Math. Ann.
DOI 10.1007/s00208-009-0378-9

Mathematische Annalen

Polarized variation of Hodge structures of Calabi–Yau type and characteristic subvarieties over bounded symmetric domains

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Received: 11 July 2007 / Revised: 5 April 2009

Abstract In this paper, we extend the construction of the canonical polarized variation of Hodge structures over tube domain considered by Gross (Math Res Lett 1:1–9, 1994) to bounded symmetric domain and introduce a series of invariants of infinitesimal variation of Hodge structures, which we call characteristic subvarieties. We prove that the characteristic subvariety of the canonical polarized variations of Hodge structures over irreducible bounded symmetric domains are identified with the characteristic bundles defined by Mok (Ann Math 125(1):105–152, 1987). We verified the generating property of Gross for all irreducible bounded symmetric domains, which was predicted in Gross (Math Res Lett 1:1–9, 1994).

1 Introduction

It has been interesting for long to find a new global Torelli theorem for Calabi-Yau manifolds, which extends the celebrated global Torelli theorem for polarized K3 surfaces. The work of Gross [6] is closely related to this problem. In fact, at the Hodge theoretical level, Gross [6] has constructed certain canonical real polarized variation

This work was supported by the SFB/TR 45 Periods, Moduli Spaces and Arithmetic of Algebraic Varieties of the DFG (German Research Foundation). M. Sheng is supported by a Postdoctoral Fellowship in the East China Normal University.

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Published online: 13 January 2010

of Hodge structures (PVHS) over each irreducible tube domain, and then asked for the possible algebraic geometrical realizations of them (cf. [6, §8]). In this paper, we introduce certain invariants, called *characteristic subvarieties*, which turned out to be nontrivial obstructions to the realization problem posed by B. Gross.

For a universal family of polarized Calabi-Yau n-folds $f : \mathcal{X} \to S$, we consider the $\mathbb{Q}$-PVHS $\mathbb{V}$ formed by the primitive middle rational cohomologies of fibers. Let (E, θ) be the system of Hodge bundles associated with $\mathbb{V}$. By the definition of Calabi-Yau manifold, we have the first property of (E, θ):

$$\operatorname{rank} E^{n,0} = 1.$$

The Bogomolov–Todorov–Tian unobstructedness theorem for the moduli space of Calabi-Yau manifolds gives us the second property of (E, θ):

$$\theta : T_S \xrightarrow{\simeq} \operatorname{Hom}(E^{n,0}, E^{n-1,1}).$$

On the other hand, we see that the canonical $\mathbb{R}$-PVHSs associated to tube domains considered by Gross [6] also have the above two properties (See [6, Propositions 4.1 and 5.2]). In this paper, we consider the following types of $\mathbb{C}$-PVHS (See Definition 4.6, [9] for the notion of $\mathbb{C}$-PVHS).

Definition 1.1 Let $\mathbb{V}$ be a $\mathbb{C}$-PVHS of weight n over complex manifold S with associated system of Hodge bundles (E, θ). We call $\mathbb{V}$ PVHS of Calabi-Yau type or *Type I* if $\mathbb{V}$ is a $\mathbb{R}$-PVHS and (E, θ) satisfies

- $\operatorname{rank} E^{n,0} = 1$;
- $\theta : T_S \xrightarrow{\simeq} \operatorname{Hom}(E^{n,0}, E^{n-1,1})$.

If $\mathbb{V}$ is not defined over $\mathbb{R}$ and (E, θ) satisfies the above two properties, then we call $\mathbb{V}$ PVHS of *Type II*.

For a type I PVHS, one has $\operatorname{rank} E^{0,n} = 1$ by the Hodge isometry. In our case it is not important to emphasize the real structure of a PVHS. Rather we will simply regard a PVHS of type I or type II as of Calabi-Yau type.

This paper consists of three parts. The first part extends the construction of PVHS by Gross [6] to each irreducible bounded symmetric domain. This is a straightforward step. In the second part, we introduce a series of invariants associated with the infinitesimal variation of Hodge structures (IVHS) (See the initiative work of IVHS in [2]). We call these invariants characteristic subvarieties. Our main result (Theorem 3.3) identifies the characteristic subvarieties of the canonical PVHS over an irreducible bounded symmetric domain with the characteristic bundles defined by Mok [7] (see [8, Chapter 6 and Appendix III] for a more expository introduction). The last part of this paper verifies the generating property predicted by Gross (cf. [6, §5]), and describes the canonical PVHSs over irreducible bounded symmetric domains in certain detail.

In a recent joint work with Gerkmann [5], we used the results of this paper to disprove modularity of the moduli space of Calabi-Yau 3-folds arising from eight planes of $\mathbb{P}^3$ in general positions.

2 The canonical PVHS over bounded symmetric domain

Let D be an irreducible bounded symmetric domain, and let G be the identity component of the automorphism group of D. We fix an origin $0 \in D$. Then the isotropy subgroup of G at 0 is a maximal compact subgroup K. By Proposition 1.2.6 [3], D determines a special node v of the Dynkin diagram of the simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}} = Lie(G) \otimes \mathbb{C}$. By the standard theory on the finite dimensional representations of semi-simple complex Lie algebras (cf. [4]), we know that the special node v also determines a fundamental representation W of $\mathfrak{g}^{\mathbb{C}}$. By the Weyl's unitary trick, W gives rise to an irreducible complex representation of G. When D is tube domain, the representation W is exactly the one considered by Gross [6] and only in this case W admits an G-invariant real form. It is helpful to illustrate the above construction in the simplest case.

Example 2.1 Let $D = SU(p,q)/S(U(p) \times U(q))$ be a type A bounded symmetric domain. Then

$$G = SU(p,q), \quad dK = S(U(p) \times U(q)), \quad \mathfrak{g}^{\mathbb{C}} = sl(p+q, \mathbb{C}).$$

The special node v of the Dynkin diagram of $sl(p+q,\mathbb{C})$ corresponding to D is the pth node. Let $\mathbb{C}^{p+q}$ be the standard representation of $Sl(p+q,\mathbb{C})$. Then the fundamental representation denoted by v is

$$W = \bigwedge^{p} \mathbb{C}^{p+q}.$$

The group G preserves a hermitian symmetric bilinear form h with signature (p,q) over $\mathbb{C}^{p+q}$. Then D is the parameter space of the h-positive p-dimensional vector subspace of $\mathbb{C}^{p+q}$. By fixing an origin $0 \in D$, we obtain an h-orthogonal decomposition

$$\mathbb{C}^{p+q} = \mathbb{C}^{p}_{+} \oplus \mathbb{C}^{q}_{-}.$$

The corresponding Higgs bundle to $\mathbb{W}$ is of the form

$$E = \bigoplus_{i+j=n} E^{i,j},$$

where $n = \min(\mathrm{p,q})$ is the rank of D. The Hodge bundle $E^{n-i,i}$ is the homogeneous vector bundle determined by, at the origin 0, the irreducible K-representation

$$(E^{n-i,i})_0 = \bigwedge^{p-i} \mathbb{C}^{p}_{+} \otimes \bigwedge^{i} \mathbb{C}^{q}_{-}.$$

Let Γ be a torsion free discrete subgroup of G, we can obtain from the representation W the complex local system

$$\mathbb{W} = W \times_{\Gamma} D$$

over the locally symmetric variety $X = \Gamma\backslash D$. By the construction on the last paragraph of §4, [9], we know that $\mathbb{W}$ is a $\mathbb{C}$-PVHS. We denote by (E, θ) the associated system of Hodge bundles with $\mathbb{W}$. With similar proofs as those in Propositions 4.1 and 5.2 of [6], or from the explicit descriptions given in the last section, we have the following

Theorem 2.2 *Let $D = G/K$ be an irreducible bounded symmetric domain of rank n and Γ be a torsion free discrete subgroup of G. Let $\mathbb{W}$ be the irreducible PVHS over the locally symmetric variety $X = \Gamma\backslash D$ constructed above. Then $\mathbb{W}$ is a weight n $\mathbb{C}$-PVHS of Calabi-Yau type.*

Following B. Gross we shall call $\mathbb{W}$ over X in the above theorem as the canonical PVHS over X.

3 The characteristic subvariety and the main result

We start with a system of Hodge bundles

$$\left(E = \bigoplus_{p+q=n} E^{p,q}, \theta = \bigoplus_{p+q=n} \theta^{p,q}\right)$$

over a complex manifold X with $\dim E^{n,0} \neq 0$. By the integrability of Higgs field θ, the k-iterated Higgs field factors as $E \to E \otimes S^k(\Omega_X)$. It induces in turn the following natural map

$$\theta^k : S^k(T_X) \to \mathrm{End}(E).$$

By the Griffiths's horizontal condition, the image of θ^k is contained in the subbundle

$$\bigoplus_{p+q=n} \mathrm{Hom}(E^{p,q}, E^{p-k,q+k}) \subset \mathrm{End}(E).$$

We are interested in the projection of θ^k into the first component of the above subbundle. Abusing the notation a little bit, we denote the composition map still by θ^k. That is, we concern the following map

$$\theta^k : S^k(T_X) \to \mathrm{Hom}(E^{n,0}, E^{n-k,k}).$$

We have a tautological short exact sequence of analytic coherent sheaves defined by the iterated Higgs field θ^k:

$$0 \to I_k \to S^k(T_X) \xrightarrow{\theta^k} J_k \to 0.$$

We define a sheaf of graded $\mathcal{O}_X$-algebras $\mathcal{J}_k$ by putting

$$\mathcal{J}_k^i = \begin{cases} S^i\Omega_X & \text{if } i < k, \\ S^i\Omega_X/Im((J_k)^* \otimes S^{i-k}\Omega_X \stackrel{mult.}{\longrightarrow} S^i\Omega_X) & \text{if } i \geq k. \end{cases}$$

Definition 3.1 For $k \geq 0$, we call

$$C_k = Proj(\mathcal{J}_{k+1})$$

the kth characteristic subvariety of (E, θ) over X.

By the definition the fiber of a characteristic subvariety over a point is the zero locus of system of polynomial equations determined by the Higgs field in the projective tangent space over that point. In a concrete situation one will be able to calculate some numerical invariants of the characteristic subvarieties. For example, for a complete smooth family of hypersurfaces in a projective space, one can use the Jacobian ring to represent the system of Hodge bundles associated with the PVHS of the middle dimensional primitive cohomolgies in a small neighborhood. In [5] one finds such a calculation in another case.

Because $\theta^{n+1} = 0$,

$$C_k = \mathbb{P}(T_X),\ k \geq n,$$

where $\mathbb{P}(T_X)$ is the projective tangent bundle of X. For $0 \leq k \leq n-1$, the natural surjective morphism of graded $\mathcal{O}_X$-algebras

$$\bigoplus_{i=0}^{\infty} S^i\Omega_X \twoheadrightarrow \mathcal{J}_k$$

gives a proper embedding over X,

$$\begin{tikzcd} C_k \arrow[rr, hook] \arrow[dr, "p_k"'] & & \mathbb{P}(T_X) \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

The next lemma gives a simple criterion to test whether a nonzero tangent vector at the point $x \in X$ has its image in $(C_k)_x = p_k^{-1}(x)$.

Lemma 3.2 *Let $v \in (T_X)_x$ be a non-zero tangent vector at x and $v^k \in (S^k(T_X))_x$ the kth symmetric tensor power of v. Then its image $[v] \in (\mathbb{P}(T_X))_x$ lies in $(C_{k-1})_x$ if and only if $v^k \in (I_k)_x$, the stalk of I_k at x.*

Proof $(C_{k-1})_x \subset (\mathbb{P}(T_X))_x$ is defined by the homogeneous elements contained in $((J_k)^*)_x$. Thus $[v] \in (C_{k-1})_x$ if and only for all $f \in ((J_k)^*)_x$, $f([v]) = 0$. Now we choose a basis $\{e_1, \ldots, e_m\}$ for $(T_X)_x$ and the dual basis $\{e_1^*, \ldots, e_m^*\}$ for $(\Omega_X)_x$.

Claim $f([v])=0$ if and only $f(v^k)=0$. In the latter, we consider f as a linear form on $(S^k(T_X))_x$.

Proof of Claim Let $I=(i_1,\dots,i_m)$ denote the multi-index with $i_j\neq 0$ for all j, and one puts

$$I! = i_1!\cdots i_m!,\quad |I| = i_1+\cdots+i_m.$$

We write $v=\sum_{i=1}^m a_i e_i$ and $f=\sum_{|I|=k} b^I e^I$. Then considering f as a polynomial of degree k on $(T_X)_x$, we have

$$f(v)=\sum_{|I|=k} b^I a^I.$$

On the other hand, we have

$$v^k = k!\sum_{|I|=k}\frac{1}{I!}a^I e^I,$$

where $a^I=a_1^{i_1}\cdots a_m^{i_m}$ etc. By Ex.B.12, [4] the canonically dual basis of $(S^k(\Omega_X))_x$ to the natural basis $\{e^I,\ |I|=k\}$ of $(S^k(T_X))_x$ is $\{\frac{1}{I!}(e^*)^I,\ |I|=k\}$. Hence, evaluating f as a linear form of $(S^k(T_X))_x$ at v^k, we obtain

$$f(v^k)=k!\left(\sum_{|I|=k} b^I a^I\right).$$

It is clear now that our claim holds.

Finally, it is easy to see that $v^k\in(S^k(T_X))_x$ lies in $(I_k)_x$ if and only if for all $f\in((J_k)^*)_x$, considered as a linear form of $(S^k(T_X))_x$, $f(v^k)=0$. Therefore, the lemma follows. □

Our main result identifies the characteristic subvarieties of the canonical PVHS over an irreducible bounded symmetric domain with the characteristic bundles defined by Mok [7].

Theorem 3.3 *Let D be an irreducible bounded symmetric domain of rank n, and let (E,θ) be the system of Hodge bundles associated to the canonical PVHS over $X=\Gamma\backslash D$ as constructed in Theorem 2.2. Then for each k with $1\le k\le n-1$ the kth characteristic subvariety C_k of (E,θ) over X coincides with the kth characteristic bundle $\mathcal{S}_k$ over X.*

By the second property of being of Calabi-Yau type, C_0 is always empty. For the self-containedness of this paper, we would like to describe briefly the notion of characteristic bundles and refer to Chapter 6 and Appendix III in [8] for a full account.

The kth characteristic bundle $\mathcal{S}_k$ over $X = \Gamma\backslash D$ is firstly defined over D. It is a projective subvariety of $\mathbb{P}(T_D)$ and homogeneous under the natural action of automorphism group G on the projective tangent bundle of D. By taking quotient under the left action of Γ, one obtains the kth characteristic bundle over X. So it suffices to describe the construction of characteristic bundle at one point of D. At the origin 0 of D, the vectors contained in the fiber $(\mathcal{S}_k)_0$ are in fact determined by a rank condition. We have the isotropy representation of K on the tangent space $(T_D)_0$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and choose a maximal set of strongly orthogonal positive non-compact roots

$$\Psi = \{\psi_1, \ldots, \psi_n\}.$$

Let e_i, $1 \le i \le n$, be a root vector corresponding to the root ψ_i. Then the set Ψ determines a distinguished polydisk

$$\triangle^n \subset D$$

passing through the origin 0, and

$$(T_{\triangle^n})_0 = \sum_{1\le i\le n} \mathbb{C}e_i \subset (T_D)_0.$$

Moreover, for any nonzero element $v \in (T_D)_0$, there exists an element $k \in K^{\mathbb{C}}$ such that

$$k(v) = \sum_{1\le i\le r(v)} e_i.$$

Such an expression for the vector v is unique and the natural number $r(v)$ is called the *rank* of v. Then, for $1 \le k \le n-1$, one defines

$$(\mathcal{S}_k)_0 = \{[v] \in (\mathbb{P}(T_D))_0 | 1 \le r(v) \le k\}.$$

By the definition, we have a natural inclusion

$$\mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{n-1} \subset \mathbb{P}(T_D),$$

We can add two trivially defined characteristic bundles by putting

$$\mathcal{S}_0 = \emptyset, \quad \mathcal{S}_n = \mathbb{P}(T_D).$$

$(\mathbb{P}(T_D))_0$ is then decomposed into a disjoint union of irreducible $K^{\mathbb{C}}$ orbits

$$(\mathbb{P}(T_D))_0 = \coprod_{1\le k\le n} \{(\mathcal{S}_k)_0 - (\mathcal{S}_{k-1})_0\}.$$

Example 3.4 Let D be the type A tube domain of rank n. Then

$$D = SU(n,n)/S(U(n) \times U(n)).$$

One classically represents D as a space of matrices

$$D = \{Z \in M_{n,n}(\mathbb{C}) | I_n - \bar{Z}^t Z > 0\}.$$

At the origin $0 \in D$,

$$(T_D)_0 \simeq M_{n,n}(\mathbb{C}).$$

The action of

$$K^{\mathbb{C}} \simeq S(Gl(n,\mathbb{C}) \times Gl(n,\mathbb{C}))$$

defined by

$$M \mapsto AMB^{-1}, \text{ for } M \in M_{n,n}(\mathbb{C}) \quad \text{and} \quad (A,B) \in Gl(n,\mathbb{C}) \times Gl(n,\mathbb{C})$$

gives the isotropy representation of $K^{\mathbb{C}}$ on $(T_D)_0$. Then the rank of a vector $M \in (T_D)_0$ defined above is just the rank of M as matrix. Let $(\tilde{\mathcal{S}}_k)_0$ be the lifting of $(\mathcal{S}_k)_0$ in $(T_D)_0$. Therefore, for $1 \leq k \leq n-1$, we have

$$(\tilde{\mathcal{S}}_k)_0 - (\tilde{\mathcal{S}}_{k-1})_0 = S(Gl(n,\mathbb{C}) \times Gl(n,\mathbb{C}))/P_k,$$

where

$$P_k = \left\{(A,B) \in S(Gl(n,\mathbb{C}) \times Gl(n,\mathbb{C})) | A \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} B \right\}.$$

One can show easily that for $1 \leq k \leq n-1$ the dimension of $(\tilde{\mathcal{S}}_k)_0 - (\tilde{\mathcal{S}}_{k-1})_0$ is $(2n-k)k$.

Now let D be an irreducible bounded symmetric domain of rank n, and let

$$i : \triangle^n = \triangle_1 \times \cdots \times \triangle_n \hookrightarrow D$$

be a polydisc embedding. We are going to study the decomposition of $i^*\mathbb{W}$ into a direct sum of irreducible PVHSs over the polydisc. The following proposition is a key ingredient in the proof of Theorem 3.3.

Proposition 3.5 *Let p_i, $1 \leq i \leq n$, be the projection of the polydisc $\triangle^n$ into the ith direct factor $\triangle_i$. Then each irreducible component contained in $i^*\mathbb{W}$ is of the form*

$$p_1^*(\mathbb{L}^{\otimes k_1}) \otimes \cdots \otimes p_n^*(\mathbb{L}^{\otimes k_n}) \otimes \mathbb{U}$$

with

$$0 \le k_i \le 1, \text{ for all } i,$$

where $\mathbb{L}$ *is the weight 1 PVHS coming from the standard representation of* $Sl(2,\mathbb{R})$ *and* $\mathbb{U}$ *is a certain unitary factor. As a consequence, there exists a unique component of the form*

$$p_1^*\mathbb{L} \otimes \cdots \otimes p_n^*\mathbb{L}$$

in $i^*\mathbb{W}$ *because* $\mathbb{W}$ *is of Calabi-Yau type.*

Proof It is known that the polydisc embedding

$$i : \triangle^n \hookrightarrow D,$$

determined by a maximal set of strongly orthogonal noncompact roots $\Psi \subset \mathfrak{h}^*$, lifts to a group homomorphism

$$\phi : Sl(2,\mathbb{R})^{\times n} \to G.$$

Our problem is to study the decomposition of W with respect to all $Sl(2,\mathbb{R})$ direct factors of ϕ.

We can in fact reduce this to the study of only one direct factor. This is because a permutation of direct factors can be induced from an inner automorphism of G, which implies the restriction to each direct factor is isomorphic to each other. Furthermore, we can assume that the highest root $\tilde{\alpha}$ appears in our chosen Ψ without loss of generality (cf. [8, Ch. 5, Proposition 1]).

Let $s_{\tilde{\alpha}}$ be the distinguished sl_2-triple in the complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ corresponding to $\tilde{\alpha}$. Let

$$W = \bigoplus_{\beta \in \Phi} W_\beta$$

be the weight decomposition of W with respect to the Cartan subalgebra $\mathfrak{h}$. Then by (14.9) [4], it is clear that all irreducible component in W with respect to $s_{\tilde{\alpha}}$ is contained in

$$W_{[\beta]} = \bigoplus_{n \in \mathbb{Z}} W_{\beta + n\tilde{\alpha}}.$$

Let $\mathrm{Conv}(\Phi)$ be the convex hull of Φ, which is a closed convex polyhedron in $\mathfrak{h}^*$. We put

$$\partial\Phi = \Phi \cap \mathrm{Conv}(\Phi).$$

Then for $\beta \in \partial\Phi$, we know by (14.10) [4] that the largest component in $W_{[\beta]}$ has dimension equal to $\beta(H_{\tilde{\alpha}}) + 1$. Our proof boils down to showing the following

Claim For all $\beta \in \partial\Phi$, we have

$$|\beta(H_{\tilde{\alpha}})| \leq 1.$$

Proof of Claim We first note that

$$|\beta(H_{\tilde{\alpha}})| = \left|\frac{2(\beta, \tilde{\alpha})}{(\tilde{\alpha}, \tilde{\alpha})}\right|$$

defines a convex function on Φ. The maximal value will be achieved for the vertices of Φ, namely, the orbit of highest weight ω of W under the Weyl group W(R). Since the Weyl group preserves the Killing form, we will show that

$$|\omega(H_{s(\tilde{\alpha})})| \leq 1, \text{ for all } s \in W(R).$$

The above inequality holds obviously for $s = id$. Let α_0 be the simple root which is the special node determined by D in the last section. By the definition of special node, in the expression of $\tilde{\alpha}$ as a linear combination of simple roots the coefficient before α_0 is one (cf. 1.2.5. [3]). Therefore,

$$\begin{aligned}\omega(H_{\tilde{\alpha}}) &= \frac{2(\omega, \tilde{\alpha})}{(\tilde{\alpha}, \tilde{\alpha})}\\ &= \frac{2(\omega, \alpha_0)}{(\tilde{\alpha}, \tilde{\alpha})}\\ &= \frac{(\alpha_0, \alpha_0)}{(\tilde{\alpha}, \tilde{\alpha})}\\ &= 1.\end{aligned}$$

Now we have to separate the exceptional cases from the ongoing proof because of the complicated description of the Weyl group in the exceptional cases. In the following, we use the same notation as the appendix of [1]. Let $\{\varepsilon_1, \ldots, \varepsilon_l\}$ be the standard basis of the Euclidean space $\mathbb{R}^l$, and σ denotes a permutation of index.

Type A_{l-1}: The highest root $\tilde{\alpha} = \varepsilon_1 - \varepsilon_l$. The Weyl group permutes the basis elements. All fundamental weights

$$\omega_i = \sum_{j=1}^{i} \varepsilon_j - \frac{i}{l+1}\sum_{j=1}^{l} \varepsilon_j, \quad 1 \leq i \leq l-1$$

correspond to a special node. Then

$$
\begin{aligned}
|\omega_i(H_{s(\tilde{\alpha})})| &= |(\omega_i, s(\tilde{\alpha}))| \\
&= \left| \left(\sum_{j=1}^{i} \varepsilon_j, \varepsilon_{\sigma(1)} - \varepsilon_{\sigma(l)} \right) \right| \\
&\leq 1.
\end{aligned}
$$

Type B_l: The highest root $\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$. The Weyl group permutes the basis elements, or acts by $\varepsilon_i \mapsto \pm\varepsilon_i$. The first fundamental weight $\omega_1 = \varepsilon_1$ corresponds to the special node. Then

$$
\begin{aligned}
|\omega_1(H_{s(\tilde{\alpha})})| &= |(\omega_1, s(\tilde{\alpha}))| \\
&= |(\varepsilon_1, \pm\varepsilon_{\sigma(1)} \pm \varepsilon_{\sigma(2)})| \\
&\leq 1.
\end{aligned}
$$

Type C_l: The highest root $\tilde{\alpha} = 2\varepsilon_1$. The Weyl group permutes the basis elements, or acts by $\varepsilon_i \mapsto \pm\varepsilon_i$. The last fundamental weight $\omega_l = \sum_{i=1}^{l} \varepsilon_i$ corresponds to the special node. Then

$$
\begin{aligned}
|\omega_l(H_{s(\tilde{\alpha})})| &= \left| \frac{1}{2}(\omega_l, s(\tilde{\alpha})) \right| \\
&= \left| \frac{1}{2} \left(\sum_{i=1}^{l} \varepsilon_i, \pm 2\varepsilon_{\sigma(1)} \right) \right| \\
&= 1.
\end{aligned}
$$

Type D_l: The highest root $\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$. The Weyl group permutes the basis elements, or acts by $\varepsilon_i \mapsto (\pm 1)_i \varepsilon_i$ with $\prod_i (\pm 1)_i = 1$. We have three special nodes in this case. It suffices to check $\omega_1 = \varepsilon_1$ and $\omega_l = \frac{1}{2}(\sum_{i=1}^{l} \varepsilon_i)$. For ω_1, we have

$$
\begin{aligned}
|\omega_1(H_{s(\tilde{\alpha})})| &= |(\omega_1, s(\tilde{\alpha}))| \\
&= \left| \left(\varepsilon_1, \pm\varepsilon_{\sigma(1)} \pm \varepsilon_{\sigma(2)}\right) \right| \\
&\leq 1.
\end{aligned}
$$

For ω_l, we have

$$
\begin{aligned}
|\omega_l(H_{s(\tilde{\alpha})})| &= |(\omega_l, s(\tilde{\alpha}))| \\
&= \left| \frac{1}{2} \left(\sum_{i=1}^{l} \varepsilon_i, \pm\varepsilon_{\sigma(1)} \pm \varepsilon_{\sigma(2)} \right) \right| \\
&\leq 1.
\end{aligned}
$$

Now we treat with the exceptional cases. In the following, we shall compute the largest value of $|\beta(H_{\tilde{\alpha}})|$ among all weights β in Φ. The results will particularly imply the claim.

Type E_6: Let $\{\alpha_1, \ldots, \alpha_6\}$ be the set of simple roots of simple Lie algebra of type E_6 and $\{\omega_1, \ldots, \omega_6\}$ be the fundamental weights. The highest root is then

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.$$

A 6-tuple $(a_1, \ldots, a_6)$ denotes the weight $\beta = \sum_{i=1}^{6} a_i \omega_i$. There are two special nodes in this case and it suffices to study either of them. The following table lists all elements of Φ for the fundamental representation ω_1:

$$\begin{array}{cccc}
(1,0,0,0,0,0) & (-1,0,1,0,0,0) & (0,0,-1,1,0,0) & (0,1,0,-1,1,0) \\
(0,1,0,0,-1,1) & (0,-1,0,0,1,0) & (0,1,0,0,0,-1) & (0,-1,0,1,-1,1) \\
(0,0,1,-1,0,1) & (0,-1,0,1,0,-1) & (1,0,-1,0,0,1) & (0,0,1,-1,1,-1) \\
(1,0,-1,0,1,-1) & (0,0,1,0,-1,0) & (-1,0,0,0,0,1) & (1,0,-1,1,-1,0) \\
(-1,0,0,0,1,-1) & (1,1,0,-1,0,0) & (-1,0,0,1,-1,0) & (1,-1,0,0,0,0) \\
(-1,1,1,-1,0,0) & (0,1,-1,0,0,0) & (-1,-1,1,0,0,0) & (0,-1,-1,1,0,0) \\
(0,0,0,-1,1,0) & (0,0,0,0,-1,1) & (0,0,0,0,0,-1). &
\end{array}$$

For an element $(a_1, \ldots, a_6)$ in the above table, we have

$$\begin{aligned}
\beta(H_{\tilde{\alpha}}) &= (\beta, \tilde{\alpha}) \\
&= \left(\sum_{i=1}^{6} a_i \omega_i, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \right) \\
&= a_1 + 2a_2 + 2a_3 + 3a_4 + 2a_5 + a_6.
\end{aligned}$$

According to this formula, it is straightforward to compute that the largest value of $|\beta(H_{\tilde{\alpha}})|$ is equal to one.

Type E_7: Let $\{\alpha_1, \ldots, \alpha_7\}$ be the set of simple roots of simple Lie algebra of type E_7 and $\{\omega_1, \ldots, \omega_7\}$ be the fundamental weights. We can choose the maximal set Ψ of the strongly orthogonal noncompact roots to be

$$\begin{aligned}
\{\psi_1 &= 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \\
\psi_2 &= \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \ \psi_3 = \alpha_7\}.
\end{aligned}$$

It is simpler to use ψ_3 to verify our statement, instead of ψ_1 which is the highest root $\tilde{\alpha}$. As in the last case, a 7-tuple $(a_1, \cdots a_7)$ denotes the weight $\beta = \sum_{i=1}^{7} a_i \omega_i$. The following table lists all elements of Φ for the fundamental representation ω_7:

(0, 0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 1, −1)	(0, 0, 0, 0, 1, −1, 0)	(0, 0, 0, 1, −1, 0, 0)
(0, 1, 1, −1, 0, 0, 0)	(1, 1, −1, 0, 0, 0, 0)	(0, −1, 1, 0, 0, 0, 0)	(1, −1, −1, 1, 0, 0, 0)
(−1, 1, 0, 0, 0, 0, 0)	(1, 0, 0, −1, 1, 0, 0)	(−1, −1, 0, 1, 0, 0, 0)	(1, 0, 0, 0, −1, 1, 0)
(−1, 0, 1, −1, 1, 0, 0)	(1, 0, 0, 0, 0, −1, 1)	(0, 0, −1, 0, 1, 0, 0)	(−1, 0, 1, 0, −1, 1, 0)
(1, 0, 0, 0, 0, 0, −1)	(0, 0, −1, 1, −1, 1, 0)	(−1, 0, 1, 0, 0, −1, 1)	(0, 1, 0, −1, 0, 1, 0)
(0, 0, −1, 1, 0, −1, 1)	(−1, 0, 1, 0, 0, 0, −1)	(0, 1, 0, −1, 1, −1, 1)	(0, 0, −1, 1, 0, 0, −1)
(0, −1, 0, 0, 0, 1, 0)	(0, 1, 0, 0, −1, 0, 1)	(0, 1, 0, −1, 1, 0, −1)	(0, −1, 0, 0, 1, −1, 1)
(0, 1, 0, 0, −1, 1, −1)	(0, −1, 0, 1, −1, 0, 1)	(0, −1, 0, 0, 1, 0, −1)	(0, 1, 0, 0, 0, −1, 0)
(0, 0, 1, −1, 0, 0, 1)	(0, −1, 0, 1, −1, 1, −1)	(1, 0, −1, 0, 0, 0, 1)	(0, 0, 1, −1, 0, 1, −1)
(0, −1, 0, 1, 0, −1, 0)	(1, 0, −1, 0, 0, 1, −1)	(0, 0, 1, −1, 1, −1, 0)	(−1, 0, 0, 0, 0, 0, 1)
(1, 0, −1, 0, 1, −1, 0)	(0, 0, 1, 0, −1, 0, 0)	(−1, 0, 0, 0, 0, 1, −1)	(1, 0, −1, 1, −1, 0, 0)
(−1, 0, 0, 0, 1, −1, 0)	(1, 1, 0, −1, 0, 0, 0)	(−1, 0, 0, 1, −1, 0, 0)	(1, −1, 0, 0, 0, 0, 0)
(−1, 1, 1, −1, 0, 0, 0)	(0, 1, −1, 0, 0, 0, 0)	(−1, −1, 1, 0, 0, 0, 0)	(0, −1, −1, 1, 0, 0, 0)
(0, 0, 0, −1, 1, 0, 0)	(0, 0, 0, 0, −1, 1, 0)	(0, 0, 0, 0, 0, −1, 1)	(0, 0, 0, 0, 0, 0, −1).

Then for an element $(a_1, \ldots, a_7)$ in the above table, we have

$$\begin{aligned}\beta(H_{\alpha_7}) &= (\beta, \alpha_7)\\ &= \left(\sum_{i=1}^{7} a_i \omega_i, \alpha_7\right)\\ &= a_7.\end{aligned}$$

It is straightforward to see the largest value of $|\beta(H_{\alpha_7})|$ is one. This completes the whole proof. □

We can now proceed to prove our main result.

Proof of Theorem 3.3 It suffices to prove the isomorphism over D, and we obtain the claimed isomorphism by taking quotient under the left action of Γ. Since the constructions on both sides are G-equivariant, it is enough to show the isomorphism at the origin of D. Over the origin 0, we have the adjoint action of $K^{\mathbb{C}}$ on the holomorphic tangent space $(T_D)_0$ and the dual action on $(\Omega_D)_0$. Since the Higgs field of a locally homogeneous VHS is G-equivariant, for each k, $(J_k)_0 \subset (S^k\Omega_D)_0$ is $K^{\mathbb{C}}$-invariant. This implies C_k is $K^{\mathbb{C}}$-invariant. So we can obtain a decomposition of $(\mathbb{P}(T_D))_0$ into disjoint union of $K^{\mathbb{C}}$ orbits as follows:

$$(\mathbb{P}(T_D))_0 = \coprod_{1\le k\le n} \{(C_k)_0 - (C_{k-1})_0\}.$$

For $1 \le k \le n$, we put

$$v_k = e_1 + \cdots + e_k.$$

It is clear that

$$v_k \in (\mathcal{S}_k)_0 - (\mathcal{S}_{k-1})_0$$

and $K^{\mathbb{C}}(v_k) = (\mathcal{S}_k)_0 - (\mathcal{S}_{k-1})_0$. Next we make the following

Claim For $1 \leq k \leq n$,

$$v_k \in (C_k)_0 - (C_{k-1})_0.$$

This claim implies the inclusion of $K^{\mathbb{C}}$ orbit for each k

$$(\mathcal{S}_k)_0 - (\mathcal{S}_{k-1})_0 \subset (C_k)_0 - (C_{k-1})_0,$$

and hence the equality for each k. Therefore, for $1 \leq k \leq n-1$,

$$(C_k)_0 = (\mathcal{S}_k)_0.$$

Proof of Claim Let $\gamma_k : \triangle \to D$ be the composition map

$$\triangle \xrightarrow{\text{diag.}} \triangle_1 \times \cdots \times \triangle_k \hookrightarrow \triangle^n \overset{i}{\hookrightarrow} D.$$

Obviously, for a suitable basis element u of $(T_\triangle)_0$ one has $(d\gamma_k)_0(u) = v_k$. By Proposition 3.5, we have the decomposition of PVHS

$$\gamma_k^* \mathbb{W} = \mathbb{L}^{\otimes k} \otimes \mathbb{U} \oplus \mathbb{V}'$$

where $\mathbb{U}$ is a unitary factor and $\mathbb{V}'$ is a PVHS with width $\leq k-1$. Let (E, θ) be the system of Hodge bundles corresponding to $\mathbb{W}$, and for $v \in (T_D)_0$, we denote by

$$\theta_v : E_0 \to E_0$$

the action of the Higgs field θ on the bundle E at the origin 0 along the tangent direction v. Since

$$\theta_{v_k}(E) = \theta_u(\gamma_k^* E),$$

we see that

$$(\theta_{v_k})^k \neq 0, \quad (\theta_{v_k})^{k+1} = 0.$$

Together with Lemma 3.2, one easily sees that the claim holds. □

4 Enumeration of canonical PVHS over irreducible bounded symmetric domain and the generating property of gross

Let (E, θ) be a system of Hodge bundles over X. We use the same notation as that in previous sections. We note that

$$I = \bigoplus_{k \geq 1} I_k$$

forms a graded ideal of the symmetric algebra

$$Sym(T_X) = \bigoplus_{k \geq 0} S^k T_X.$$

It is trivial to see that

$$I_k = S^k(T_X), \quad \text{for } k \geq n+1.$$

In [6, §5], Gross suspected if I is generated by I_2 for the canonical PVHS over an irreducible tube domain. We can assert this generating property for the canonical PVHS over an irreducible bounded symmetric domain in general.

Theorem 4.1 *We use the same notation as Theorem 2.2. Then the graded ideal I, formed by the kernel of iterated Higgs field, is generated by the degree 2 graded piece I_2. That is, the multiplication map*

$$I_2 \otimes S^{k-2}(T_X) \to I_k$$

is surjective for all $k \geq 2$.

It suffices to prove the surjectivity for $k \leq n+1$ where $n = \operatorname{rank}(D)$. In fact, for $k \geq n+2$, we have

$$\begin{aligned} I_2 \otimes S^{n-1}(T_X) \otimes (T_X)^{\otimes k-n-1} &\twoheadrightarrow I_{n+1} \otimes (T_X)^{\otimes k-n-1} \\ &= S^{n+1}(T_X) \otimes (T_X)^{\otimes k-n-1} \\ &\twoheadrightarrow S^k(T_X) = I_k. \end{aligned}$$

By the integrality of Higgs field, the above surjective map factors through $I_2 \otimes S^{k-2}(T_X)$. As the proof of Theorem 3.3, we can work on the level of bounded symmetric domain and prove the statement at the origin as K-representations. The theorem will be proved case by case. In the classical case, we shall also describe the system of Hodge bundles (E, θ) associated with the canonical PVHS $\mathbb{W}$ using the Grassmannian description of classical symmetric domain.

Let D be an irreducible bounded symmetric domain. By fixing an origin of D, we obtain an equivalence of categories of homogeneous vector bundles and finite dimensional complex representations of K. Since K has one dimensional center, a finite dimensional complex K-representation is written as $\mathbb{C}(l) \otimes V$ where V is a representation of the semisimple part K' of K and is determined by the induced action of the complexified Lie algebra $\mathfrak{k}'^{\mathbb{C}}$. In the following, the same notation for a K-representation and the corresponding homogeneous vector bundle will be used when the context causes no confusion. All the isomorphisms are isomorphisms between homogeneous bundles. A highest weight representation of $sl(n, \mathbb{C})$ will be denoted interchangeably by $\Gamma_{a_1,\ldots,a_{n-1}}$ and $\mathbb{S}_\lambda(\mathbb{C}^n)$ (cf. [4, Sect. 15.3]).

4.1 Type A

The irreducible bounded symmetric domain of type A is $D^I_{p,q} = G/K$ where

$$G = SU(p,q), \quad K = S(U(p) \times U(q)).$$

Let $V = \mathbb{C}^{p+q}$ be a complex vector space equipped with a Hermitian symmetric bilinear form h of signature (p,q). Then $D^I_{p,q}$ parameterizes the dimension p complex vector subspaces $U \subset V$ such that

$$h|_U : U \times U \to \mathbb{C}$$

is positive definite. This forms the tautological subbundle $S \subset V \times D$ of rank p and denote by Q the tautological quotient bundle of rank q. We have the natural isomorphism of holomorphic vector bundles

$$T_{D^I_{p,q}} \simeq \mathrm{Hom}(S, Q). \tag{4.1.1}$$

The standard representation V of G gives rise to a weight 1 PVHS $\mathbb{V}$ over $D^I_{p,q}$, and its associated Higgs bundle

$$F = F^{1,0} \oplus F^{0,1}, \quad \eta = \eta^{1,0} \oplus \eta^{0,1}$$

is determined by

$$F^{1,0} = S, \quad F^{0,1} = Q, \quad \eta^{0,1} = 0,$$

and $\eta^{1,0}$ is defined by the above isomorphism. The canonical PVHS is

$$\mathbb{W} = \bigwedge^p \mathbb{V}$$

and its associated system of Hodge bundles (E, θ) is then

$$(E, \theta) = \bigwedge^p (F, \eta).$$

Since

$$\mathfrak{k}'^{\mathbb{C}} = sl(p, \mathbb{C}) \oplus sl(q, \mathbb{C}),$$

by Schur's lemma, a finite dimensional irreducible complex representation of $\mathfrak{k}'^{\mathbb{C}}$ is of the form

$$\Gamma_{a_1,\ldots,a_{p-1}} \otimes \Gamma'_{b_1,\ldots,b_{q-1}}.$$

We put $V_1 = \mathbb{C}^p$ to be the representation space $\Gamma_{0,\dots,0,1}$ of $sl(p,\mathbb{C})$ and $V_2 = \mathbb{C}^q$ the representation space $\Gamma'_{0,\dots,0,1}$ of $sl(q,\mathbb{C})$. In the remaining subsection, we shall assume $p \le q$ in order to simplify the notations in the argument.

Lemma 4.2 *We have isomorphism*

$$T_{D^I_{p,q}} \simeq V_1 \otimes V_2.$$

Then, for $k \ge 2$, we have isomorphism

$$S^k(T_{D^I_{p,q}}) \simeq \bigoplus_{\lambda} \mathbb{S}_\lambda(V_1) \otimes \mathbb{S}_\lambda(V_2),$$

where λ runs through all partitions of k with at most p rows. Under this isomorphism, the kth iterated Higgs field for $k \le p$,

$$\theta^k : S^k(T_{D^I_{p,q}}) \to \mathrm{Hom}(E^{p,0}, E^{p-k,k})$$

is identified with the projection map onto the irreducible component

$$\bigoplus_{\lambda} \mathbb{S}_\lambda(V_1) \otimes \mathbb{S}_\lambda(V_2) \twoheadrightarrow \mathbb{S}_{\lambda^0}(V_1) \otimes \mathbb{S}_{\lambda^0}(V_2),$$

where $\lambda^0 = (1, \dots, 1)$.

Proof By the isomorphism 4.1.1, we have isomorphism

$$T_{D^I_{p,q}} \simeq V_1 \otimes V_2.$$

The formula in Ex.6.11 [4] gives the decomposition of $S^k(V_1 \otimes V_2)$ with respect to $sl(p,\mathbb{C}) \oplus sl(q,\mathbb{C})$:

$$S^k(V_1 \otimes V_2) = \bigoplus_{\lambda} \mathbb{S}_\lambda(V_1) \otimes \mathbb{S}_\lambda(V_2),$$

where λ runs through all partitions of k with at most p rows. Since the center of K acts on $(T_{D^I_{p,q}})_0$ trivially, it acts on $(S^k(T_{D^I_{p,q}}))_0$ trivially too. Hence the second isomorphism of the statement follows. For the last statement, it suffices to show θ^k is a non-zero map because $\mathrm{Hom}(E^{p,0}, E^{p-k,k})$ is irreducible. But this follows directly from the definition of the Higgs field θ as pth wedge power of η. The lemma is proved. □

From the lemma, we know that

$$\theta^2 \simeq pr : \Gamma_{0,\dots,2} \otimes \Gamma'_{0,\dots,2} \oplus \Gamma_{0,\dots,0,1,0} \otimes \Gamma'_{0,\dots,0,1,0} \to \Gamma_{0,\dots,0,1,0} \otimes \Gamma'_{0,\dots,0,1,0}.$$

So by definition,

$$I_2 \simeq \Gamma_{0,\ldots,0,2} \otimes \Gamma'_{0,\ldots,0,2}.$$

Proof of Theorem 4.1 for Type A Now we proceed to prove that $I_2 \otimes S^{k-2}(T_{D^I_{p,q}})$ generates I_k. By the above lemma and the Formula 6.8 [4], we have

$$\begin{aligned}
I_2 \otimes S^{k-2}(T_{D^I_{p,q}}) &\simeq S^2(V_1) \otimes S^2(V_2) \otimes S^{k-2}(V_1 \otimes V_2)\\
&\simeq \bigoplus_{\mu} (S^2(V_1) \otimes \mathbb{S}_{\mu}(V_1)) \otimes (S^2(V_2) \otimes \mathbb{S}_{\mu}(V_2))\\
&= \bigoplus_{\mu} \left[\left(\bigoplus_{\nu^1_\mu} \mathbb{S}_{\nu^1_\mu}(V_1) \right) \otimes \left(\bigoplus_{\nu^2_\mu} \mathbb{S}_{\nu^2_\mu}(V_2) \right) \right]\\
&= \bigoplus_{\mu} \bigoplus_{\nu^1_\mu, \nu^2_\mu} (\mathbb{S}_{\nu^1_\mu}(V_1) \otimes \mathbb{S}_{\nu^2_\mu}(V_2)),
\end{aligned}$$

where μ runs through all partitions of $k-2$ with at most p rows, and for a fixed μ, $\nu^i_\mu, i = 1, 2$ runs through those Young diagrams by adding two boxes to different columns of the Young diagram of μ. Let λ be a Young diagram corresponding to a direct factor of I_k under the isomorphism in the above lemma. Since

$$\bigoplus_{\mu} \bigoplus_{\nu_\mu} (\mathbb{S}_{\nu_\mu}(V_1) \otimes \mathbb{S}_{\nu_\mu}(V_2)) \subset \bigoplus_{\mu} \bigoplus_{\nu^1_\mu, \nu^2_\mu} (\mathbb{S}_{\nu^1_\mu}(V_1) \otimes \mathbb{S}_{\nu^2_\mu}(V_2)),$$

it is enough to show that λ can be obtained by a Young diagram μ by adding two boxes to different columns of μ. Actually, by the above lemma the partition λ of I_k has the property that, either for some $1 \leq i_0 \leq p-1$,

$$\lambda_{i_0} > \lambda_{i_0+1} \geq 1,$$

or for some $1 \leq i_0 \leq p$,

$$2 \leq \lambda_1 = \cdots = \lambda_{i_0} > \lambda_{i_0+1} \geq 0.$$

In the first case, we can choose μ as

$$\mu_i = \begin{cases} \lambda_i - 1 & \text{if } i = i_0, i_0 + 1, \\ \lambda_i & \text{otherwise.} \end{cases}$$

In the second case, we choose μ as

$$\mu_i = \begin{cases} \lambda_i - 2 & \text{if } i = i_0, \\ \lambda_i & \text{otherwise.} \end{cases}$$

The proof of Theorem 4.1 in the type A case is therefore completed. □

4.2 Type B, Type $D^{\mathbb{R}}$

For $n \geq 3$, we let

$$G = Spin(2, n), \quad K = Spin(2) \times_{\mu_2} Spin(n).$$

Then $D_n^{IV} = G/K$ is the bounded symmetric domain of type B when n is odd, of type $D^{\mathbb{R}}$ when n even. Let $(V_{\mathbb{R}}, Q)$ be a real vector space of dimension $n+2$ equipped with a symmetric bilinear form of signature $(2, n)$. Then D_n^{IV} is one of connected components parameterizing all Q-positive two dimensional subspace of $V_{\mathbb{R}}$. In order to see clearer the complex structure of D_n^{IV}, we complexify $(V_{\mathbb{R}}, Q)$, to obtain $(V = V_{\mathbb{R}} \otimes \mathbb{C}, Q)$. Then it is known that D_n^{IV} is an open submanifold of the quadratic hypersurface defined by $Q = 0$ in $\mathbb{P}(V) \simeq \mathbb{P}^{n+1}$, which is just the compact dual of D_n^{IV}. For a Q-isotropic line $L \subset V$, we define its polarization hyperplane to be

$$P(L) = \{v \in V | Q(L, v) = 0\}.$$

So for each point of D_n^{IV}, we obtain a natural filtration of V by

$$L \subset P(L) \subset V.$$

Varying the points on D_n^{IV}, the above filtration yields a filtration of homogeneous bundles

$$S \subset P(S) \subset V \times D_n^{IV}.$$

On the other hand, we have a commutative diagram

$$\begin{array}{ccc} T_{D_n^{IV}} & \xrightarrow{\simeq} & \mathrm{Hom}\left(L, \frac{P(L)}{L}\right) \\ \cap\downarrow & & \downarrow\cap \\ T_{\mathbb{P}(V),[L]} & \xrightarrow{\simeq} & \mathrm{Hom}\left(L, \frac{V}{L}\right), \end{array}$$

whose top horizontal line gives the isomorphism of tangent bundle

$$T_{D_n^{IV}} \simeq \mathrm{Hom}\left(S, \frac{P(S)}{S}\right). \tag{4.2.1}$$

We also notice that Q descends to a non-degenerate bilinear form on $\frac{P(L)}{L}$, so that we have a natural isomorphism

$$\left(\frac{P(S)}{S}\right)^* \simeq \frac{P(S)}{S}. \tag{4.2.2}$$

Now we put

$$E^{2,0} = S, \quad E^{1,1} = \frac{P(S)}{S}, \quad E^{0,2} = \frac{V \times D_n^{IV}}{P(S)},$$

and

$$\theta^{2,0} : E^{2,0} \to E^{1,1} \otimes \Omega_{D_n^{IV}},$$
$$\theta^{1,1} : E^{1,1} \to E^{0,2} \otimes \Omega_{D_n^{IV}}$$

are determined by the isomorphisms 4.2.1 and 4.2.2, and $\theta^{0,2} = 0$. The Higgs bundle

$$\left(E = \bigoplus_{p+q=2} E^{p,q}, \theta = \bigoplus_{p+q=2} \theta^{p,q}\right)$$

is the associated system of Hodge bundles with the canonical PVHS $\mathbb{W}$.

Let $m = [\frac{n}{2}]$ be the rank of $\mathfrak{so}(\mathfrak{n})$, and $\Gamma_{a_1,\ldots,a_m}$ denotes a highest weight representation of $\mathfrak{so}(\mathfrak{n})$. In terms of this notation, we have

$$E^{2,0} \simeq \mathbb{C}(-2) \otimes \Gamma_{0,\ldots,0}, \quad E^{1,1} \simeq \mathbb{C} \otimes \Gamma_{1,0,\ldots,0}, \quad E^{0,2} \simeq \mathbb{C}(2) \otimes \Gamma_{0,\ldots,0}.$$

The following easy lemma makes Theorem 4.1 in the cases of type B and type $D^{\mathbb{R}}$ clear.

Lemma 4.3 *We have isomorphisms*

$$\begin{aligned}
T_{D_n^{IV}} &\simeq \mathbb{C}(2) \otimes \Gamma_{1,0,\ldots,0},\\
S^2(T_{D_n^{IV}}) &\simeq \mathbb{C}(4) \otimes \Gamma_{2,0,\ldots,0} \oplus \mathbb{C}(4) \otimes \Gamma_{0,\ldots,0},\\
I_2 &\simeq \mathbb{C}(4) \otimes \Gamma_{2,0,\ldots,0},\\
I_2 \otimes T_{D_n^{IV}} &\simeq \mathbb{C}(6) \otimes \Gamma_{3,0,\ldots,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,0,\ldots,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,1,0,\ldots,0},\\
S^3(T_{D_n^{IV}}) &\simeq \mathbb{C}(6) \otimes \Gamma_{3,0,\ldots,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,0,\ldots,0}.
\end{aligned}$$

4.3 Type C

We fix $n \geq 2$. Let

$$G = Sp(2n, \mathbb{R}), \quad K = U(n).$$

Then $D_n^{III} = G/K$ is the bounded symmetric domain of type C. D_n^{III} is known as the Siegel space of degree n. Let $(V_{\mathbb{R}}, \omega)$ be a real vector space of dimension $2n$ equipped with a skew symmetric bilinear form ω. As before, we denote also by (V, ω) the complexification, and $h(u, v) = i\omega(u, \bar{v})$ defines a hermitian symmetric bilinear form over V. Then D_n^{III} parameterizes the maximal ω-isotropic and h-positive complex subspaces of V. The standard representation V of G gives a weight 1 $\mathbb{R}$-PVHS $\mathbb{V}$ over D_n^{III}. Let (F, η) be the associated Higgs bundle with $\mathbb{V}$. Then $F^{1,0}$ is simply the tautological subbundle over D_n^{III} and $F^{0,1}$ is the h-orthogonal complement of $F^{1,0}$. Clearly, we have a natural embedding of bounded symmetric domains

$$\iota : D_n^{III} \hookrightarrow D_{n,n}^{I}.$$

It induces a commutative diagram:

$$\begin{array}{ccc} T_{D_n^{III}} & \xrightarrow{\simeq} & S^2(F^{0,1}) \\ \cap\downarrow & & \downarrow\cap \\ \iota^*(T_{D_{n,n}^{I}}) & \xrightarrow{\simeq} & (F^{0,1})^{\otimes 2}. \end{array}$$

The Higgs field $\eta^{1,0}$ is defined by the composition of maps

$$T_{D_n^{III}} \simeq S^2(F^{0,1}) \hookrightarrow (F^{0,1})^{\otimes 2} \simeq \mathrm{Hom}(F^{1,0}, F^{0,1}). \tag{4.3.1}$$

The canonical PVHS $\mathbb{W}$ is the unique weight n sub-PVHS of $\wedge^n(\mathbb{V})$. In fact, we have a decomposition of $\mathbb{R}$-PVHS

$$\bigwedge^n(\mathbb{V}) = \mathbb{W} \oplus \mathbb{V}',$$

where $\mathbb{V}'$ is a weight $n-2$ $\mathbb{R}$-PVHS. Therefore the corresponding Higgs bundle (E, θ) to $\mathbb{W}$ is a sub-Higgs bundle of $\wedge^n(F, \eta)$.

Let $V_1 = (F^{0,1})_0$ be the standard representation of K. It is straightforward to obtain the following

Lemma 4.4 *We have isomorphism*

$$T_{D_n^{III}} \simeq \mathbb{S}_{(2)}(V_1).$$

Then, for $k \geq 2$, we have isomorphism

$$S^k(T_{D_n^{III}}) \simeq \bigoplus_{\lambda} \mathbb{S}_{\lambda}(V_1),$$

where $\lambda = \{\lambda_1, \ldots, \lambda_l\}$ runs through all partitions of $2k$ with each λ_i even and $l \leq n$. Under this isomorphism, for $k \leq n$, the kth iterated Higgs field θ^k is identified with the projection map onto the irreducible component $\mathbb{S}_{\lambda^0}(V_1)$ where $\lambda^0 = (2, \ldots, 2)$.

Proof of Theorem 4.1 for Type C By the last lemma, we know that

$$\theta^2 \simeq pr : \mathbb{S}_{(4)}(V_1) \oplus \mathbb{S}_{(2,2)}(V_1) \to \mathbb{S}_{(2,2)}(V_1)$$

and then $I_2 \simeq \mathbb{S}_{(4)}(V_1)$. Applying the Formula 6.8, [4] to decompose $I_2 \otimes S^{k-2}(T_{D_n^{III}})$, we obtain

$$\begin{aligned} I_2 \otimes S^{k-2}(T_{D_n^{III}}) &\simeq \mathbb{S}_{(4)}(V_1) \otimes \bigoplus_{\mu} \mathbb{S}_{\mu}(V_1) \\ &\simeq \bigoplus_{\mu} (\mathbb{S}_{(4)}(V_1) \otimes \mathbb{S}_{\mu}(V_1)) \\ &= \bigoplus_{\mu} \left[\left(\bigoplus_{\nu_\mu} \mathbb{S}_{\nu}(V_1) \right) \right], \end{aligned}$$

where μ runs through all partitions of $2(k-2)$ with the property as that in Lemma 4.4, and for a fixed μ, ν_μ runs through those Young diagrams by adding four boxes to different columns of the Young diagram μ. The partition λ of an irreducible component in I_k is of the form

$$\lambda_1 \geq \cdots \geq \lambda_s > \lambda_{s+1} = \cdots = \lambda_l \geq 2.$$

We may then take μ to be

$$\mu_i = \begin{cases} \lambda_i - 2 & \text{if } i = s, l, \\ \lambda_i & \text{otherwise.} \end{cases}$$

Then we define ν_μ by adding two boxes to λ_s and λ_l in μ respectively to obtain the starting λ. Therefore Theorem 4.1 in the type C case is proved. □

4.4 Type $D^{\mathbb{H}}$

For $n \geq 3$, we let

$$G = SO^*(2n), \quad K = U(n).$$

Then $D_n^{II} = G/K$ is the bounded symmetric domain of type $D^{\mathbb{H}}$. We recall that

$$G \simeq \{M \in Sl(2n, \mathbb{C}) | M I_{n,n} M^* = I_{n,n},\ M S_n M^\tau = S_n\},$$

where $I_{n,n}$ denotes the matrix $\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$ and S_n denotes the matrix $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Let (V, h, S) be a complex vector space of dimension $2n$ equipped with a hermitian symmetric form h and symmetric bilinear form S, where, under the identification $V \simeq \mathbb{C}^{2n}$, h is defined by the matrix $I_{n,n}$ and S is defined by the matrix S_n. Then D_n^{II}

parameterizes all n-dimensional S-isotropic and h-positive complex subspaces of V. The standard representation V of G determines a weight 1 PVHS $\mathbb{V}$. Its associated Higgs bundle (F, η) is determined in a similar manner as type C case. Namely, $F^{1,0}$ is simply the tautological subbundle and $F^{0,1}$ is its h-orthogonal complement. The natural embedding

$$\iota' : D_n^{II} \hookrightarrow D_{n,n}^{I}$$

induces a commutative diagram:

$$\begin{array}{ccc} T_{D_n^{II}} & \xrightarrow{\simeq} & \bigwedge^2(F^{0,1}) \\ \cap\big\downarrow & & \big\downarrow\cap \\ \iota'^*(T_{D_{n,n}^{I}}) & \xrightarrow{\simeq} & (F^{0,1})^{\otimes 2}, \end{array}$$

and the Higgs field $\eta^{1,0}$ is induced by the composition of maps

$$T_{D_n^{II}} \simeq \bigwedge^2(F^{0,1}) \hookrightarrow (F^{0,1})^{\otimes 2} \simeq \mathrm{Hom}(F^{1,0}, F^{0,1}). \tag{4.4.1}$$

The canonical PVHS $\mathbb{W}$ comes from a half spin representation. We write the corresponding Higgs bundle as

$$\left(E = \bigoplus_{p+q=[\frac{n}{2}]} E^{p,q}, \theta = \bigoplus_{p+q=[\frac{n}{2}]} \theta^{p,q}\right).$$

Then the Hodge bundle is

$$E^{p,q} = \bigwedge^{n-2q} F^{1,0},$$

and the Higgs field $\theta^{p,q}$ is induced by the natural wedge product map

$$\bigwedge^2 F^{0,1} \otimes \bigwedge^{2q} F^{0,1} \to \bigwedge^{2q+2} F^{0,1}.$$

While type $D^{\mathbb{H}}$ case enjoys many similarity with type C case, there is one difference we would like to point out. That is, the canonical PVHS $\mathbb{W}$ is not a sub-PVHS of $\wedge^n\mathbb{V}$. In fact, the PVHS $\wedge^n\mathbb{V}$ is the direct sum of two irreducible PVHSs. One of them, say $\mathbb{V}'$, has

$$\bigwedge^n(F^{1,0}) \otimes \bigwedge^0(F^{0,1}) \simeq \left(\bigwedge^n(F^{1,0})\right)^{\otimes 2}$$

as the first Hodge bundle. For this irreducible $\mathbb{V}'$, we have an inclusion of PVHS

$$\mathbb{V}' \subset Sym^2(\mathbb{W}).$$

Let $V_1 = (F^{0,1})_0$ be the dual of the standard representation of K. It is straightforward to obtain the following

Lemma 4.5 *We have isomorphism*

$$T_{D_n^{II}} \simeq \mathbb{S}_{(1,1)}(V_1).$$

Then, for $k \geq 2$, we have isomorphism

$$S^k(T_{D_n^{II}}) \simeq \bigoplus_{\lambda} \mathbb{S}_\lambda(V_1),$$

where $\lambda = \{\lambda_1, \ldots, \lambda_l\}$ runs through all partitions of $2k$ with $l \leq n$ and each entry of the conjugate λ' of λ even. Under this isomorphism, for $k \leq [\frac{n}{2}]$, the kth iterated Higgs field θ^k is identified with the projection map onto the irreducible component $\mathbb{S}_{\lambda^0}(V_1)$ with $\lambda^0 = (k, k)$.

By the above lemma, we know that

$$\theta^2 \simeq pr : \mathbb{S}_{(1,1,1,1)}(V_1) \oplus \mathbb{S}_{(2,2)}(V_1) \to \mathbb{S}_{(2,2)}(V_1).$$

Thus we have isomorphism $I_2 \simeq \mathbb{S}_{(1,1,1,1)}(V_1)$. By Formula 6.9, [4], we have

$$\begin{aligned} I_2 \otimes S^{k-2}(T_{D_n^{II}}) &\simeq \mathbb{S}_{(1,1,1,1)}(V_1) \otimes \bigoplus_{\mu} \mathbb{S}_\mu(V_1) \\ &\simeq \bigoplus_{\mu} (\mathbb{S}_{(1,1,1,1)}(V_1) \otimes \mathbb{S}_\mu(V_1)) \\ &= \bigoplus_{\mu} \left[\left(\bigoplus_{\nu_\mu} \mathbb{S}_\nu(V_1) \right) \right], \end{aligned}$$

where μ runs through all partitions of $2(k-2)$ with the property as that in Lemma 4.5, and for a fixed μ, ν_μ runs through those Young diagrams by adding four boxes to different rows of the Young diagram of μ. We observe that we are in the conjugate case of that of type C. Theorem 4.1 in type $D^{\mathbb{H}}$ case follows easily.

4.5 Type E

There are two exceptional irreducible bounded symmetric domains. We first discuss the E_6 case. In this case,

$$G = E_{6,2}, \quad K = U(1) \times_{\mu_4} Spin(10).$$

Then $D^V = G/K$ is a 16-dimensional bounded symmetric domain of rank 2. There are two special nodes in the Dynkin diagram of E_6. But they induce isomorphic bounded symmetric domains. We take the first node so that the fundamental representation corresponding to this special node is W_{27}. Let $(E = \oplus_{p+q=2} E^{p,q}, \theta)$ be the corresponding Higgs bundle to $\mathbb{W}$. Then we have isomorphism

$$E^{2,0} \simeq \mathbb{C}(-2), \quad E^{1,1} \simeq \mathbb{C} \otimes \Gamma_{0,0,0,1,0}, \quad E^{0,2} \simeq \mathbb{C}(2) \otimes \Gamma_{1,0,0,0,0}.$$

Furthermore, it is straightforward to obtain the following

Lemma 4.6 *We have following isomorphisms*:

$$\begin{aligned}
T_X &\simeq \mathbb{C}(2) \otimes \Gamma_{0,0,0,1,0}, \\
S^2(T_X) &\simeq \mathbb{C}(4) \otimes \Gamma_{0,0,0,2,0} \oplus \mathbb{C}(4) \otimes \Gamma_{1,0,0,0,0}, \\
I_2 &\simeq \mathbb{C}(4) \otimes \Gamma_{0,0,0,2,0}, \\
I_2 \otimes T_X &\simeq \mathbb{C}(6) \otimes \Gamma_{0,0,0,3,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,0,0,1,0} \oplus \mathbb{C}(6) \otimes \Gamma_{0,0,1,1,0}, \\
S^3(T_X) &\simeq \mathbb{C}(6) \otimes \Gamma_{0,0,0,3,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,0,0,1,0}.
\end{aligned}$$

We continue to discuss the remaining case, which has already appeared in [6]. Let

$$G = E_{7,3}, \quad K = U(1) \times_{\mu_3} E_6.$$

Then $D^{VI} = G/K$ is of dimension 27 and rank 3. We refer the reader to §4 [6] for the description of Hodge bundles. The lemma corresponding to Lemma 4.6 is the following

Lemma 4.7 *We have the following isomorphisms*:

$$\begin{aligned}
T_X &\simeq \mathbb{C}(2) \otimes \Gamma_{1,0,0,0,0,0}, \\
S^2(T_X) &\simeq \mathbb{C}(4) \otimes \Gamma_{2,0,0,0,0,0} \oplus \mathbb{C}(4) \otimes \Gamma_{0,0,0,0,0,1}, \\
I_2 &\simeq \mathbb{C}(4) \otimes \Gamma_{2,0,0,0,0,0}, \\
I_2 \otimes T_X &\simeq \mathbb{C}(6) \otimes \Gamma_{1,0,0,0,0,1} \oplus \mathbb{C}(6) \otimes \Gamma_{3,0,0,0,0,0} \oplus \mathbb{C}(6) \otimes \Gamma_{1,0,1,0,0,0}, \\
S^3(T_X) &\simeq \mathbb{C}(6) \otimes \Gamma_{1,0,0,0,0,1} \oplus \mathbb{C}(6) \otimes \Gamma_{3,0,0,0,0,0} \oplus \mathbb{C}(6) \otimes \Gamma_{0,0,0,0,0,0}, \\
I_3 &\simeq \mathbb{C}(6) \otimes \Gamma_{1,0,0,0,0,1} \oplus \mathbb{C}(6) \otimes \Gamma_{3,0,0,0,0,0}, \\
I_2 \otimes S^2(T_X) &\simeq \mathbb{C}(8) \otimes \Gamma_{4,0,0,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{2,0,0,0,0,1} \oplus \mathbb{C}(8) \otimes \Gamma_{0,0,0,0,0,2} \\
&\quad \oplus \mathbb{C}(8) \otimes \Gamma_{2,0,1,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{0,0,2,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{0,0,1,0,0,1} \\
&\quad \oplus \mathbb{C}(8) \otimes \Gamma_{1,0,0,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{1,1,0,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{2,0,0,0,0,1}, \\
S^4(T_X) &\simeq \mathbb{C}(8) \otimes \Gamma_{4,0,0,0,0,0} \oplus \mathbb{C}(8) \otimes \Gamma_{2,0,0,0,0,1} \oplus \mathbb{C}(8) \otimes \Gamma_{0,0,0,0,0,2} \\
&\quad \oplus \mathbb{C}(8) \otimes \Gamma_{1,0,0,0,0,0}.
\end{aligned}$$

Lemmas 4.6 and 4.7 make it clear that the generating property of Gross also holds for the exceptional cases. Then the proof of Theorem 4.1 is completed.

Acknowledgments The authors would like to thank Ngaiming Mok for his explanation of the notion of characteristic bundles, and thank Eckart Viehweg for his interests and helpful discussions on this work.

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