathbb{V}$ *does not factor canonically.*

By this we mean the following. Associated to a Hermitian symmetric domain $D_0 = G_0/K_0$ there is a special PVHS $\mathbb{W}$ on D_0 coming from the representation $\rho_{can} : G_0 \to GL(W)$, which is called the canonical PVHS by B. Gross [12]. We say that $\mathbb{V}$ *factors canonically* if the period map of $\mathbb{V}$ factors through the one determined by a certain canonical PVHS (D_0, ρ_{can}). For a precise definition we refer the reader to Sect. 3.

A way to exclude this happening consists of picking an appropriate point $s \in S$, computing an appropriate *characteristic subvariety* of $\mathbb{V}$ in $\mathbb{P}(\mathcal{T}_{S,s})$ and comparing it with the corresponding object for $\mathbb{W}$. If these varieties are not isomorphic we are done.

Our example is a member of a well-known infinite series of CY n-folds coming from double covers of generic arrangements of $2n + 2$ hyperplanes in $\mathbb{P}^n$. For $n = 1$ one has the classical theory of four points in $\mathbb{P}^1$ and the associated elliptic curves and their modulus. The paper [18] and the book [34] were devoted to $n = 2$ case where we have K3-surfaces that are double covers of six lines in the plane. In [18] it was shown among other things that in this case there is a natural good family whose associated weight 2 PVHS factors canonically. It was asked by I. Dolgachev ([2]) if the cases with $n \geq 3$ are canonical with respect to $(D_{n,n}^I, \rho_{can})$, where $\rho_{can} : SU(n, n) \xrightarrow{\wedge^n} Sp\left(\binom{2n}{n}, \mathbb{R}\right)$ is the indicated representation of real Lie groups. Motivated by this question, it was checked in [27] that the primitive Hodge numbers of CY n-folds are exactly the same as predicted by Dolgachev for all $n \in \mathbb{N}$.

In the pioneering work [26] a result similar to Theorem 1.1 for more general moduli spaces of configurations was given, but the methods used there are completely different from ours. Our method is based on classical Hodge theory (see for example [10]) and can be applied to many other concrete moduli spaces, for example the moduli spaces of the CY varieties in toric varieties.

Using Theorem 1.1 we will prove the

Theorem 1.2 *Let $s \in S$ be a base point and let*

$$\tau : \pi_1(S, s) \to Sp(20, \mathbb{Q})$$

be the monodromy representation associated to $\mathbb{V}$. Then the image of τ is Zariski dense.

Using results of C. Schoen and P. Deligne the above theorem implies:

Corollary 1.3 *The special Mumford–Tate group of a general member in $\mathfrak{M}_{CY}$ is $Sp(20, \mathbb{Q})$.*

However, there exists an interesting subvariety of $\mathfrak{M}_{CY}$ where the special Mumford–Tate groups are proper subgroups of $Sp(20, \mathbb{Q})$. Generalizing a construction from [18], we define a five-dimensional subvariety $\mathfrak{H}_{CY} \subset \mathfrak{M}_{CY}$ that we call the *hyperelliptic locus*. Over it, the Hodge structure is isomorphic to $\wedge^3$ of a $H^1(C)$, where C is a hyperelliptic curve of genus three. It is natural to ask if this decomposition can be extended to a larger variety $\mathfrak{H}$ that contains $\mathfrak{H}_{CY}$. Using a calculation of characteristic subvarieties we arrive at a negative answer.

Theorem 1.4 *Let $\mathfrak{H}_{CY}$ be the hyperelliptic locus of $\mathfrak{M}_{CY}$ and $\mathfrak{H}$ be any irreducible subvariety of $\mathfrak{M}_{CY}$ which strictly contains $\mathfrak{H}_{CY}$. Let $f : \mathcal{X} \to S$ be a good family for $\mathfrak{M}_{CY}$ whose moduli map $S \to \mathfrak{M}_{CY}$ is dominant over $\mathfrak{H}$. Then the restriction of $\mathbb{V}$ to the inverse image of $\mathfrak{H}$ does not factor through $(D_3^{III}, \wedge^3)$.*

As a corollary we have the following

Corollary 1.5 *The special Mumford–Tate group of the Calabi–Yau threefolds in $\mathfrak{H}_{CY}$ is a subgroup of $Sp(6, \mathbb{Q})$. Furthermore $\mathfrak{H}_{CY}$ is maximal with this property. That is, for any irreducible subvariety $\mathfrak{H}$ of $\mathfrak{M}_{CY}$ which strictly contains $\mathfrak{H}_{CY}$, the special Mumford–Tate group of a general closed point in $\mathfrak{H}$ is not contained in $Sp(6, \mathbb{Q})$.*

The proof of Theorem 1.2 relies on new results of Hodge-theoretical nature. There is Theorem 1.6 on the tensor product decomposition of $\mathbb{C}$-PVHS, parallel to the direct sum decomposition of $\mathbb{C}$-PVHS due to P. Deligne (cf. [5]). Let $\bar{S}$ denote a projective manifold, Z a simple divisor with normal crossing and $S = \bar{S} \backslash Z$. Let $\mathbb{V}$ denote an irreducible $\mathbb{C}$-PVHS over S with quasi-unipotent local monodromy around each component of Z. Fix a base point $s \in S$ let

$$\rho : \pi_1(S, s) \longrightarrow \mathrm{GL}(\mathbb{V}_s)$$

denote the representation of the fundamental group associated to the underlying local system of $\mathbb{C}$-vector spaces, where $\mathbb{V}_s$ denotes the fibre of $\mathbb{V}$ over s. Let G be the Zariski closure of the image of ρ inside $\mathrm{GL}(\mathbb{V}_s)$. Assume that G decomposes into a direct product $G_1 \times \cdots \times G_k$ of simple Lie groups. Then by Schur's lemma, we obtain a decomposition of local systems $\mathbb{V} \cong \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_k$.

Theorem 1.6 *Each local system $\mathbb{V}_i$ admits the structure of a $\mathbb{C}$-PVHS such that the induced $\mathbb{C}$-PVHS on the tensor product $\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_k$ coincides with the given $\mathbb{C}$-PVHS on $\mathbb{V}$.*

The proof of the theorem is independent of the other results in this paper. We expect the result to be useful in other situations. In this article it helps to prove the following classification result.

Theorem 1.7 *Let S be a smooth quasi-projective algebraic variety and $\mathbb{V}$ be a weight 3 $\mathbb{Z}$-PVHS over S such that $\mathbb{V} \otimes \mathbb{C}$ is an irreducible complex local system. If the Hodge numbers of $\mathbb{V}$ are $(1, 9, 9, 1)$, then after a possible finite étale base change the connected component of the real Zariski closure of the monodromy group of $\mathbb{V}$ is one of the following:*

(A) $SU(1, 1) \times SO_0(2, 8)$,
(B) $SU(3, 3)$,
(C) $Sp(20, \mathbb{R})$.

Using this theorem and Theorem 1.1 to exclude cases (A) and (B) one easily obtains Theorem 1.2.

2 Two Calabi–Yau threefolds

An arrangement $\mathfrak{A} = (H_1, \ldots, H_8)$ of eight planes in $\mathbb{P}^3$ can be given by a matrix $A \in M(8 \times 4, \mathbb{C})$, the ith row corresponding to the defining equation

$$\sum_{j=1}^{4} a_{ij} x_j = 0$$

of the hyperplane H_i. We say that $\mathfrak{A}$ is *in general position* if no four of the planes intersect in a point. In terms of the matrix A this means that each (4×4)-minor is non-zero. We now describe two closely related CY threefolds associated to such an arrangement $\mathfrak{A}$ in general position.

2.1 The double octic

The planes of the arrangement $\mathfrak{A}$ determine a divisor $R = \sum_{i=1}^{8} H_i$ on $\mathbb{P}^3$. As the degree of R is even and the Picard group of $\mathbb{P}^3$ has no torsion, there exists a *unique* double cover $\pi : X \to \mathbb{P}^3$ that ramifies over R. The singular locus of such a double octic X is precisely the preimage of the singular locus of R. Its irreducible components are given by the lines $H_{ij} = H_i \cap H_j$ for $1 \leq i < j \leq 8$. We fix an ordering of the index set $I = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i < j \leq 8\}$ and let $\phi : \widetilde{\mathbb{P}}^3 \to \mathbb{P}^3$ denote the composition of blow-ups whose centers are the strict transforms of H_{ij}, taken in the chosen order. The normalization $\widetilde{X}$ of the fibre product $X \times_{\mathbb{P}^3} \widetilde{\mathbb{P}}^3$ sits in a commutative diagram

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\psi} & X \\
\widetilde{\pi} \downarrow & & \downarrow \pi \\
\widetilde{\mathbb{P}}^3 & \xrightarrow{\phi} & \mathbb{P}^3
\end{array}
$$

In case we start with an arrangement in *general position*, the variety $\widetilde{X}$ thus obtained is a smooth CY threefold. Note however that a different ordering of I yields a different birational minimal model of the singular variety X.

Lemma 2.1 *The space of infinitesimal deformations of $\widetilde{X}$ is naturally isomorphic to the space of infinitesimal deformations of $\mathfrak{A}$.*

Proof This follows from the description of the infinitesimal deformation space of a double cover obtained in [3]. Let $\mathcal{L}$ (resp. $\tilde{\mathcal{L}}$) be the line bundle on $\mathbb{P}^3$ (resp. $\widetilde{\mathbb{P}}^3$) in the decomposition

$$\pi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{L}^{-1} \left(\text{resp. } \pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{\mathbb{P}}^3} \oplus \tilde{\mathcal{L}}^{-1} \right).$$

It satisfies $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_{\mathbb{P}^3}(R)$ (resp. $\tilde{\mathcal{L}}^{\otimes 2} \simeq \mathcal{O}_{\widetilde{\mathbb{P}}^3}(\widetilde{R})$ for $\widetilde{R}$ the strict transform of R under ϕ). One has the decomposition of the direct image of the tangent bundle

$$\pi_* \mathcal{T}_{\widetilde{X}} = \mathcal{T}_{\mathbb{P}^3} \left(- \log \widetilde{R} \right) \oplus \mathcal{T}_{\widetilde{\mathbb{P}}^3} \otimes \tilde{\mathcal{L}}^{-1}.$$

It follows that one has a natural isomorphism (see also Proposition 2.1 [3])

$$H^1 \left(\mathcal{T}_{\widetilde{X}} \right) \simeq H^1 \left(\mathcal{T}_{\mathbb{P}^3} \left(- \log \widetilde{R} \right) \right) \oplus H^1 \left(\mathcal{T}_{\widetilde{\mathbb{P}}^3} \otimes \tilde{\mathcal{L}}^{-1} \right).$$

By Corollary 4.3 in [3], the space $H^1(\mathcal{T}_{\mathbb{P}^3}(- \log \widetilde{R}))$ is naturally isomorphic to the space of *equisingular deformations* of R in $\mathbb{P}^3$. Since R is a divisor with normal crossings, this space is isomorphic to the space of infinitesimal deformations of the arrangement $\mathfrak{A}$ in $\mathbb{P}^3$. Furthermore, the space of transverse deformations $H^1(\mathcal{T}_{\widetilde{\mathbb{P}}^3} \otimes \tilde{\mathcal{L}}^{-1})$ has dimension (after Proposition 5.1 [3])

$$
\begin{aligned}
h^1(\mathcal{T}_{\widetilde{\mathbb{P}}^3} \otimes \tilde{\mathcal{L}}^{-1}) &= h^1(\mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{L}^{-1}) + \sum_{(i,j) \in I} h^0(K_{H_{ij}}) \\
&= h^1(\mathcal{T}_{\mathbb{P}^3} \otimes \mathcal{O}(-3)) + \sum_{(i,j) \in I} h^0(K_{\mathbb{P}^1}) \\
&= 0,
\end{aligned}
$$

where the vanishing of the first summand follows from Bott's vanishing theorem on homogenous vector bundles over a projective space. The lemma thus follows. □

From now on we let $\mathfrak{M}_{AR}$ denote the coarse moduli space of arrangements $\mathfrak{A}$ of eight planes in $\mathbb{P}^3$ in general position, and let $\mathfrak{M}_{CY}$ denote the coarse moduli space of $\widetilde{X}$.

Corollary 2.2 *The moduli map $\mathfrak{M}_{AR} \to \mathfrak{M}_{CY}$ is étale.*

There are many ways to construct a good family for $\mathfrak{M}_{AR}$. Here is one. Let $\mathfrak{A}$ be an arrangement in general position. It is easy to verify that under the automorphism group of $\mathbb{P}^3$ one can transform in a unique way the ordered first five planes $(H_1, \ldots, H_5)$ of $\mathfrak{A}$ into the ordered five planes in $\mathbb{P}^3$, that are given by the first five rows in the following matrix A. Hence the moduli point of $\mathfrak{A}$ in $\mathfrak{M}_{AR}$ can be uniquely represented by the matrix A of the form:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & * & * & * \\
0 & 0 & 1 & 0 & 1 & * & * & * \\
0 & 0 & 0 & 1 & 1 & * & * & *
\end{pmatrix}^t
$$

Conversely a matrix A in the above form whose all 4×4 minors are nonzero represents an arrangement $\mathfrak{A}$ in general position. Thus, $\mathfrak{M}_{AR}$ can be realized as an open subvariety of the affine space $\mathbb{C}^9$ and it admits a natural good family $f_0 : \mathcal{X} \to \mathfrak{M}_{AR}$, where $\mathcal{X}$ is obtained by the simultaneous resolution of the singular double octic over $\mathfrak{M}_{AR}$. Note also that a good family for $\mathfrak{M}_{AR}$ gives rise to a good family for $\mathfrak{M}_{CY}$.

Remark 2.3 This construction and the above lemma can actually be generalized to arrangements of $2n + 2$ hyperplanes in $\mathbb{P}^n$. It yields a coarse moduli space of smooth CY n-folds whose primitive Hodge numbers are given by $h_{pr}^{p,n-p}(\widetilde{X}) = \binom{n}{p}^2$. For details, we refer the reader to [27].

Now we proceed with the construction of another CY threefold Y for a given arrangement $\mathfrak{A}$.

2.2 The Kummer cover

Let $A = (a_{ij})$ denote a (8×4)-matrix associated with $\mathfrak{A}$ as described above. Furthermore let $B = (b_{ij})$ denote a matrix in $M(4 \times 8, \mathbb{C})$ such that the sequence

$$0 \longrightarrow \mathbb{C}^4 \xrightarrow{A} \mathbb{C}^8 \xrightarrow{B} \mathbb{C}^4 \longrightarrow 0$$

is exact. We let Y denote the complete intersection of the four quadrics in $\mathbb{P}^7$ defined by the four equations

$$b_{i1}y_1^2 + b_{i2}y_2^2 + \cdots + b_{i8}y_8^2 = 0, \quad 1 \leq i \leq 4.$$

In case $\mathfrak{A}$ is in general position, the space Y is smooth (see Proposition 3.1.2 in [29]). There is a simple relation between the singular double octic X and Y. To describe it, let $G_1 = \mathbb{F}_2^8$ denote the elementary abelian 2-group of order 256. For $a = (a_1, \ldots, a_8) \in G_1$ we define an automorphism $\sigma_a : \mathbb{P}^7 \to \mathbb{P}^7$ by

$$\sigma_a(x_1 : \cdots : x_i : \cdots : x_8) = ((-1)^{a_1}x_1 : \cdots : (-1)^{a_i}x_i : \cdots : (-1)^{a_8}x_8).$$

The group G_1 contains a distinguished normal subgroup $N_1 \triangleleft G_1$ of index two, the kernel of the map $a \mapsto \sum_{i=1}^8 a_i$.

Lemma 2.4 *The map $\pi_1 : \mathbb{P}^7 \to \mathbb{P}^7$, $(y_1 : \cdots : y_8) \mapsto (y_1^2 : \cdots : y_8^2)$ defines a cover $Y \to \mathbb{P}^3$ of degree 2^7.*

Proof The matrix A defines a linear embedding $j : \mathbb{P}^3 \to \mathbb{P}^7$ of projective spaces. Obviously an arbitrary point $y = (y_1 : \cdots, : y_8) \in \mathbb{P}^7$ is contained in the complete intersection Y if and only if the corresponding vector in $\mathbb{C}^8$ lies in the kernel of B. By exactness, this implies that $\pi_1(y)$ is contained in $j(\mathbb{P}^3)$. $\square$

Proposition 2.5 $X \cong Y/N_1$

Proof As $\ker(B) = \operatorname{im}(A)$, the map $\pi_1 : \mathbb{P}^7 \to \mathbb{P}^7$, $(y_1 : \cdots : y_8) \mapsto (y_1^2 : \cdots : y_8^2)$ maps Y onto the image of j and realizes $\mathbb{P}^3$ as the quotient of Y by G_1. The quotient map π_1 factors over Y/N_1, and the degree of Y/N_1 over $\mathbb{P}^3$ is two. The ramification locus of the resulting map $\alpha : Y/N_1 \to \mathbb{P}^3$ is precisely R. Indeed, the nontrivial element in G_1/N_1 is represented by any vector $e_i = (0, \ldots, 1, \ldots, 0)$ in the standard bases of $\mathbb{F}_2^8$, $1 \leq i \leq 8$. Consequently, the ramification locus consists of all points in $\mathbb{P}^7$

with one coordinate zero. The embedding $j : \mathbb{P}^3 \to \mathbb{P}^7$ maps the planes $H_1, \ldots, H_8$ onto the intersection of $j(\mathbb{P}^3)$ with the coordinate hyperplanes in $\mathbb{P}^7$. As the double octic X is uniquely determined by R, it follows that $X \cong Y/N_1$. □

This geometric relation between Y, X and $\widetilde{X}$ immediately leads to the following isomorphism of $\mathbb{Q}$-Hodge structures.

Proposition 2.6 *For a given arrangement $\mathfrak{A}$ in general position there exists a natural isomorphism*

$$\mathrm{H}^3(Y, \mathbb{Q})^{N_1} \cong \mathrm{H}^3(X, \mathbb{Q}) \cong \mathrm{H}^3(\widetilde{X}, \mathbb{Q})$$

of rational polarized Hodge structures, where $\mathrm{H}^3(Y, \mathbb{Q})^{N_1}$ denotes the subspace of invariants under N_1.

Proof As $X \cong Y/N_1$, one has immediately $\mathrm{H}^3(Y, \mathbb{Q})^{N_1} \cong \mathrm{H}^3(X, \mathbb{Q})$ (see 5.2.4 [13]), showing that $\mathrm{H}^3(X, \mathbb{Q})$ carries a pure Hodge structure. Let $\psi : \widetilde{X} \longrightarrow X$ be the resolution map described above. As in general the kernel of the map $\psi^* : \mathrm{H}^3(X, \mathbb{Q}) \longrightarrow \mathrm{H}^3(\widetilde{X}, \mathbb{Q})$ is the part of smaller Hodge-weight (see Corollary 5.42 [22]), we conclude that ψ^* is injective. As the dimensions agree, ψ^* is an isomorphism. (This, of course, can also directly be seen from the Leray spectral sequence for ψ.) □

2.3 The hyperelliptic locus

There exist an interesting locus in $\mathfrak{M}_{AR}$ where the Hodge-structure is a third exterior power:

$$\mathrm{H}^3(\widetilde{X}, \mathbb{Q}) = \wedge^3 \mathrm{H}^1(C, \mathbb{Q})$$

where C is a hyperelliptic curve of genus 3. Such a C is obtained by a two-fold cover of $\mathbb{P}^1$ ramified over eight points. There exists a natural Galois covering with Galois group S_3, the permutation group of three letters:

$$\gamma : (\mathbb{P}^1)^3 \longrightarrow \mathrm{Sym}^3(\mathbb{P}^1) = \mathbb{P}^3.$$

Here the identification attaches to a divisor of degree 3 the ray of its equation in $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(3))$.

Lemma 2.7 *Let $(p_1, \ldots, p_8)$ denote a collection of eight distinct points in $\mathbb{P}^1$, and put $H_i = \gamma(\{p_i\} \times \mathbb{P}^1 \times \mathbb{P}^1)$. Then $\mathfrak{A} = (H_1, \ldots, H_8)$ is an arrangement of planes in general position.*

Proof The divisors of degree 3 in $\mathbb{P}^1$ containing a given point form a hyperplane. Moreover a divisor of degree 3 can not contain 4 distinct points. This means that no four hyperplanes in $\mathfrak{A}$ do meet. □

Let C be the hyperelliptic curve of genus 3 branched at $p_1, \ldots, p_8 \in \mathbb{P}^1$, and let $q : C \to \mathbb{P}^1$ denote the corresponding covering map. The threefold product

$$h : C^3 \xrightarrow{q^3} (\mathbb{P}^1)^3 \xrightarrow{\gamma} \mathbb{P}^3$$

is a Galois covering of degree $2^3 \cdot 6 = 48$. Its Galois group G_2 is isomorphic to a semi-direct product $N \rtimes S_3$, where $N = \langle \iota_1, \iota_2, \iota_3 \rangle$ is the group generated by the hyperelliptic involution. We let N_2 denote the index two subgroup $N' \rtimes S_3$ of G_2, where N' is the kernel of $N \cong \mathbb{F}_2^3 \xrightarrow{\Sigma} \mathbb{F}_2$, where the isomorphism sends ι_1 to $(1, 0, 0)$ etc. We factor h over the set of N_2-orbits and obtain a commutative diagram

$$
\begin{array}{ccc}
C^3 & \xrightarrow{\delta} & C^3/N_2 \\
q^3 \downarrow & & \downarrow \pi \\
(\mathbb{P}^1)^3 & \xrightarrow{\gamma} & \mathbb{P}^3
\end{array}
$$

Lemma 2.8 *The double cover $\pi : C^3/N_2 \to \mathbb{P}^3$ branches along the union of the hyperplanes $H_1, \ldots, H_8$ associated to the points $p_1, \ldots, p_8$. So $C^3/N_2 \cong X$, where X is the double octic determined by $H_1, \ldots, H_8$.*

Proof The Galois group of π is isomorphic to G_2/N_2, and it is generated by the image of ι_1. Hence the ramification locus of π is the image of the fixed locus $L_1 \subseteq C^3$ of ι_1 under $\pi \circ \delta$. By the commutativity of the above diagram, it coincides with the image of L_1 under $\gamma \circ q^3$. Obviously we have

$$q^3(L_1) = \{p_1, \ldots, p_8\} \times \mathbb{P}^1 \times \mathbb{P}^1$$

so that our claim follows from Lemma 2.7. As C^3/N_2 and X are both double covers of $\mathbb{P}^3$ with the same ramification divisor, they are isomorphic. $\qquad\square$

Proposition 2.9 *Let $p_1, \ldots, p_8$ be eight distinct points in $\mathbb{P}^1$, $\mathfrak{A}$ the associated arrangement of hyperplanes and $\widetilde{X}$ the CY manifold which corresponds to this arrangement. Furthermore, let $q : C \to \mathbb{P}^1$ denote the hyperelliptic curve which ramifies at $p_1, \ldots, p_8$. Then we have an isomorphism of rational polarized Hodge structures*

$$H^3(\widetilde{X}, \mathbb{Q}) \cong H^3(X, \mathbb{Q}) \cong \wedge^3 H^1(C, \mathbb{Q}).$$

Proof As $X \cong C^3/N_2$, we have

$$H^3(X, \mathbb{Q}) \cong H^3(C^3, \mathbb{Q})^{N_2}.$$

Since S_3 is contained in N_2, we have an inclusion

$$H^3(C^3, \mathbb{Q})^{N_2} \hookrightarrow H^3(C^3, \mathbb{Q})^{S_3} \cong H^3(\text{Sym}^3(C), \mathbb{Q}).$$

Here $\mathrm{Sym}^3(C)$ denotes the symmetric threefold product of C, i.e. the quotient space C^3/S_3 where S_3 acts on C^3 by permutation of the factors. Let J denote the three dimensional Jacobian of C. By the Abel-Jacobi theorem, the natural map $\varphi : \mathrm{Sym}^3(C) \to J$ is birational, and it induces an isomorphism between the middle cohomology spaces. Thus,

$$\mathrm{H}^3(\mathrm{Sym}^3(C), \mathbb{Q}) \cong \mathrm{H}^3(J, \mathbb{Q}) \cong \wedge^3 \mathrm{H}^1(J, \mathbb{Q}) \cong \wedge^3 \mathrm{H}^1(C, \mathbb{Q}).$$

Now the latter space is 20 dimensional, which is also the dimension of $\mathrm{H}^3(\widetilde{X}, \mathbb{Q})$. This shows that the inclusion $\mathrm{H}^3(\widetilde{X}, \mathbb{Q}) \hookrightarrow \wedge^3 \mathrm{H}^1(C, \mathbb{Q})$ we constructed is actually an isomorphism. $\qquad\square$

Let $\mathfrak{M}_8$ denote the moduli space of eight points in $\mathbb{P}^1$, which is five dimensional. Lemma 2.7 shows that there exist a natural embedding $\mathfrak{M}_8 \hookrightarrow \mathfrak{M}_{AR}$. We denote its image by $\widetilde{\mathfrak{H}}_{CY}$ and its image in $\mathfrak{M}_{CY}$ under the map $\mathfrak{M}_{AR} \to \mathfrak{M}_{CY}$ by $\mathfrak{H}_{CY}$. We call this the *hyperelliptic sublocus*.

Remark 2.10 It is worthwhile to remark that the construction generalizes to all $n \geq 2$. It produces a $2n - 1$ dimensional hyperelliptic locus in the n^2 dimensional moduli of CY manifolds, over which the primitive middle dimensional rational Hodge structure is the nth exterior power of a weight one Hodge structure. For any n the hyperelliptic locus corresponds to hyperplane arrangements osculating a twisted rational curve of degree n. It was shown in [18] that the hyperelliptic locus for $n = 2$ is an arithmetic quotient of Siegel space of degree 2. This fact motivated Theorem 1.4.

3 Classifying spaces and canonical variations

We briefly recall some basic facts on Hodge structures and their classifying spaces. Let V denote a real vector space, $n \in \mathbb{N}$ and $b : V \times V \to \mathbb{R}$ a non-degenerate bilinear form which is symmetric if n is even and skew-symmetric if n is odd. Furthermore, let $\{h^{p,q}\}$ denote a collection of non-negative integers parameterized by $(p, q) \in \mathbb{N}_0^2$ such that

$$h^{p,q} \neq 0 \quad \text{only if} \quad p + q = n \quad \text{and} \quad h^{q,p} = h^{p,q} \quad \text{for all} \quad p, q \in \mathbb{Z}.$$

The set D of all real Hodge structures of type $\Phi = (V, \{h^{p,q}\}, b)$ is equipped with a natural structure of a complex manifold, called the *classifying space* of Hodge structures of type Φ (see Chapter I, [10]). It is a homogeneous space of the form $D = G/K$, where G denotes the real Lie subgroup of $GL(V)$ consisting of all $\mathbb{R}$-linear automorphisms fixing b and where K denotes a compact subgroup of G.

Now let S be a complex manifold and fix a base point $s \in S$. Every $\mathbb{R}$-PVHS $\mathbb{V}$ of type Φ over S gives rise to a map

$$\phi : S \to \Gamma \backslash D,$$

called the *period map* associated to $\mathbb{V}$. Here Γ is the image of the monodromy representation

$$\tau : \pi_1(S, s) \to G$$

defined by $\mathbb{V}$ considered as a local system of real vector spaces. We refer to Chapter I in [10] for more details.

In practice one often encounters situations where the real structure is lost. A typical example arises from the eigenspace decomposition of the complexification of the real cohomology of a cyclic cover with respect to the cyclic group action (see for example [1,6]). Complex polarized variations of Hodge structures are also natural objects in Simpson's correspondence (see [24], Sect. 4). We will use this slightly generalized notion of $\mathbb{C}$-PVHS in this paper and we refer the reader to Sect. 1 in [5] for the notions of complex Hodge structure ($\mathbb{C}$-HS), complex polarized Hodge structure ($\mathbb{C}$-PHS) and complex polarized variations ($\mathbb{C}$-PVHS). In [5], the classifying space of $\mathbb{C}$-PHSs of with given Hodge numbers was defined. The corresponding classifying spaces of $\mathbb{C}$-PHSs are also of form $D = G/K$ where G is a real Lie group and K is a compact subgroup of G. The difference from that of $\mathbb{R}$-PHS is in that G is not necessarily $Sp(2n, \mathbb{R})$ or $SO(r, s)$, but can also be a special unitary group $SU(p, q)$.

3.1 Locally homogenous and canonical variations

Let $D_0 = G_0/K_0$ be a Hermitian symmetric domain (HSD) with G_0 the connected component of the automorphism group of D_0 and K_0 a maximal compact subgroup of G_0. After E. Cartan there are four infinite series of classical domains:

(I) $D_{p,q}^I = \frac{SU(p,q)}{S(U(p) \times U(q))}, \quad p \geq q \geq 1,$

(II) $D_n^{II} = \frac{SO^*(2n)}{U(n)}, \quad n \geq 5,$

(III) $D_n^{III} = \frac{Sp(2n,\mathbb{R})}{U(n)}, \quad n \geq 2,$

(IV) $D_n^{IV} = \frac{SO_0(2,n)}{SO(2) \times SO(n)}, \quad n \geq 5.$

Let Γ_0 be a torsion free discrete subgroup of G_0. Let $\rho_0 : G_0 \to GL(F)$ be a finite dimensional complex representation of G_0. S. Zucker has shown in Sect. 4 of [35], that the complex local system $\mathbb{F}_{\Gamma_0} = (F \times_{\Gamma_0} D_0)$ over the complex manifold $\Gamma_0 \backslash D_0$ admits naturally a structure of $\mathbb{C}$-PVHS, which we will call, following [35], a locally homogenous PVHS. By construction, it is clear that only when ρ is defined over $\mathbb{R}$, $\mathbb{F}_{\Gamma_0}$ is a $\mathbb{R}$-PVHS.
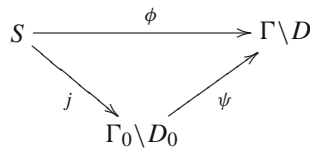
The following example of locally homogenous PVHS was considered by B. Gross in [12] and also appeared in [28] in connection with moduli spaces of CY varieties. As explained in Proposition 1.2.6 in [5], in the case that D_0 is irreducible, D_0 determines a special node of the Dynkin diagram of the simple complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = Lie(G_0) \otimes \mathbb{C}$. By the standard theory of the finite dimensional representations of semi-simple complex Lie algebras, this node also determines a fundamental representation W of $\mathfrak{g}_{\mathbb{C}}$, and W integrates to an irreducible complex representation ρ_{can} of G_0. When D_0 is of *tube type*, the representation W is exactly the one considered

by B. Gross in [12] and only in this case W does admit a G_0-invariant real form. The locally homogenous PVHS by the above construction are of CY-type (see Sect. 1 in [28]). Following Gross in [12], we call them canonical PVHS over $\Gamma_0 \backslash D_0$. One uses the tensor product to define the notion of canonical PVHS for a product of locally HSDs.

Definition 3.1 Let (D_0, ρ_0) be a pair consisting of a HSD D_0 and a homomorphism $\rho_0 : G_0 \to G$ of real Lie groups and let $\tilde{\psi} : D_0 \to D$ the map induced from ρ_0 and

$$\psi : \Gamma_0 \backslash D_0 \to \Gamma \backslash D, \quad \Gamma_0 := \rho_0^{-1}(\Gamma)$$

(1) We say that a $\mathbb{C}$-PVHS $\mathbb{V}$ over S *factors* with respect to (D_0, ρ_0), when the period map $\phi : S \to \Gamma \backslash D$ of $\mathbb{V}$ factors over ψ. That is, the following diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\phi} & \Gamma \backslash D \\
 & \searrow{\scriptstyle j} \quad \nearrow{\scriptstyle \psi} & \\
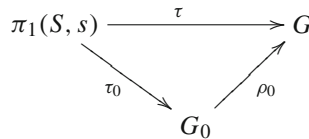 & \Gamma_0 \backslash D_0 &
\end{array}
$$

commutes.

(2) For a $\mathbb{C}$-PVHS $\mathbb{V}$ of CY-type we say that $\mathbb{V}$ factors *canonically* if it factors with respect to a certain (D_0, ρ_{can}).

The following lemma follows from the definition:

Lemma 3.2 *If $\mathbb{V}$ factors with respect to (D_0, ρ_0), then j induces an isomorphism of $\mathbb{C}$-PVHS $\mathbb{V} \simeq j^*\mathbb{F}$, where $\mathbb{F}$ is the locally homogeneous PVHS induced by ρ_0. Moreover the monodromy representation $\tau : \pi_1(S, s) \to G$ of $\mathbb{V}$ factorizes over ρ_0, i.e. there exists a homomorphism $\tau_0 : \pi_1(S, s) \to G_0$ such that the diagram*

$$
\begin{array}{ccc}
\pi_1(S, s) & \xrightarrow{\tau} & G \\
 & \searrow{\scriptstyle \tau_0} \quad \nearrow{\scriptstyle \rho_0} & \\
 & G_0 &
\end{array}
$$

commutes.

Proposition 3.3 *Let $\mathbb{V}$ be the $\mathbb{C}$-PVHS associated with a good family for the coarse moduli space $\mathfrak{M}_{CY}$. If $\mathbb{V}$ factors canonically, then it must factor with respect to either $(D_{3,3}^I, \wedge^3)$ or $(D_{1,1}^I \times D_8^{IV}, id \otimes id)$.*

Proof Assume $\mathbb{V}$ factors with respect to (D_0, ρ_{can}) for certain HSD D_0. By the local Torelli theorem for CY manifolds we know that D_0 is at least 9 dimensional. We write $D_0 = D_1 \times \cdots \times D_k$ be the decomposition into product of irreducible HSDs. Then $\rho_{can} = \rho_{can,1} \otimes \cdots \otimes \rho_{can,k}$ with $\rho_{can,i}$ canonical PVHS over D_i. By Schur's lemma $\mathbb{V} \simeq \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_k$ decomposes accordingly. Recall that the canonical PVHS $\mathbb{V}_i$ over

the irreducible HSD D_i has the property that the weight of $\mathbb{V}_i$ is equal to the rank of D_i, and $\mathbb{V}_i$ is real iff D_i is of tube type. Since the weight of $\mathbb{V}$ is 3, it follows that $k \leq 3$. There are three cases to consider:

Case $k = 1$. Then D_0 must be of tube type and $\mathbb{W}$ associated with $\rho_{can,1}$ must be the canonical $\mathbb{R}$-PVHS. This excludes for example the possibility of D_0 being a ball $D_{n,1}^I$, $n \geq 2$. By the classification in [12], the pair $(D_{3,3}^I, \wedge^3)$ is the only possibility with correct Hodge numbers.

Case $k = 2$. Since D_0 has rank 3, we can assume that D_1 has rank 1. So D_1 is the unit disk. Then it follows that the Hodge numbers of $\mathbb{V}_1$ is 1, 1. Since $\mathbb{V}$ and $\mathbb{V}_1$ have real structure and $\mathbb{V}_2 \subset \mathbb{V} \otimes \mathbb{V}_1^*$ is a direct factor, $\mathbb{V}_2$ has also real structure. As D_2 supports the canonical $\mathbb{R}$-PVHS $\mathbb{V}_2$ of weight 2, it must be a type IV domain. By checking the Hodge numbers, one sees immediately the pair $(D_{1,1}^I \times D_8^{IV}, \text{id} \otimes \text{id})$ is the unique possibility.

Case $k = 3$. It follows that each D_i, $i = 1, 2, 3$ has rank 1 and so the dimension of $D_1 \times D_2 \times D_3$ is less than 9. This case is impossible. $\qquad\square$

4 Characteristic subvariety

We refer the reader to Sect. 3 in [11] and references therein for an account of the theory of *infinitesimal variations of Hodge structures*, in short IVHS, initiated by P. Griffiths. There is an important series of invariants of IVHS of a $\mathbb{C}$-PVHS $\mathbb{V}$ of CY-type over S, namely the *characteristic subvarieties* which are contained in the projectivized tangent bundle $\mathbb{P}(\mathcal{T}_S)$. The basic theory of characteristic subvarieties is developed in [28]. We recall the definition.

Definition 4.1 Let $\mathbb{V}$ a $\mathbb{C}$-PVHS of weight n over S of weight n and (E, θ) the associated Higgs bundle. For every k with $1 \leq k \leq n$, the kth iterated Higgs field defines a morphism

$$\theta^k : \text{Sym}^k(\mathcal{T}_S) \longrightarrow \text{Hom}(E^{n,0}, E^{n-k,k})$$

with the dual map $(\theta^k)^* : \text{Hom}(E^{n,0}, E^{n-k,k})^* \rightarrow \text{Sym}^k(\Omega_S)$. Let $\mathfrak{a}_k$ denote the ideal generated by the image of $(\theta^k)^*$ in the symmetric algebra $\text{Sym}^\bullet(\Omega_S)$. Then for every k with $1 \leq k \leq n - 1$, the projective variety

$$\mathcal{C}_k = \text{Proj}\left(\text{Sym}^\bullet\left(\Omega_S^1\right) \Big/ \mathfrak{a}_{k+1}\right) \subset \mathbb{P}(\mathcal{T}_S)$$

over S is called the *kth characteristic subvariety* of $\mathbb{V}$.

Remark 4.2 For a proper smooth family $f : \mathcal{X} \rightarrow S$ of CY n-folds the Yukawa coupling of f is the section of $\text{Sym}^n(\Omega_S^1) \otimes (R^n f_* \mathcal{O}_\mathcal{X})^{\otimes 2}$ defined by the nth iterated Kodaira-Spencer maps of f. It has significance in physics and is an important invariant in the study of geometry on moduli spaces of CY manifolds. Note that the $(n-1)$th characteristic subvariety is just the vanishing locus of the Yukawa coupling. The motivation of introducing the notion of the characteristic subvariety in [28] comes

from the study of $\mathfrak{M}_{CY}$, the coarse moduli space of the paper. It was also inspired by the notion of characteristic bundle introduced by N. Mok in his fundamental work on the metric rigidity theorem of locally HSD [16]. It is E. Viehweg who pointed out to us that the iterated Kodaira-Spencer maps of $\mathfrak{M}_{CY}$ are surjective. This fact is also shared by those of canonical PVHSs in Proposition 3.3, and this actually led us to a closer investigation of the geometry contained in the kernels of iterated Kodaira-Spencer maps.

Let $s \in S$ and $(\mathcal{C}_k)_s$ be the fiber of $\mathcal{C}_k$ over s, which is a subvariety of $\mathbb{P}(\mathcal{T}_{S,s})$. The following simple lemma characterizes the tangent vectors at s whose classes lie in the reduced subvariety $(\mathcal{C}_k)_s^{red}$.

Lemma 4.3 (Lemma 3.2 in [28]) *Let $v \in \mathcal{T}_{S,s}$ be a non-zero tangent vector, $[v]$ its class in $\mathbb{P}(\mathcal{T}_{S,s})$ and $v^{k+1} \in \mathrm{Sym}^{k+1}(\mathcal{T}_{S,s})$ the $k+1$th symmetric tensor power of v. Then:*

$$[v] \in (\mathcal{C}_k)_s^{red} \subset \mathbb{P}(\mathcal{T}_{S,s}) \quad \text{if and only if} \quad v^{k+1} \in \ker(\theta^{k+1}).$$

Corollary 4.4 *Let $\mathfrak{M}$ be a coarse moduli space of polarized smooth CY n-folds and $f : \mathcal{X} \to S$ be a good family for it. Let $s \in S$ be a point in the ramification locus of the moduli map $S \to \mathfrak{M}$ of f. Then there is a projective linear subspace in $(\mathcal{C}_k)_s^{red}$ for $1 \leq k \leq n-1$.*

Proof The question is analytically local. Let X be the fiber of f over s and $[s]$ be the image of s in $\mathfrak{M}$. By the Bogomolov-Tian-Todorov theorem the differential of the moduli map of f at s is naturally identified with the Kodaira-Spencer map:

$$\rho_{f,s} : \mathcal{T}_{S,s} \to \mathrm{H}^1(X, \mathcal{T}_X).$$

Let $v \in \mathcal{T}_{S,s}$ be a nonzero tangent vector and ω be a generator of $\mathrm{H}^0(X, K_X)$. By Griffiths (see [9] Sect. 2) we have the formula for the Higgs field action:

$$\theta_v(\omega) = \rho_{f,s}(v) \cup \omega,$$

where the cup product induces an isomorphism $\mathrm{H}^1(X, \mathcal{T}_X) \otimes \mathrm{H}^0(X, K_X) \simeq \mathrm{H}^1(X, \Omega_X^{n-1})$ for the CY manifold X. So it is clear that the kernel of $\rho_{f,s}$ is exactly the kernel of the Higgs field θ at s. Thus, the corollary follows from Lemma 4.3. $\square$

The main result in [28] identifies the characteristic subvarieties of the canonical PVHS over an irreducible HSD with the characteristic bundles.

Theorem 4.5 (Theorem 3 in [28]) *Let $D_0 = G_0/K_0$ be an irreducible HSD of rank n, Γ_0 be a torsion free discrete subgroup of G_0 and let (E, θ) be the system of Hodge bundles associated to the canonical PVHS over $T := \Gamma_0 \backslash D_0$. Then for each k with $1 \leq k \leq n-1$ the kth characteristic subvariety $\mathcal{C}_k$ of (E, θ) over T coincides with the kth characteristic bundle $\mathcal{S}_k$ over T.*

The characteristic bundles are first defined over D_0 and invariant under the G_0-action. By taking the quotient under the group Γ_0 one obtains the characteristic bundles on T. For each irreducible HSD the characteristic bundles are explicitly described in Appendix (III.3) in [17]. For the purpose of this article we need only the information of the first characteristic bundle for the HSDs $D_{3,3}^I$, D_3^{III}, D_8^{IV}. For the convenience of the reader, we state them here. The following proposition is a direct consequence of Theorem 4.5.

Proposition 4.6 *Notations as Theorem 4.5 and $t \in T$ an arbitrary point. Then $(\mathcal{C}_1)_t$ as subvariety of $\mathbb{P}(\mathcal{T}_{T,t})$ is isomorphic to*

(i). $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ *with the Segre embedding, when $D_0 = D_{3,3}^I$,*

(ii). $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ *with the Veronese embedding, when $D_0 = D_3^{III}$,*

(iii). $Q \hookrightarrow \mathbb{P}^7$ *with Q a smooth quadric, when $D_0 = D_8^{IV}$.*

For the application we also need to work out the first characteristic subvariety in the reducible case $D_0 = D_{1,1}^I \times D_8^{IV}$.

Proposition 4.7 *Notations as last proposition. Let D_0 be the HSD $D_{1,1}^I \times D_8^{IV}$. Then for each point of $t \in T$, $(\mathcal{C}_1)_t$ as subvariety of $\mathbb{P}(\mathcal{T}_{T,t})$ is isomorphic to a disjoint union of a point P with a smooth quadric Q in a hyperplane of $\mathbb{P}^8$ away from P. In particular, it is not equidimensional.*

Proof We put $D_1 = D_{1,1}^I$ and $D_2 = D_8^{IV}$ and fix the base point $0 \in D_0$. Then the canonical PVHS $\mathbb{W}$ over D_0 is given by tensor product $\mathbb{W}_1 \otimes \mathbb{W}_2$ of the canonical PVHS $\mathbb{W}_i$ over D_i, $i = 1, 2$. Let (E_i, θ_i) be the corresponding Higgs-bundle to $\mathbb{W}_i$. Then the corresponding Higgs-bundle (E, θ) to $\mathbb{W}$ is given by $(E_1 \otimes E_2, \theta_1 \otimes id + id \otimes \theta_2)$. Let $p_i : D_0 \to D_i$, $i = 1, 2$ be the natural projection. Then one has natural isomorphism $\mathcal{T}_{D_0} \simeq p_1^*(\mathcal{T}_{D_1}) \oplus p_2^*(\mathcal{T}_{D_2})$. Under this isomorphism, we represent a tangent vector $v \in \mathcal{T}_{D_0,0}$ by a pair (v_1, v_2) with $v_i \in \mathcal{T}_{D_i,0}$. Take a nonzero vector $e = e_1 \otimes e_2 \in (E_1^{1,0} \otimes E_2^{2,0})_0$. Then by Lemma 4.3, the fiber over 0 of the first characteristic subvariety of D_0 is determined by

$$\left\{ [v] \in \mathbb{P}(\mathcal{T}_{D_0,0}) | (\theta_v)^2(e) = 0 \right\}.$$

A simple calculation shows that

$$(\theta_v)^2(e) = 2((\theta_1)_{v_1}(e_1) \otimes (\theta_2)_{v_2}(e_2)) + e_1 \otimes ((\theta_2)_{v_2})^2(e_2).$$

Hence v is a characteristic vector if and only if $v_2 = 0$ or, $v_1 = 0$ and v_2 is a characteristic vector in $\mathcal{T}_{D_2,0}$. Thus, the fiber over 0 of the first characteristic subvariety is the disjoint union of a point with the fiber over 0 of the first characteristic subvariety of D_2, which is isomorphic to a smooth quadric in $\mathbb{P}^7$ by Proposition 4.6 (iii). □

Corollary 4.8 *If $\mathbb{V}$ factors canonically, then for any $s \in S$ away from the ramification locus of the moduli map $S \to \mathfrak{M}_{CY}$, then $(\mathcal{C}_1)_s \subset \mathbb{P}(\mathcal{T}_{S,s})$ is isomorphic to either $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ in Proposition 4.6 or $P \cup Q \subset \mathbb{P}^8$ in Proposition 4.7.*

Proof This is a consequence of Proposition 3.3 and Propositions 4.6, 4.7. □

5 Explicit infinitesimal variation of Hodge structures of the Calabi–Yau threefolds

In those cases where there is an explicit description of the cohomology with the help of a jacobian ring, it is usually also possible to construct the infinitesimal invariants of the corresponding variation. In particular, it will be possible to compute the characteristic subvarieties. We will illustrate this for the local system $\mathbb{V}$ for a good family of double octics $f : \mathcal{X} \to S$ with fibres $\widetilde{X}$ covered by Y.

5.1 Jacobian rings

As Y is a complete intersection, one can find a description of the cohomology in terms of a certain Jacobian ring with the aid of the *Cayley-trick*. Let

$$R_{8,4} = \mathbb{C}[x_1, \ldots, x_8, y_1, \ldots, y_4]$$

denote the polynomial ring over $\mathbb{C}$ in 12 variables. The four quadrics defining Y are given by

$$f_i = \sum_{j=1}^{8} b_{ij} x_j^2,$$

and one defines

$$F = \sum_{i=1}^{4} y_i f_i \in R_{8,4}.$$

Let $\mathfrak{J}_\mathfrak{A}$ denote the homogenous ideal generated by the twelve partial derivatives $\frac{\partial F}{\partial x_j}$ and $\frac{\partial F}{\partial y_i}$, and put

$$R_Y = R_{8,4}/\mathfrak{J}_\mathfrak{A}.$$

There is a natural bigrading on $R_{8,4}$ which assigns the value $(0, 1)$ to each of the variables x_i and $(1, -2)$ to each of y_i. It induces a bigrading on R_Y. For $0 \leq p \leq 3$ we let $R_Y^{(p)}$ denote the subspace generated by monomials of bidegree $(p, 0)$, that is, monomials whose total degree in the x_js is $2p$ and whose total degree in the y_is is p.

Proposition 5.1 *There is an isomorphism*

$$H^3(Y, \mathbb{C}) \cong R_Y$$

of $\mathbb{C}$-vector spaces which identifies

$$H^{3-p,p}(Y) \cong R_Y^{(p)}, \qquad 0 \leq p \leq 3$$

Proof See [20], Prop. 2.2.10 on page 40 or [30]. □

In Sect. 2.2 we defined the group G_1 of sign-changes and its subgroup N_1 of index two. These groups naturally act on the polynomial ring $R_{8,4}$ by sign-changes on the x_j, $1 \leq j \leq 8$ and trivially on the y_i, $1 \leq i \leq 4$. There is an induced action of N_1 on the quotient R_Y. We let $\widetilde{R}_Y$ denote the subring $R_Y^{N_1}$ of elements in R_Y fixed by every $\sigma \in N_1$.

Corollary 5.2 *It induces an isomorphism between the subspaces*

$$\mathrm{H}^3(\widetilde{\mathrm{X}}, \mathbb{C}) \cong \widetilde{R}_Y$$

which is also compatible with the Hodge decomposition and the total grading.

Proof By Proposition 2.6, $\mathrm{H}^3(\widetilde{\mathrm{X}}, \mathbb{Q})$ is the subspace of $\mathrm{H}^3(Y, \mathbb{Q})$ invariant under N_1-action and moreover is a sub PHS of $\mathrm{H}^3(Y, \mathbb{Q})$. So one has

$$\mathrm{F}^p\mathrm{H}^3(\widetilde{\mathrm{X}}, \mathbb{C}) = \mathrm{F}^p\mathrm{H}^3(Y, \mathbb{C}) \cap \mathrm{H}^3(\widetilde{\mathrm{X}}, \mathbb{C}) = (\mathrm{F}^p\mathrm{H}^3(Y, \mathbb{C}))^{N_1}.$$

So the assertion follows. □

5.2 Multiplication and cup-product

Proposition 5.3 *For $0 \leq p \leq 2$ the following diagram commutes*

$$
\begin{array}{ccc}
\mathcal{T}_{S,s} \otimes \mathrm{H}^{3-p,p}(\widetilde{\mathrm{X}}) & \xrightarrow{\theta_s} & \mathrm{H}^{2-p,p+1}(\widetilde{\mathrm{X}}) \\
\cong \downarrow & & \downarrow \cong \\
\widetilde{R}_Y^{(1)} \otimes \widetilde{R}_Y^{(p)} & \xrightarrow{\mu} & \widetilde{R}_Y^{(p+1)}.
\end{array}
$$

Here the vertical arrows are provided by Corollary 5.2, and the lower horizontal arrow is the ring multiplication map.

Proof First we recall that in Sect. 3 we associate each element in S with a smooth complete intersection of four quadrics Y in $\mathbb{P}^7$. Let $g_0 : \mathcal{Z} \to Z$ be a good family over the coarse moduli space of smooth complete intersections of four quadrics in $\mathbb{P}^7$. So one has the natural closed embedding $S \hookrightarrow Z$. Put

$$h_0 = g_0|_S : \mathcal{Y} = \mathcal{Z}|_{g_0^{-1}(S)} \to S.$$

Globalizing the construction in Proposition 2.6 to h_0, one obtains a new family $\tilde{h}_0 : \widetilde{\mathcal{Y}} \to S$ admitting N_1-action over S, and by taking quotient of it under N_1-action one recovers the family $f_0 : \mathcal{X} \to S$. In summary, one has the following commutative diagrams

$$\mathcal{X} \xleftarrow{\pi} \tilde{\mathcal{Y}} \xrightarrow{\sigma} \mathcal{Y} \xhookrightarrow{} \mathcal{Z}$$

$$f_0 \downarrow \qquad \tilde{h}_0 \downarrow \qquad h_0 = g_0|_S \downarrow \qquad g_0 \downarrow$$

$$S \xleftarrow{=} S \xrightarrow{=} S \xhookrightarrow{} Z.$$

Let (F, η) be the system of Hodge bundles associated with the PVHS $\mathbb{W} = R^3 g_{0*}\mathbb{Q}_{\mathcal{Z}}$. By construction, it is clear that $\mathbb{V} \simeq (\mathbb{W}|_S)^{N_1}$ as PVHS. It follows that for $v \in \mathcal{T}_{S,s} \subset \mathcal{T}_{Z,s}$ one has a natural identification

$$\theta_s(v)(\alpha) = \eta_s(v)(\alpha),$$

where $\alpha \in H^{3-p,p}(\tilde{X}) \simeq H^{3-p,p}(Y)^{N_1} \subset H^{3-p,p}(Y)$.

Furthermore, for the IVHS of $\mathbb{W}$ at s, one has the following commutative diagram (see Proposition 2.6 in [30])

$$\begin{array}{ccc}
\mathcal{T}_{Z,s} & \xrightarrow{\eta_s} & \bigoplus_p \mathrm{Hom}(H^{3-p,p}(Y), H^{2-p,p+1}(Y)) \\
\cong \downarrow & & \downarrow \cong \\
R_Y^{(1)} & \longrightarrow & \bigoplus_p \mathrm{Hom}\left(R_Y^{(p)}, R_Y^{(p+1)}\right),
\end{array}$$

where the lower horizontal arrow is induced by the ring multiplication map.

Finally one has the following commutative diagrams by the construction of the families $h_0, \tilde{h}_0$ and f_0:

$$\begin{array}{ccc}
\mathcal{T}_{S,s} & \xrightarrow{\rho_{f_0,s}} H^1(\tilde{X}, \mathcal{T}_{\tilde{X}}) \xrightarrow{\simeq} & H^{2,1}(\tilde{X}) \\
\cap \downarrow & \downarrow & \downarrow \\
\mathcal{T}_{Z,s} & \xrightarrow{\rho_{g_0,s}} H^1(Y, \mathcal{T}_Y) \xrightarrow{\simeq} & H^{2,1}(Y).
\end{array}$$

This shows that under the left vertical isomorphism $\mathcal{T}_{Z,s} \to R_Y^{(1)}$ (in the second paragraph), the image of $\mathcal{T}_{S,s}$ is exactly $\tilde{R}_Y^{(1)}$. The proposition follows by putting everything above together. $\qquad\square$

5.3 Calculation

Now we proceed to describe the computation of the characteristic subvarieties introduced in Sect. 4. According to Proposition 5.3, the action of Higgs field on the cohomology classes along a given tangent vector is equivalent to the multiplication of corresponding elements in $\tilde{R}_Y$ with some fixed element in $\tilde{R}_Y^{(1)}$. Furthermore, since $\tilde{R}_Y^{(0)}$ is one-dimensional, there exists an isomorphism $\mathrm{Hom}(\tilde{R}_Y^{(0)}, \tilde{R}_Y^{(k)}) \cong \tilde{R}_Y^{(k)}$. Hence in our case the kth iterated Higgs fields

$$(\theta_s)^k : \mathrm{Sym}^k(T_{S,s}) \longrightarrow \mathrm{Hom}\left(H^{3,0}\left(\tilde{X}\right), H^{3-k,k}\left(\tilde{X}\right)\right)$$

is given by the multiplication map

$$\mu_k : \mathrm{Sym}^k\left(\widetilde{R}_Y^{(1)}\right) \longrightarrow \widetilde{R}_Y^{(k)}.$$

It follows that the kth characteristic subvariety at s $(\mathcal{C}_k)_s$ is isomorphic to

$$\mathbb{P}\left(\mathrm{Sym}^{\cdot}\left(\left(\widetilde{R}_Y^{(1)}\right)^* / \mathfrak{a}_{k+1}\right)\right),$$

where $\mathfrak{a}_{k+1}$ denotes the image of the dual $\mu_{k+1}^* : (\widetilde{R}_Y^{(k+1)})^* \to \mathrm{Sym}^{k+1}((\widetilde{R}_Y^{(1)})^*)$ of the multiplication map. By choosing a basis of $\widetilde{R}_Y^{(1)}$ one obtains an isomorphism $\mathbb{P}(\mathrm{Sym}^{\cdot}((\widetilde{R}_Y^{(1)})^*)) \cong \mathbb{P}^8$. This shows that the first characteristic subvariety $(\mathcal{C}_1)_s$ can be computed by the following steps. Let $\mathcal{B}_1 = \{u_1, \ldots, u_9\}$ (resp. $\mathcal{B}_2 = \{v_1, \ldots, v_9\}$ be a basis of $\widetilde{R}_Y^{(1)}$ (resp. $\widetilde{R}_Y^{(2)}$).

(1) Fix a bijection

$$\phi : \{(i, j) \in \mathbb{N}^2 \mid 1 \le i \le j \le 9\} \xrightarrow{\sim} \{1, \ldots, 45\},$$

and define a basis $\mathcal{B} = \{w_1, \ldots, w_{45}\}$ of $\mathrm{Sym}^2(\widetilde{R}_Y^{(1)})$ by $w_{\phi(i,j)} = u_i u_j$. Compute the representation matrix $C \in M(9 \times 45, \mathbb{C})$ of the multiplication map μ_2 with respect to $\mathcal{B}$ and $\mathcal{B}_2$.

(2) Let $\mathcal{B}_1^* = (u_1^*, \ldots, u_9^*)$ denote the dual basis of $\mathcal{B}_1$, and let the basis $\widetilde{\mathcal{B}} = (\widetilde{w}_1, \ldots, \widetilde{w}_{45})$ of $\mathrm{Sym}^2((\widetilde{R}_Y^{(1)})^*)$ be defined by $\widetilde{w}_{\phi(i,j)} = u_i^* u_j^*$. Determine the representation matrix $D = (d_{ij}) \in M(45 \times 9, \mathbb{C})$ of the dualized multiplication μ_2^* with respect to $\mathcal{B}_2^*$ and $\widetilde{\mathcal{B}}$.

(3) For the kth column of D define a polynomial $f_k \in \mathbb{C}[z_1, \ldots, z_9]$ by

$$f_k = \sum_{i=1}^{9}\sum_{j=i}^{9} d_{\phi(i,j)k} z_i z_j.$$

Let $\mathfrak{a}_2$ be the ideal generated by $f_1, \ldots, f_9$. Then $(\mathcal{C}_1)_s$ is isomorphic to the projective subvariety in $\mathbb{P}^8$ which corresponds to $\mathfrak{a}_2$.

Remark 5.4 Step (1) can be carried out in practice using computer algebra. For each pair (i, j), one computes the product $g_i g_j$ of the polynomials $g_i, g_j \in R_{8,4}$ which correspond to the basis vectors $w_i, w_j \in \mathcal{B}_1$ and reduces it with respect to the Jacobian ideal $\mathfrak{J}_{\mathfrak{A}}$. The result can be expressed as a linear combination of the elements in $\mathcal{B}_2$. For step (2) notice that $\widetilde{\mathcal{B}}$ is related to the dual basis $\widetilde{\mathcal{B}}^*$ of $\mathcal{B}$ by

$$\widetilde{w}_{\phi(i,j)} = \begin{cases} \widetilde{w}_{\phi(i,j)}^* & \text{if } i = j \\ \dfrac{1}{2}\widetilde{w}_{\phi(i,j)}^* & \text{if } i \ne j. \end{cases}$$

By a similar procedure, we can compute the characteristic subvariety $(\mathcal{C}_2)_s$. As it will not be used in the sequel, we skip the details but refer the reader to [8].

5.4 A special point

Here we summarize the results of the calculations that can be performed by the method explained above. First, we choose the special point $s_0 \in \widetilde{\mathfrak{H}}_{CY}$ represented by the matrix $A \in M(8 \times 4, \mathbb{C})$ with

$$
A^t = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 \\
1 & 8 & 27 & 64 & 125 & 216 & 343 & 512
\end{pmatrix}
$$

One determines the homogenous ideal

$$
\mathfrak{a}_2 \subset \mathbb{C}[z_1, \ldots, z_9]
$$

generated by nine quadratic polynomials: The explicit forms are given at the end of the section.

Proposition 5.5 *The characteristic subvariety* $(\mathcal{C}_1)_{s_0}^{red}$ *consists of two irreducible surfaces of degree* 6, *each spanning the same* $\mathbb{P}^7 \subset \mathbb{P}^8$.

Remark 5.6 For a generic configuration $\mathfrak{A}$ giving a general point $\eta_0 \in \mathfrak{M}_{CY}$, a Gröbner-basis calculation shows that the characteristic subvariety $(\mathcal{C}_1)_{\eta_0}$ is empty set (see [8] for details). One could also take η_0 in the proof of Theorem 6.1, for which the idea of the proof might be clearer. Since we can use the point s_0 to show both Theorems 1.1 and 1.4, we avoid giving the explicit equations of $(\mathcal{C}_1)_{\eta_0}$ here. For the point s_0 we analyzed the ideal $\mathfrak{a}_2$ using the computer algebra system SINGULAR. Via a Poincaré-series computation, one shows that the variety in $\mathbb{P}^8$ has dimension two and is of degree 12. In order to find the irreducible components of this surface, one has to make a primary decomposition of the ideal $\mathfrak{a}_2$. For this we had to use several tricks.

By elimination of the first five variables we obtain a hypersurface of degree 8 in variables z_6, z_7, z_8, z_9. By looking at reductions modulo various primes p, it was observed that the octic factors as product of two quartics over $\mathbb{F}_p$ in case $p \equiv 1 \mod 4$. With some more work one finds a factorization of the octic into two quartics over the field $\mathbb{Q}(\sqrt{-1})$.

Both quartics have a smooth twisted cubic as singular locus. Because the generic plane section is a three nodal quartic, we see that these two quartic surfaces are *irreducible* over $\mathbb{C}$. The decomposition of the degree 8 surface into two quartics gives a splitting of the degree 12 surface in $\mathbb{P}^8$ into two components, which are surfaces of degree 6.

The change in degree from 12 to 8 is due to the fact the projecting out the first five variables is *non-generic*. If instead we eliminate the variables z_1, z_5, z_6, z_8, z_9 one finds that the degree 6 components project to sextic surfaces, whose plane section is

a 10-nodal sextic, hence rational. Such surfaces are in fact ruled and can be obtained as join in $\mathbb{P}^7$ of corresponding points on a conic and rational normal curve of degree four and are cut out by 15 quadrics in $\mathbb{P}^7$. Indeed, it turns out that both components are contained in the hyperplane of $\mathbb{P}^8$ given by

$$2269z_1 - 378z_2 + 21z_3 - 1029z_4 + 147z_5 - 7z_6 + 192z_7 - 24z_8 + z_9 = 0$$

By taking ideal quotients one can find a complete primary decomposition of the ideal $\mathfrak{a}_2$. There are embedded components of dimension one contained in the union of the two scrolls.

Nine quadratic equations:

$$f_1 = 5184z_1^2 - 864z_1z_2 - 2028z_1z_4 + 156z_1z_5 + 36z_2^2 + 156z_2z_4 - 12z_2z_5$$
$$+196z_4^2 - 28z_4z_5 + z_5^2,$$
$$f_2 = 114048z_1^2 - 11232z_1z_2 - 864z_1z_3 - 43680z_1z_4 + 1836z_1z_5 + 156z_1z_6$$
$$+144z_2^2 + 72z_2z_3 + 1836z_2z_4 - 24z_2z_5 - 12z_2z_6 + 156z_3z_4 - 12z_3z_5$$
$$+4116z_4^2 - 294z_4z_5 - 28z_4z_6 + 2z_5z_6,$$
$$f_3 = 627264z_1^2 - 19008z_1z_2 - 9504z_1z_3 - 235200z_1z_4 + 1680z_1z_5 + 1680z_1z_6$$
$$+144z_2^2 + 144z_2z_3 + 1680z_2z_4 - 12z_2z_5 - 12z_2z_6 + 36z_3^2 + 1680z_3z_4$$
$$-12z_3z_5 - 12z_3z_6 + 21609z_4^2 - 294z_4z_6 + z_6^2,$$
$$f_4 = 36288z_1^2 - 6048z_1z_2 - 7098z_1z_4 + 546z_1z_5 - 2352z_1z_7 + 168z_1z_8 + 252z_2^2$$
$$+546z_2z_4 - 42z_2z_5 + 168z_2z_7 - 12z_2z_8 + 450z_4z_7 - 30z_4z_8 - 30z_5z_7 + 2z_5z_8$$
$$f_5 = 870912z_1^2 - 90720z_1z_2 - 6048z_1z_3 - 167076z_1z_4 + 7518z_1z_5 + 546z_1z_6$$
$$-54432z_1z_7 + 2112z_1z_8 + 168z_1z_9 + 1512z_2^2 + 504z_2z_3 + 7518z_2z_4$$
$$-168z_2z_5 - 42z_2z_6 + 2112z_2z_7 - 24z_2z_8 - 12z_2z_9 + 546z_3z_4 - 42z_3z_5$$
$$+168z_3z_7 - 12z_3z_8 + 10140z_4z_7 - 338z_4z_8 - 30z_4z_9 - 338z_5z_7 + 2z_5z_9$$
$$-30z_6z_7 + 2z_6z_8,$$
$$f_6 = 5225472z_1^2 - 217728z_1z_2 - 72576z_1z_3 - 983178z_1z_4 + 12852z_1z_5$$
$$+6426z_1z_6 - 314928z_1z_7 + 1944z_1z_8 + 1944z_1z_9 + 2268z_2^2 + 1512z_2z_3$$
$$+12852z_2z_4 - 168z_2z_5 - 84z_2z_6 + 1944z_2z_7 - 12z_2z_8 - 12z_2z_9 + 252z_3^2$$
$$+6426z_3z_4 - 84z_3z_5 - 42z_3z_6 + 1944z_3z_7 - 12z_3z_8 - 12z_3z_9 + 57122z_4z_7$$
$$-338z_4z_9 - 338z_6z_7 + 2z_6z_9,$$
$$f_7 = 63504z_1^2 - 10584z_1z_2 - 8232z_1z_7 + 588z_1z_8 + 441z_2^2 + 588z_2z_7 - 42z_2z_8$$
$$+256z_7^2 - 32z_7z_8 + z_8^2,$$
$$f_8 = 1651104z_1^2 - 179928z_1z_2 - 10584z_1z_3 - 206976z_1z_7 + 8568z_1z_8 + 588z_1z_9$$
$$+3528z_2^2 + 882z_2z_3 + 8568z_2z_7 - 168z_2z_8 - 42z_2z_9 + 588z_3z_7 - 42z_3z_8$$
$$+6144z_7^2 - 384z_7z_8 - 32z_7z_9 + 2z_8z_9,$$

$$f_9 = 10732176z_1^2 - 550368z_1z_2 - 137592z_1z_3 - 1300992z_1z_7 + 14784z_1z_8$$
$$+ 7392z_1z_9 + 7056z_2^2 + 3528z_2z_3 + 14784z_2z_7 - 168z_2z_8 - 84z_2z_9 + 441z_3^2$$
$$+ 7392z_3z_7 - 84z_3z_8 - 42z_3z_9 + 36864z_7^2 - 384z_7z_9 + z_9^2.$$

Remark 5.7 We remind the reader that the Segre variety in Proposition 4.6 (i) is given by a special set of nine quadratic polynomials, which are 2×2 minors of a 3×3 matrix. The above calculation of $(\mathcal{C}_1)_{s_0}$ suggests that there is a nice geometry here. We hope to take up this issue on a later occasion.

6 Proof of the Main Theorems

Theorem 6.1 *Let $f : \mathcal{X} \to S$ be a good family of $\mathfrak{M}_{CY}$ and $\mathbb{V}$ be the associated weight 3 PVHS. Then $\mathbb{V}$ does not factor canonically.*

Proof Assume the contrary. By Corollary 4.8, for any $s \in S$ away from the ramification locus of the moduli map $S \to \mathfrak{M}_{CY}$ the first characteristic subvariety $(\mathcal{C}_1)_s$ is then isomorphic to either $\mathbb{P}^2 \times \mathbb{P}^2$ or $P \cup Q$. In particular, in both cases there exists an irreducible component in $(\mathcal{C}_1)_s$ whose dimension is greater than 2. By Proposition 5.5 the fiber of the first characteristic subvariety at the point $[s_0]$, the image of s_0 in $\mathfrak{M}_{CY}$, is two dimensional. By semi-continuity, there is an open neighborhood of $[s_0]$ in $\mathfrak{M}_{CY}$ such that the fibers of the first characteristic subvariety over the closed points in it are of dimension ≤ 2. Therefore, there exists also a closed point in S away from the ramification locus of the moduli map of f, over which the fiber of the first characteristic subvariety is of dimension ≤ 2. This gives a contradiction. □

Our next aim is to prove Theorem 6.3. For the proof of it, we use the notation as in Sect. 1. We first prove the following

Lemma 6.2 *Each factor $\mathbb{V}_i$ has quasi-unipotent local monodromy around each irreducible component of Z.*

Proof We put $\mathbb{V}' = \mathbb{V}_2 \otimes \cdots \otimes \mathbb{V}_k$ and $n = \operatorname{rank}\mathbb{V}'$. One considers the $\mathbb{C}$-PVHS $\bigwedge^n \mathbb{V} = \bigwedge^n (\mathbb{V}_1 \otimes \mathbb{V}')$. By Exercise 6.11 (b) in [7] $\operatorname{Sym}^n\mathbb{V}_1$ is a direct factor of $\bigwedge^n \mathbb{V}$. Since $\mathbb{V}$ is of quasi-unipotent local monodromy, the same holds for each direct factor of $\bigwedge^n \mathbb{V}$, in particular for $\operatorname{Sym}^n(\mathbb{V}_1)$. Thus, one induces that $\mathbb{V}_1$ itself is also of quasi-unipotent local monodromy, and therefore so is $\mathbb{V}'$. By induction on the number of factors k in the tensor decomposition of $\mathbb{V}$, one concludes that each factor has quasi-unipotent local monodromy. □

Theorem 6.3 *Each local system $\mathbb{V}_i$ admits the structure of a $\mathbb{C}$-PVHS such that the induced $\mathbb{C}$-PVHS on the tensor product $\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_k$ coincides with the given $\mathbb{C}$-PVHS on $\mathbb{V}$.*

Proof Let $s \in S$ be a base point and let $\rho_i : \pi_1(S, s) \to G_i$ be the monodromy representation of $\mathbb{V}_i$. We put V_1 to be the fiber of $\mathbb{V}_1$ at s. Since ρ_i is a Zariski dense representation into the simple Lie group G_i with quasi-unipotent local monodromy

around Z, by [14] there exists a pluri-harmonic metric on the flat bundle $\mathbb{V}_i$ with finite energy, which makes $\mathbb{V}_i$ into a Higgs bundle (E_i, θ_i) over S. Furthermore in [15] Mochizuki has analyzed the singularity of this harmonic metric in detail and has shown that (E_i, θ_i) admits a logarithmic extension $(\bar{E}_i, \bar{\theta}_i)$ over $\bar{S}$, i.e. a vector bundle $\bar{E}_i$ over $\bar{S}$ which extends E_i and a map

$$\bar{\theta} : \bar{E}_i \longrightarrow \bar{E}_i \otimes \Omega^1_{\bar{S}}(\log Z)$$

which coincides with θ over S. Such a pluri-harmonic metric is called tame. In this case the residue of $\bar{\theta}$ along Z is unipotent.

From the proof of Lemma 6.2 we know that one finds a non-trivial component $\mathrm{Sym}^n(\mathbb{V}_1)$ in $\bigwedge^n(\mathbb{V}_1 \otimes \mathbb{V}')$. Since ρ_1 is Zariski dense in G_1 and G_1 is simple, one finds a suitable Schur functor $\mathbb{S}_\mu$ such that $\mathbb{S}_\mu(\rho_1)$ is a nontrivial irreducible direct factor of $\bigwedge^n(\mathbb{V}_1 \otimes \mathbb{V}')$. Since $\bigwedge^n(\mathbb{V}_1 \otimes \mathbb{V}')$ is semi-simple, there exists a decomposition

$$\bigwedge^n(\mathbb{V}_1 \otimes \mathbb{V}') = \bigoplus_{i=1}^m \mathbb{W}_{i1} \otimes \mathbb{W}_{i2}$$

where the $\mathbb{W}_{i1}$ are irreducible and the $\mathbb{W}_{i2}$ are trivial. By Proposition 1.13 in [5], there exist uniquely determined $\mathbb{C}$-PVHS on the $\mathbb{W}_{i1}$ and complex Hodge structures on the $\mathbb{W}_{i2}$ such that the direct sum of the tensor products of them coincides with the $\mathbb{C}$-PVHS on $\bigwedge^n(\mathbb{V}_1 \otimes \mathbb{V}')$. In particular, there exists a $\mathbb{C}$-PVHS on $\mathbb{S}_\mu(\mathbb{V}_1)$.

By the uniqueness of such pluri-harmonic metrics, the induced pluri-harmonic metric on $\mathbb{S}_\mu(\bar{E}_1, \bar{\theta}_1)$ coincides with that of the $\mathbb{C}$-PVHS on $\mathbb{S}_\mu(\mathbb{V}_1)$. Hence $\mathbb{S}_\mu(\bar{E}_1, \bar{\theta}_1)$ is a fixed point of the $\mathbb{C}^\times$-action. The Schur functor $\mathbb{S}_\mu$ induces a nontrivial morphism $G_1 \to GL(\mathbb{S}_\mu(V_1))$, which is injective since G_1 is simple. It induces the morphism

$$\phi_\mu : \mathfrak{M}(\pi_1(S), G_1)^{\mathrm{ss}} \longrightarrow \mathfrak{M}(\pi_1(S), \mathrm{GL}(\mathbb{S}_\mu(V_1)))^{\mathrm{ss}}$$

between the corresponding moduli spaces of semi-simple representations. By Corollary 9.16 in [25], the morphism ϕ_μ is finite.

If $Z = \emptyset$, then $\mathbb{C}^\times$ acts on both moduli spaces continuously via the Hermitian Yang-Mills metric on poly-stable Higgs bundles $(E, t\theta)$, and this action is compatible with ϕ_μ. Since $\mathbb{S}_\mu(\rho_1)$ is a fixed point of the $\mathbb{C}^\times$-action, the representation ρ_1 itself is a fixed point of the $\mathbb{C}^\times$-action. Hence (E_1, θ_1) is a $\mathbb{C}$-PVHS on $\mathbb{V}_1$. Now we consider the general situation $Z \neq \emptyset$. Let $C \subseteq \bar{S}$ denote a curve which is a complete intersection of ample hypersurfaces, and define $C_0 = C \backslash Z$. Taking the restriction

$$\rho_1|_{C_0} \in \mathfrak{M}(\pi_1(C_0), G_1)^{\mathrm{ss}},$$

we have $\mathbb{S}_\mu(\rho_1)|_{C_0} \in \mathfrak{M}(\pi_1(C_0), \mathrm{GL}(\mathbb{S}_\mu(V_1)))^{\mathrm{ss}}$. Now we consider the map

$$\phi_\mu : \mathfrak{M}(\pi_1(C_0), G_1)^{\mathrm{ss}} \longrightarrow \mathfrak{M}(\pi_1(C_0), \mathrm{GL}(\mathbb{S}_\mu(V_1)))^{\mathrm{ss}}.$$

By Simpson's main theorem in [23], there exist Hermitian Yang-Mills metrics on poly-stable Higgs bundles on C with logarithmic poles of the Higgs field on $C \cap Z$.

The $\mathbb{C}^\times$-action can be defined on both spaces of semi-simple representations on C_0 via a Hermitian Yang-Mills metric on $(\bar{E}, t\bar{\theta})$. Applying the same arguments as above to the compact case, we show that the pullback of $(\bar{E}_1, \bar{\theta}_1)$ to C_0 is a fixed point of the $\mathbb{C}^\times$-action. If we choose C_0 sufficiently ample, then $(\bar{E}_1, \bar{\theta}_1)$ is also a fixed point of the $\mathbb{C}^\times$-action. Again by [23], $(\bar{E}_1, \bar{\theta}_1)$ is a $\mathbb{C}$-PVHS on $\mathbb{V}_1$.

Since $\mathbb{V}'$ is a direct factor $\mathbb{V}_1^* \otimes \mathbb{V}_1 \otimes \mathbb{V}' = \mathbb{V}_1^* \otimes \mathbb{V}$, $\mathbb{V}'$ admits a $\mathbb{C}$-PVHS as well. The tensor product of $\mathbb{C}$-PVHS on $\mathbb{V}_1$ and $\mathbb{V}'$ is a $\mathbb{C}$-PVHS on $\mathbb{V}_1 \otimes \mathbb{V}'$. By Deligne's uniqueness theorem on $\mathbb{C}$-PVHS on irreducible local systems, the tensor product coincides with the original $\mathbb{C}$-PVHS on $\mathbb{V}_1 \otimes \mathbb{V}' = \mathbb{V}$. By induction on the number of factors k, we conclude the proof of the theorem. $\qquad\square$

Theorem 6.4 *Let S be a smooth quasi-projective algebraic variety and $\mathbb{V}$ be a weight 3 $\mathbb{Z}$-PVHS over S such that $\mathbb{V} \otimes \mathbb{C}$ is irreducible as local system. If the Hodge numbers of $\mathbb{V}$ are $(1, 9, 9, 1)$, then after a possible finite étale base change the connected component of the real Zariski closure of the monodromy group of $\mathbb{V}$ is one of the following:*

(A) $SU(1, 1) \times SO_0(2, 8)$,
(B) $SU(3, 3)$,
(C) $Sp(20, \mathbb{R})$.

Proof Let $s \in S$ be a base point of S and V be the fiber of $\mathbb{V}$ at s. Let $\tau : \pi_1(S, s) \longrightarrow GL(V \otimes_\mathbb{Z} \mathbb{R})$ be the monodromy representation of $\mathbb{V} \otimes \mathbb{R}$ and G be the Zariski closure of τ. So we have the factorization

$$\tau : \pi_1(S, s) \to G \xrightarrow{\rho} GL(V_\mathbb{R})$$

where ρ is a morphism of real algebraic groups. Since $\mathbb{V}$ is of polarized and of $\mathbb{Z}$-coefficients, G is semi-simple by Deligne (cf. Corollary 4.2.9 in [4]). Since $\mathbb{V}$ is of weight 3 and the dimension of V is 20, G is a semi-simple real Lie subgroup of $Sp(20, \mathbb{R})$. Let G^0 be the connected component of G. The classification of G^0 consists of several steps.

Step 1. Let $\mathfrak{g}$ be the Lie algebra of G^0 and $\chi : \mathfrak{g}_\mathbb{C} \longrightarrow \mathfrak{sp}_{20}\mathbb{C}$ be the complexification of the differential of ρ. Then the pair $(\mathfrak{g}_\mathbb{C}, \chi)$ is one of the following list:

(1) $(\mathfrak{sl}(2), \Gamma_{19})$
(2) $(\mathfrak{so}(5), \Gamma_{03})$
(3) $(\mathfrak{sl}(6), \Gamma_{00100})$
(4) $(\mathfrak{sp}(20), \Gamma_{1000000000})$
(5) $(\mathfrak{sl}(2) \oplus \mathfrak{sl}(2), \Gamma_3 \otimes \Gamma_4)$
(6) $(\mathfrak{sl}(2) \oplus \mathfrak{so}(5), \Gamma_3 \otimes \Gamma_{10})$
(7) $(\mathfrak{so}(5) \oplus \mathfrak{sl}(2), \Gamma_{01} \otimes \Gamma_4)$
(8) $(\mathfrak{so}(5) \oplus \mathfrak{so}(5), \Gamma_{10} \otimes \Gamma_{02})$
(9) $(\mathfrak{sl}(2) \oplus \mathfrak{so}(5), \Gamma_1 \otimes \Gamma_{02})$
(10) $(\mathfrak{sl}(2) \oplus \mathfrak{so}(10), \Gamma_1 \otimes \Gamma_{10000})$

Here we use the notations as given in [7]. The list results from a rather standard calculation in the representation theory of semi-simple complex Lie algebras. As $\mathfrak{g}_\mathbb{C}$ is semi-simple, we can write $\mathfrak{g}_\mathbb{C} = \oplus_{i=1}^m \mathfrak{g}_i$ into direct sum of simple Lie algebras. By Schur's lemma and since χ is irreducible, we have the tensor decomposition of $\chi = \otimes \chi_i$ into irreducible representations. Then one has particularly $\prod_{i=1}^m d_i = 20$, where d_i is the dimension of the representation space of χ_i. It is straightforward to

write down a complete list of irreducible representations of complex simple Lie algebras whose dimensions divide 20. For each pair $(\mathfrak{g}_\mathbb{C}, \chi)$ where $\chi : \mathfrak{g}_\mathbb{C} \to \mathfrak{sl}(20)$ in this preliminary list, it appears in the final list, i.e. χ factors through $\mathfrak{sp}(20) \subset \mathfrak{sl}(20)$ if and only if $\wedge^2 \chi$ contains a trivial representation. This can be easily checked by using the plethysm of semi-simple complex Lie algebras.

Step 2. Let $G_\mathbb{C}$ be the complexification of G^0. Then by Lemma 4.4 in [24], G^0 is a real form of $G_\mathbb{C}$ which is of Hodge type (see Sect. 4 [24] for the definition). We shall make use of the list of all simple real Lie groups of Hodge type on page 50 [24]. In connection with the list in Step 1, one can immediately exclude the case where $\mathfrak{g}$ is simple but $\mathfrak{g}_\mathbb{C}$ is not simple by Proposition 4.4.10 in [24]. Hence the number of irreducible factors in $\mathfrak{g}$ is equal to that of $\mathfrak{g}_\mathbb{C}$.

Step 3. We start with two factors. Namely, $G^0 = G_1 \times G_2$ with G_i simple real Lie groups. It induces the tensor decomposition of real local systems $\mathbb{V}_\mathbb{R} = \mathbb{V}_1 \otimes \mathbb{V}_2$. By Theorem 6.3 there exist $\mathbb{C}$-PVHS structures on $\mathbb{V}_1$ and $\mathbb{V}_2$ such that the induced PVHS on $\mathbb{V}$ coincides with the original one. It follows that the Lie groups G_1 and G_2 are also of Hodge type. By Lemma 5.5 in [24] $\mathbb{V}_i$, $i = 1, 2$ underlies $\mathbb{R}$-PVHS structure. Recall that the weight of $\mathbb{V}$ is three and its Hodge numbers are 1, 9, 9, 1. Then after a possible permutation of factors the PVHS $\mathbb{V}_1$ must be of weight 1 with Hodge numbers 1, 1 and $\mathbb{V}_2$ is of weight 2 with Hodge numbers 1, 8, 1. This implies $G_1 = SU(1, 1)$ and $G_2 \subseteq SO(2, 8)$. This excludes immediately the case (8) of the list in Step 1. It excludes also the cases (5–7) by checking the dimension of representation on $\mathfrak{sl}(2)$-factor. Let us now consider the case (9). Note that the representation Γ_{02} is simply the second wedge power of the standard representation in this case. By the list in [24] G_2 can be one of the groups $SO(5)$, $SO(1, 4)$ or $SO(2, 3)$. Since $SO(5)$ is compact, it is mapped into the compact form $SO(10)$ of $SO(10, \mathbb{C})$ under $\wedge^2$ of the standard representation of $SO(5, \mathbb{C})$. Also one checks that under the same representation the other two real forms $SO(1, 4)$ and $SO(2, 3)$ are mapped into the real forms $SO(6, 4)$ and $SO(4, 6)$, respectively. Thus, case (9) can be excluded. So it remains (10) for the non-simple case. Obviously $G_2 = SO_0(2, 8)$ and this gives the case (A).

Step 4. We treat the case that $\mathfrak{g}_\mathbb{C}$ is simple. By the list in [24], G^0 can be $SU(1, 1)$ in case (1), $SO(5)$, $SO(2, 3)$ and $SO(1, 4)$ in case (2), $SU(p, 6 - p)$ in case (3), $Sp(20, \mathbb{R})$ in case (4). Note that except for $SO(1, 4)$ the remaining groups are of Hermitian type. By consideration of weight and Hodge numbers as in Step 3 we can exclude all cases except $SU(3, 3)$ and $Sp(20, \mathbb{R})$, which give case (B) and (C), respectively. Finally one can check directly that the real form $SO(1, 4)$ of $SO(5, \mathbb{C})$ does not map into the split form $Sp(20, \mathbb{R})$ of $Sp(20, \mathbb{C})$ under the third wedge power. For this, one can consult Example 3, Sect. 7 in [21]. This completes the classification. □

Theorem 6.5 *Let* $f : \mathcal{X} \to S$ *be a good family of* $\mathfrak{M}_{CY}$ *and* $\mathbb{V}$ *be the associated weight 3* $\mathbb{Z}$-*PVHS. Let* $s \in S$ *be a base point and let*

$$\tau : \pi_1(S, s) \to Sp(20, \mathbb{R})$$

be the monodromy representation associated to $\mathbb{V} \otimes \mathbb{R}$. *Then the image of* τ *is Zariski dense in* $Sp(20, \mathbb{R})$.

Proof Let G be the real Zariski closure of τ. Note that the property of being Zariski dense is not changed under a finite étale base change. By Theorem 6.4 we have to show for $\mathbb{V}$ in the statement the connected component G^0 of G can not be the case (A) or (B). Assuming the contrary, we will deduce a contradiction in the following. Note that by the proof of Theorem 6.4, the inclusion $G \to Sp(20, \mathbb{R})$ comes with a uniquely determined morphism of real algebraic groups $\rho : G \to Sp(20, \mathbb{R})$, which restricts to ρ_0 on G^0. Actually ρ_0 has already appeared in Proposition 3.3 implicitly. It is easy to verify that ρ_0 in each case maps the maximal compact group K^0 of G^0 into the compact subgroup $U(1) \times U(9)$ of $Sp(20, \mathbb{R})$. Let Γ be the monodromy group of τ and $\Gamma_0 = \rho^{-1}(\Gamma) \subset G$. So we get a factorization

$$\tau : \pi_1(S, s) \xrightarrow{\tau_0} \Gamma_0 \xrightarrow{\rho} \Gamma.$$

This gives a factorization of the period map of $\mathbb{V}$

$$\phi : S \xrightarrow{j} \Gamma_0 \backslash G / K \xrightarrow{\psi} \Gamma \backslash Sp(20, \mathbb{R}) / U(1) \times U(9)$$

where K is the maximal compact group of G. Since S is connected, the morphism j factors though $\Gamma_0' \backslash G^0 / K^0 \subset \Gamma_0 \backslash G / K$ for $\Gamma_0' = \Gamma_0 \cap G^0$. Thus, we arrive at the factorization in Definition 3.1. Since $\rho_0 : G^0 \to Sp(20, \mathbb{R})$ gives rise to the canonical PVHS, the factorization contradicts the assertion of Theorem 6.1. The proof is complete. □

Corollary 6.6 *The special Mumford–Tate group of a general member in $\mathfrak{M}_{CY}$ is $Sp(20, \mathbb{Q})$.*

Proof Let $f : \mathcal{X} \to S$ be a good family for $\mathfrak{M}_{CY}$. Let $\mathbb{V}$ be the associated weight 3 $\mathbb{Q}$-PVHS of f and $\tau : \pi_1(S, s) \to Sp(20, \mathbb{Q})$ be the monodromy representation. By Deligne and Schoen (see for example Lemma 2.4 [32]), the connected component of the $\mathbb{Q}$-Zariski closure of the monodromy group is a normal subgroup of the special Mumford–Tate group $Hg(\mathbb{V})$ of $\mathbb{V}$, which is equal to the special Mumford–Tate group of a general closed fiber of f. By Theorem 6.5 the $\mathbb{Q}$-Zariski closure of the monodromy group of $\mathbb{V}$ has to be the whole symplectic group $Sp(20, \mathbb{Q})$, the corollary follows since the moduli map of f is dominant. □

Theorem 6.7 *Let $\mathfrak{H}_{CY}$ be the hyperelliptic locus of $\mathfrak{M}_{CY}$ and $\mathfrak{H}$ be any irreducible subvariety of $\mathfrak{M}_{CY}$ which strictly contains $\mathfrak{H}_{CY}$. Let $f : \mathcal{X} \to S$ be a good family of $\mathfrak{M}_{CY}$ whose moduli map $S \to \mathfrak{M}_{CY}$ is dominant over $\mathfrak{H}$. Then the restriction of $\mathbb{V}$ to the inverse image of $\mathfrak{H}$ does not factor through $(D_3^{III}, \wedge^3)$.*

Proof We first prove the statement for the good family f_0 over $\mathfrak{M}_{AR}$. Assume that an extension $\mathfrak{H}$ for f_0 as in the theorem does exist. We can assume $\dim \mathfrak{H} = 6$ without loss of generality. Let $\widetilde{\mathfrak{H}}$ be the inverse image of $\mathfrak{H}$ in $\mathfrak{M}_{AR}$ and $f_0' = f_0|_{\widetilde{\mathfrak{H}}}$ the restriction of f_0 to $\widetilde{\mathfrak{H}} \subset \mathfrak{M}_{AR}$. Let ϕ be the period map of f_0 and ϕ' that of f_0'. By assumption one has the factorization

$$\phi' : \widetilde{\mathfrak{H}} \xrightarrow{j} \Gamma_0 \backslash Sp(6, \mathbb{R}) / U(3) \xrightarrow{\wedge^3} \Gamma \backslash Sp(20, \mathbb{R}) / U(1) \times U(9).$$

By Corollary 2.2 and the local Torelli theorem for CY manifolds, ϕ is étale over its image. So is the restriction ϕ'. Because $\Gamma_0 \backslash Sp(6, \mathbb{R})/U(3)$ is of six dimensional, j is then étale. We derive the contradiction at the special point $s_0 \in \widetilde{\mathfrak{H}}_{CY} \subset \widetilde{\mathfrak{H}}$. We denote by $\mathcal{C}_1$ the first characteristic subvariety of f_0 and $\mathcal{C}_1'$ that of f_0'. By Proposition 4.6 (ii), the fiber $(\mathcal{C}_1')_{s_0}$ as subvariety of $\mathbb{P}(\mathcal{T}_{\widetilde{\mathfrak{H}}, s_0})$ is isomorphic to the $\mathbb{P}^2$ into $\mathbb{P}^5$ via the Veronese embedding. In particular $(\mathcal{C}_1')_{s_0}$ is reduced. On the other hand, by Lemma 4.3 we know that

$$(\mathcal{C}_1')_{s_0} = ((\mathcal{C}_1)_{s_0} \cap \mathbb{P}(\mathcal{T}_{\widetilde{\mathfrak{H}}, s_0}))^{red},$$

where the scheme-theoretical intersection of the right hand side is taken in the projective space $\mathbb{P}(\mathcal{T}_{\mathfrak{M}_{AR}, s_0})$. Now by Proposition 5.5, $(\mathcal{C}_1)_{s_0}$ is two dimensional and has two irreducible components which are not contained in any linear projective subspace of dimension ≤ 6. It follows that $((\mathcal{C}_1)_{s_0} \cap \mathbb{P}(\mathcal{T}_{\widetilde{\mathfrak{H}}, s_0}))^{red}$ is of dimension ≤ 1. A contradiction. Thus, such an extension $\mathfrak{H}$ for f_0 does not exist.

As a consequence we get the maximal property of $\mathfrak{H}_{CY}$ stated in Corollary 1.5. This fact shows in turn the non-extension property for other good families in the theorem. Let f be such a good family. As above we can assume $\dim \mathfrak{H} = 6$. We put S' to be inverse image of $\mathfrak{H}$ under the moduli map $S \to \mathfrak{M}_{CY}$. The factorization of the period map gives the morphism $j : S' \to \Gamma_0 \backslash Sp(6, \mathbb{R})/U(3)$ and it induces an isomorphism $\mathbb{V}|_{S'} \simeq j^* \mathbb{W}$ with $\mathbb{W}$ the canonical $\mathbb{R}$-PVHS over $\Gamma_0 \backslash Sp(6, \mathbb{R})/U(3)$. Let $\mathbb{V}_{\mathbb{Q}}$ be the $\mathbb{Q}$-PVHS associated to f. The canonical PVHS $\mathbb{W}$ over $\Gamma_0 \backslash Sp(6, \mathbb{R})/U(3)$ has also a natural $\mathbb{Q}$-structure $\mathbb{W}_{\mathbb{Q}}$ such that over the points in $\mathfrak{H}_{CY}$ the isomorphism between $\mathbb{V}$ and $j^* \mathbb{W}$ is defined over $\mathbb{Q}$. Thus, we have actually an isomorphism $(\mathbb{V}_{\mathbb{Q}})|_{S'} \simeq j^* \mathbb{W}_{\mathbb{Q}}$. This implies that the special Mumford–Tate group of a general closed point in S' and hence in $\mathfrak{H}$ is contained in $Sp(6, \mathbb{Q})$. This contradicts Corollary 1.5. □

References

1. Allcock, D., Carlson, J., Toledo, D.: The complex hyperbolic geometry of the moduli space of cubic surfaces. J. Algebraic Geom. **11**(4), 659–724 (2002)
2. Borcherds, R.: The moduli space of Enriques surfaces and the fake monster Lie superalgebra. Topology **35**, 699–710 (1996)
3. Cynk, S., van Straten, D.: Infinitesimal deformations of double covers of smooth algebraic varieties. Math. Nachr. **279**(7), 716–726 (2006)
4. Deligne, P.: Théorie de Hodge II. Publ. Math. I.H.E.S. **40**, 5–57 (1971)
5. Deligne, P.: Un théorème de finitude pour la monodromie. Discrete Groups in Geometry and Analysis, Birkhauser, pp. 1–19 (1987)
6. Deligne, P., Mostow, G.D.: Monodromy of hypergeometric functions and non-lattice integral monodromy. Publ. Math. I.H.E.S. Tome **63**, 5–89 (1986)
7. Fulton, W., Harris, J.: Representation theory, A first course. GTM 129
8. Gerkmann, R., Sheng, M., Zuo, K.: Computational details on the disproof of modularity. arXiv:0709.1054 (2007)

9. Griffiths, P.: Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping. Publ. Math. I.H.E.S., **38**, 125–180 (1970)
10. Griffiths, P.: Topics in Transcendental Algebraic Geometry. Annals of Mathematics Studies, vol. 106. Princeton University Press (1984)
11. Griffiths, P.: Hodge theory and geometry. Bull. Lond. Math. Soc. **36**, 721–757 (2004)
12. Gross, B.: A remark on tube domains. Math. Res. Lett. **1**, 1–9 (1994)
13. Grothendieck, A.: Sur quelques points d'algèbre homologique. Tôhoku Math. J. **9**, 119–221 (1957)
14. Jost, J., Zuo, K.: Harmonic maps and Sl(r, C)-representations of fundamental groups of quasiprojective manifolds. J. Algebraic Geom. **5**(1), 77–106 (1996)
15. Mochizuki, T.: Asymptotic behaviour of tame nilpotent harmonic bundles with trivial parabolic structure. J. Differ. Geom. **62**(3), 351–559 (2002)
16. Mok, N.: Uniqueness theorems of Hermitian metrics of seminegative curvature on quotients of bounded symmetric domains. Ann. Math. **125**(1), 105–152 (1987)
17. Mok, N.: Metric Rigity Theorems on Hermitian Locally Symmetric Manifolds. Series in Pure Mathmatics, vol. 6. World Scientific Publishing Co., Inc., Teaneck (1989)
18. Matsumoto, K., Sasaki, T., Yoshida, M.: The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type (3,6). Int. J. Math. **3**(1), 1–164 (1992)
19. Moeller, M., Viehweg, E., Zuo, K.: Stability of Hodge bundles and a numerical characterization of Shimura varieties. arXiv AG/07063462
20. Nagel, J.: The Image of the Abel-Jacobi Map for Complete Intersections. Ph.D Thesis, Rijksuniversiteit Leiden (1997)
21. Onishchik, L.: Lectures on Real Semisimple Lie Algebras and Their Representations. ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich (2004)
22. Peters, C., Steenbrink, J.: Mixed Hodge structures. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge 52, Springer, Berlin (2008)
23. Simpson, C.: Harmonic bundles on noncompact curves. J. Amer. Math. Soc. **3**(3), 713–770 (1990)
24. Simpson, C.: Higgs bundles and local systems. Publ. Math. I.H.E.S., **75**, 5–95 (1992)
25. Simpson, C.: Moduli of representations of the fundamental group of a smooth projective varieites II. Publ. Math. I.H.E.S., **80**, 5–79 (1994)
26. Sasaki, T., Yamaguchi, K., Yoshida, M.: On the ridity of differential systems modelled on Hermitian symmetric spaces and disproofs of a conjecture concerning modular interpretations of configuration spaces. Adv. Stud. Pure Math. **25**, 318–354 (1997)
27. Sheng, M.: On the Geometric Realizations of Hermitian Symmetric Domains. Ph.D Thesis, The Chinese University of Hong Kong (2006)
28. Sheng, M., Zuo, K.: Polarized variation of Hodge structures of Calabi–Yau type and characteristic subvarieties over bounded symmetric domains. Math. Ann. **348**, 211–236 (2010)
29. Terasoma, T.: Complete intersections of hypersurfaces—the Fermat case and the quadric case. Jpn. J. Math. **14**(2), 309–384 (1988)
30. Terasoma, T.: Infinitesimal variation of Hodge structures and the weak global Torelli theorem for complete intersections. Ann. Math. **132**(2), 213–225 (1990)
31. Viehweg, E.: Quasi-projective moduli for polarized manifolds. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 30. Springer, Berlin (1995)
32. Viehweg, E., Zuo, K.: A characterization of certain Shimura curves in the moduli stack of abelian varieties. J. Differ. Geom. **66**(2), 233–287 (2004)
33. Viehweg, E., Zuo, K.: Arakelov inequalities and the uniformization of certain rigid Shimura varieties. J. Differ. Geom. **77**(2), 291–352 (2007)
34. Yoshida, M.: Hypergeometric Functions, My Love: Modular Interpretations of Configuration Spaces. Aspects of Mathematics, E32. Friedr. Vieweg & Sohn, Braunschweig (1997)
35. Zucker, S.: Locally homogenous variations of Hodge structures. Enseign. Math. **27**(3–4), 243–276 (1982)