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The monodromy groups of Dolgachev's CY moduli spaces are Zariski dense ☆

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A B S T R A C T

Let $\mathcal{M}_{n,2n+2}$ be the coarse moduli space of CY manifolds arising from a crepant resolution of double covers of $\mathbb{P}^n$ branched along $2n+2$ hyperplanes in general position. We show that the monodromy group of a good family for $\mathcal{M}_{n,2n+2}$ is Zariski dense in the corresponding symplectic or orthogonal group if $n \geq 3$. In particular, the period map does not give a uniformization of any partial compactification of the coarse moduli space as a Shimura variety whenever $n \geq 3$. This disproves a conjecture of Dolgachev. As a consequence, the fundamental group of the coarse moduli space of m ordered points in $\mathbb{P}^n$ is shown to be large once it is not a point. Similar Zariski-density result is obtained for moduli spaces of CY manifolds arising from cyclic covers of $\mathbb{P}^n$ branched along m hyperplanes in general position. A classification towards the geometric realization problem of B. Gross for type A bounded symmetric domains is given.

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1. Introduction

Among all moduli spaces of algebraic varieties, the moduli spaces of hyperplane arrangements in a projective space make a classical object of study (see [13] for a nice account from the point of view of GIT). The simplest nontrivial example of this class is the moduli space of four points in $\mathbb{P}^1$, that is well-known to be identified with the moduli space of elliptic curves with level two structure. An elliptic curve is obtained by taking the double cover of $\mathbb{P}^1$ branched at four distinct points and, via this construction, the variation of the four points in $\mathbb{P}^1$ is reflected into the variation of the Hodge structures attached to elliptic curves. The notion of variation of Hodge structure (or equivalently period map from the analytic point of view) as introduced by P. Griffiths in general has proven to be quite effective in several important geometric questions on moduli spaces of algebraic varieties. It is well known that the period map associating the normalized period of the corresponding elliptic curve to the cross ratio of a four pointed set in $\mathbb{P}^1$ is a modular form. This fascinating idea is very successful in showing some (partial compactifications of) moduli spaces are Shimura varieties (see e.g. [2,11,12,22,23]). The current paper concerns the following

Conjecture 1.1. *(See I. Dolgachev [3].) The period space of a family of CY n-folds which is obtained by a resolution of double covers of $\mathbb{P}^n$ branched along $2n+2$ hyperplanes in general position is the complement of a $\Gamma = GL(2n, \mathbb{Z}[i])$-automorphic form on the type A tube domain $D^I_{n,n}$.*

This conjecture is naturally connected with the geometric realization problem posed by B. Gross in Section 8 of [20] on the canonical polarized variation of Hodge structure (abbreviated as PVHS) over a tube domain. Let $\mathfrak{M}_{AR}$ be the coarse moduli space of ordered $2n+2$ hyperplane arrangements in $\mathbb{P}^n$ in general position and $\tilde{\mathcal{X}}_{AR} \xrightarrow{\tilde{f}} \mathfrak{M}_{AR}$ be the family of CY n-folds which is obtained by a resolution of double covers of $\mathbb{P}^n$ branched along $2n+2$ hyperplanes in general position. This family gives a weight n $\mathbb{Q}$-PVHS $\tilde{\mathbb{V}}_{AR} = (R^n\tilde{f}_*\mathbb{Q})_{pr}$ over $\mathfrak{M}_{AR}$. Let $\mathbb{V}_{can}$ be the canonical $\mathbb{C}$-PVHS over the type A tube domain $D^I_{n,n}$. Then we disprove Conjecture 1.1 in $n \geq 3$ cases by showing the following

Theorem 1.2. *If $n \geq 3$, $\tilde{\mathbb{V}}_{AR}$ does not factor through $\mathbb{V}_{can}$.*

By this we mean that there does not exist a nonempty analytically open subset $U \subset \mathfrak{M}_{AR}$ and a holomorphic map $j: U \to D^I_{n,n}$, such that $\tilde{\mathbb{V}}_{AR} \otimes \mathbb{C}|_U \simeq j^*\mathbb{V}_{can}$ as $\mathbb{C}$-PVHS.

The $n = 1$ case is the classical modular family of elliptic curves and the $n = 2$ case is treated in [24], where it was shown among other things that the weight two PVHS, modulo the constant part, factors through $\mathbb{V}_{can}$ canonically. The $n = 3$ case is the turning point of Conjecture 1.1 which was shown to be false in [17]. The idea is to compare the characteristic subvarieties associated to $\tilde{\mathbb{V}}_{AR}$ and $\mathbb{V}_{can}$. However, the required information on the characteristic subvariety of $\tilde{\mathbb{V}}_{AR}$ was attained only with the aid of a computational commutative algebra program. This obvious drawback prevented us from proceeding further in the general case. One valuable computation made in the current paper is that the characteristic subvariety of $\tilde{\mathbb{V}}_{AR}$ can be determined by hand at the generic point of the underlying moduli space. See Section 7.2 for details.

Surprisingly, building upon Theorem 1.2, we can conclude further that the monodromy group of $\tilde{\mathbb{V}}_{AR}$ is actually Zariski dense. Take an $s \in \mathfrak{M}_{AR}$ to be the base point and let

$$\rho : \pi_1(\mathfrak{M}_{AR}, s) \to Aut(V, Q)$$

be the monodromy representation associated to $\tilde{\mathbb{V}}_{AR}$, where $V = \tilde{\mathbb{V}}_{AR,s}$ is the fiber of $\tilde{\mathbb{V}}_{AR}$ over s, and Q is the bilinear form on V induced by the natural polarization. We denote the Zariski closure of the image of ρ by Mon. Let Mon^0 and $Aut^0(V,Q)$ be the identity component of Mon and $Aut(V,Q)$ respectively. We will prove

Theorem 1.3. *If $n \geq 3$, then $Mon^0 = Aut^0(V,Q)$. That is, the monodromy group of $\tilde{\mathbb{V}}_{AR}$ is Zariski dense in $Aut^0(V,Q)$.*

Using results of C. Schoen and P. Deligne the above theorem implies:

Corollary 1.4. *Let $\mathcal{M}_{n,2n+2}$ be the coarse moduli space of the CY n-folds obtained by a resolution of double covers of $\mathbb{P}^n$ branched along $2n+2$ hyperplanes in general position. Then for $n \geq 3$ the special Mumford–Tate group of a general member in $\mathcal{M}_{n,2n+2}$ is $Aut^0(V,Q)$.*

This result will imply in turn that any good family for the moduli space $\mathcal{M}_{n,2n+2}$ has the Zariski dense monodromy group.

As a consequence of Theorem 1.3, we can actually show that the fundamental group of the coarse moduli space $\mathfrak{M}_{n,m}, m \geq n+3$ of m ordered hyperplanes in $\mathbb{P}^n$ in general position is large. More precisely, we have

Corollary 1.5. *The fundamental group $\pi_1(\mathfrak{M}_{n,m}, s)$ is large, that is, there is a homomorphism of $\pi_1(\mathfrak{M}_{n,m}, s)$ to a noncompact semisimple real algebraic group which has Zariski-dense image.*

Proof. One can deduce easily from Theorem 5.4 that the real monodromy representation $\rho_{\mathbb{R}} : \pi_1(\mathfrak{M}_{AR}, s) \to Aut(V \otimes \mathbb{R}, Q)$ has Zariski-dense image, and $Aut^0(V \otimes \mathbb{R}, Q)$ is a noncompact semisimple real algebraic group. Hence $\pi_1(\mathfrak{M}_{n,2n+2}, s) = \pi_1(\mathfrak{M}_{AR}, s)$ is large.

If $m \geq 2n+2$, we consider the configuration space $X(n,m) := \{(H_1, \ldots, H_m) \in (\hat{\mathbb{P}}^n)^m \mid H_1, \ldots, H_m$ are in general position$\}$, where $\hat{\mathbb{P}}^n$ is the dual projective space of $\mathbb{P}^n$. Obviously $\mathfrak{M}_{n,m} = X(n,m)/PGL(n,\mathbb{C})$ and the quotient map $X(n,m) \xrightarrow{\pi_{n,m}} \mathfrak{M}_{n,m}$ is a $PGL(n,\mathbb{C})$ principle bundle. Since $m \geq 2n+2$, we can define the natural forgetful map $\tilde{f} : X(n,m) \to X(n,2n+2)$, which sends an ordered hyperplane $(H_1, \ldots, H_m)$ to

$(H_1, \ldots, H_{2n+2})$. It can be seen easily that $\tilde{f}$ descends to a forgetful map $f : \mathfrak{M}_{n,m} \to \mathfrak{M}_{n,2n+2}$ and we have a commutative diagram

$$\begin{array}{ccc} X(n,m) & \xrightarrow{\tilde{f}} & X(n,2n+2) \\ \downarrow{\scriptstyle \pi_{n,m}} & & \downarrow{\scriptstyle \pi_{n,2n+2}} \\ \mathfrak{M}_{n,m} & \xrightarrow{f} & \mathfrak{M}_{n,2n+2} \end{array}$$

By [39, Corollary 5.6], $\tilde{f}$ induces a surjective homomorphism from the fundamental group of $X(n,m)$ to the fundamental group of $X(n,2n+2)$. Since $X(n,2n+2)$ is a principle $PGL(n,\mathbb{C})$ bundle over $\mathfrak{M}_{n,2n+2}$, the map $\pi_{n,2n+2}$ also induces a surjective map between fundamental groups. Then we deduce the map $\pi_1(\mathfrak{M}_{n,m}, s) \xrightarrow{f_*} \pi_1(\mathfrak{M}_{n,2n+2}, f(s))$ is surjective, and $\pi_1(\mathfrak{M}_{n,m}, f(s))$ is large because of the largeness of $\pi_1(\mathfrak{M}_{n,2n+2}, s)$.

If $m \leq 2n+2$, then $m \geq 2(m-n-2)+2$. We have the association isomorphism (Ch. III of [13]): $\mathfrak{M}_{n,m} \simeq \mathfrak{M}_{m-n-2,m}$. Then the largeness of $\pi_1(\mathfrak{M}_{n,m}, s)$ follows from the largeness of $\pi_1(\mathfrak{M}_{m-n-2,m}, s)$, which has been verified in the last paragraph. □

Large groups are infinite and, moreover, always contain a free group of rank two. This corollary can be viewed as a degenerate case of a result of Carlson and Toledo in [5], where they considered the fundamental groups of parameter spaces of hypersurfaces in projective spaces, and showed that except several obvious cases, the kernels of monodromy representations are always large.

A remark on the methodology of the paper before explaining our strategy in detail: A usual method to show the Zariski-density of the monodromy group of a family of algebraic varieties is to show the existence of enough Lefschetz degenerations, which is based on the work of Deligne (see Proposition 5.3, Theorem 5.4 in [8, I] and Lemma 4.4.2 in [8, II]). Instead of seeking for such degenerations towards the boundary of the moduli space,[1] we work on an invariant of the infinitesimal variation of Hodge structure (abbreviated as IVHS), more precisely the first characteristic subvariety introduced in [33], at a general interior point of the moduli space. The general notion of the IVHS was first introduced by P. Griffiths and his collaborators as a surrogate of the theta divisor in a Jacobian (see [4,18] and also Ch. III in [19] for a nice exposition); though related, our first characteristic subvariety however does not belong to the collection of the Griffiths determinantal varieties (see Section 5 of [18]). The idea of IVHS has been proven to be very successful in establishing Torelli-type results for algebraic varieties (see e.g. Ch. XII of [19], [37]). The result of this paper shows that one can also use IVHS to obtain some important topological assertion on a moduli space of algebraic varieties. In particular, our method can be used to show also the Zariski-density of monodromy groups of good families for moduli spaces of smooth hypersurfaces in projective spaces. Now we proceed

[1] We conjecture that the moduli space $\mathcal{M}_{n,2n+2}, n \geq 3$ admits no Lefschetz degeneration at all.

to explain the strategy of the proof of our main result Theorem 1.3 in the following three steps.

Step 1: We show that the moduli space $\mathfrak{M}_{hp}$ of ordered $2n+2$ distinct points on $\mathbb{P}^1$ can be embedded into $\mathfrak{M}_{AR}$, and under this embedding, the induced VHS $\tilde{\mathbb{V}}_{AR}|_{\mathfrak{M}_{hp}}$ by restriction is isomorphic to the n-th wedge product of the weight one $\mathbb{Q}$-PVHS $\mathbb{V}_C$ associated to a good family of hyperelliptic curves over $\mathfrak{M}_{hp}$. Then we have a commutative diagram of monodromy representations

$$\begin{array}{ccc} \pi_1(\mathfrak{M}_{hp}, s) & \xrightarrow{\tau} & Aut(\mathbb{V}_{C,s}, Q) \\ \downarrow & & \downarrow \rho_{\wedge^n} \\ \pi_1(\mathfrak{M}_{AR}, s) & \xrightarrow{\rho} & Aut(V, Q) \end{array}$$

where $\rho_{\wedge^n}$ is the homomorphism induced by the n-th wedge product of the standard representation of $Aut(\mathbb{V}_{C,s}, Q) = Sp(2n, \mathbb{Q})$.

By Theorem 1 of [1], $\tau(\pi_1(\mathfrak{M}_{hp}, s))$ is Zariski dense in $Sp(2n, \mathbb{Q})$. So we get the commutative diagram of homomorphisms

$$\begin{array}{ccc} Sp(2n, \mathbb{Q}) & \xrightarrow{\rho_{\wedge^n}} & Aut(V, Q) \\ & \searrow \quad \nearrow & \\ & Mon & \end{array}$$

Step 2: Define the complex simple Lie algebra:

$$\mathfrak{g}_n = \begin{cases} \mathfrak{sp}_{\binom{2n}{n}}\mathbb{C}, & n \text{ odd}; \\ \mathfrak{so}_{\binom{2n}{n}}\mathbb{C}, & n \text{ even}. \end{cases}$$

Then we argue that the following classification result can be applied to the commutative diagram of homomorphisms in **Step 1**, so that we get either $Mon^0 = Aut^0(V, Q)$, or (after a possible finite étale base change) there exists a local system of complex vector spaces of rank $2n$ over $\mathfrak{M}_{AR}$, saying $\mathbb{W}$, such that we have an isomorphism of local systems $\tilde{\mathbb{V}}_{AR} \otimes \mathbb{C} \simeq \bigwedge^n \mathbb{W}$.

Proposition 1.6. *The n-th wedge product $V = \bigwedge^n \mathbb{C}^{2n}$ of the standard representation of $\mathfrak{sp}_{2n}\mathbb{C}$ induces an embedding $\mathfrak{sp}_{2n}\mathbb{C} \hookrightarrow \mathfrak{g}_n$. Suppose $\mathfrak{g}$ is a complex semi-simple Lie algebra lying between $\mathfrak{sp}_{2n}\mathbb{C}$ and $\mathfrak{g}_n$ such that the induced representation of $\mathfrak{g}$ on V is irreducible, then $\mathfrak{g}$ is one of the following:*

(1) $\mathfrak{g}_n$,
(2) $\mathfrak{sl}_{2n}\mathbb{C}$, *in which case the induced representation of $\mathfrak{g}$ on V is isomorphic to the n-th wedge product of the standard representation on $\mathbb{C}^{2n}$.*

Step 3: This is the essential step which proves particularly Theorem 1.2. Indeed, assuming the case $\tilde{\mathbb{V}}_{AR} \otimes \mathbb{C} \simeq \bigwedge^n \mathbb{W}$, the next proposition will imply this isomorphism is in fact an isomorphism of $\mathbb{C}$-PVHS, for a suitable $\mathbb{C}$-PVHS structure on $\mathbb{W}$, and this would imply that $\tilde{\mathbb{V}}_{AR}$ factors through $\mathbb{V}_{can}$ over $D^I_{n,n}$. A contradiction with Theorem 1.2. So the only possibility is $Mon^0 = Aut^0(V, Q)$ after Proposition 1.6, and we are done. As already explained above, Theorem 1.2 will be achieved by showing that the characteristic subvarieties attached to $\tilde{\mathbb{V}}_{AR}$ and $\mathbb{V}_{can}$ are non-isomorphic.

Proposition 1.7. *Let $\bar{S}$ denote a projective manifold, Z a simple divisor with normal crossing and $S = \bar{S}\backslash Z$. Let $\mathbb{V}$ denote an irreducible $\mathbb{C}$-PVHS over S with quasi-unipotent local monodromy around each component of Z. Suppose $\mathbb{W}$ is a rank $2n$ local system over S and we have an isomorphism of local systems $\mathbb{V} \simeq \bigwedge^n \mathbb{W}$, then $\mathbb{W}$ admits the structure of a $\mathbb{C}$-PVHS such that the induced $\mathbb{C}$-PVHS on the wedge product $\bigwedge^n \mathbb{W}$ coincides with the given $\mathbb{C}$-PVHS on $\mathbb{V}$.*

2. Calabi–Yau manifolds coming from hyperplane arrangements

Throughout this paper we use the following notation:

Let M be a $\mathbb{C}$-linear space, or a sheaf of $\mathbb{C}$-linear spaces on a scheme, on which the group $\mathbb{Z}/r\mathbb{Z} = \langle \sigma \rangle$ acts. Let ζ be a primitive r-th root of unit. For $i \in \mathbb{Z}/r\mathbb{Z}$ we write $M_{(i)} := \{x \in M \mid \sigma(x) = \zeta^i x\}$, which in the sheaf case has to be interpreted on the level of local sections. We refer to $M_{(i)}$ as the i-eigenspace of M. We have $M = \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} M_{(i)}$.

An ordered arrangement $\mathfrak{A} = (H_1, \ldots, H_{2n+2})$ of $2n+2$ hyperplanes in $\mathbb{P}^n$ can be given by a matrix $A \in M((n+1) \times (2n+2), \mathbb{C})$, the j-th column corresponding to the defining equation

$$\sum_{i=0}^{n} a_{ij} x_i = 0$$

of the hyperplane H_j. Here $[x_0 : \cdots : x_n]$ are the homogeneous coordinates on $\mathbb{P}^n$. We say that $\mathfrak{A}$ is in general position if no $n+1$ of the hyperplanes intersect in a point. In terms of the matrix A this means that each $(n+1) \times (n+1)$-minor is non-zero.

2.1. The double cover of $\mathbb{P}^n$ and its crepant resolution

The hyperplanes of the arrangement $\mathfrak{A}$ determine a divisor $H = \sum_{i=1}^{2n+2} H_i$ on $\mathbb{P}^n$. As the degree of H is even and the Picard group of $\mathbb{P}^n$ has no torsion, there exists a unique double cover $\pi : X \to \mathbb{P}^n$ that ramifies over R. By a direct computation using the adjunction formula, the canonical line bundle of X is trivial. The singular locus of such a double cover X is precisely the preimage of the singular locus of H. Fix an order of irreducible components of singularities of H, say $Z_1, \ldots, Z_N$. The canonical resolution of X according to that order is the following commutative diagram:

$$
\begin{array}{ccccccccccc}
X & = & X_0 & \xleftarrow{\tau_1} & X_1 & \xleftarrow{\tau_2} & \cdots & \xleftarrow{\tau_N} & X_N & = & \tilde{X} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \tilde{\pi}} \\
\mathbb{P}^n & = & \mathbb{P}_0 & \xleftarrow{\sigma_1} & \mathbb{P}_1 & \xleftarrow{\sigma_2} & \cdots & \xleftarrow{\sigma_N} & \mathbb{P}_N & = & \tilde{\mathbb{P}^n}
\end{array}
$$

Here, inductively on i, $\mathbb{P}_i \xrightarrow{\sigma_i} \mathbb{P}_{i-1}$ is the blow-up of $\mathbb{P}_{i-1}$ along the smooth center $(\sigma_{i-1} \circ \cdots \circ \sigma_1)^{-1}(Z_i)$, and X_i is the normalization of the fiber product of X_{i-1} and $\mathbb{P}_i$ over $\mathbb{P}_{i-1}$.

Lemma 2.1. *The space of infinitesimal deformations of $\tilde{X}$ is naturally isomorphic to the space of infinitesimal deformations of $\mathfrak{A}$.*

Proof. For the $n = 3$ case, see Lemma 2.1 in [17], whose proof is based on [6]. This proof goes through in general case verbatim. □

Proposition 2.2. *Let X and $\tilde{X}$ be as above. Then*

$$\dim H^{p,q}_{prim}(\tilde{X}) = \binom{n}{p}^2, \quad p + q = n.$$

The proof of this proposition is postponed to Section 7.1.

Throughout this paper except Section 6 $\mathfrak{M}_{AR}$ is denoted for the coarse moduli space of ordered arrangements of $2n + 2$ hyperplanes in $\mathbb{P}^n$ in general position. Let $\mathcal{M}_{n,2n+2}$ denote the coarse moduli space of $\tilde{X}$ and call $\mathcal{M}_{n,2n+2}$ the Dolgachev moduli spaces.

Let $\mathfrak{A}$ be an ordered arrangement in general position. It is easy to verify that under the automorphism group of $\mathbb{P}^n$ one can transform in a unique way the ordered first $n+2$ hyperplanes $(H_1, \ldots, H_{n+2})$ of $\mathfrak{A}$ into the ordered $n+2$ hyperplanes in $\mathbb{P}^n$, that are given by the first $n + 2$ columns in the following matrix A. Hence the moduli point of $\mathfrak{A}$ in $\mathfrak{M}_{AR}$ can be uniquely represented by the matrix A of the form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & & 0 & 1 & a_{11} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & & & 1 & 1 & a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Conversely a matrix A in the above form whose all $(n+1) \times (n+1)$-minors are non-zero represents an arrangement $\mathfrak{A}$ in general position. Thus $\mathfrak{M}_{AR}$ can be realized as an open subvariety of the affine space $\mathbb{C}^{n^2}$ and it admits a natural family $f : \mathcal{X}_{AR} \to \mathfrak{M}_{AR}$, where each fiber $f^{-1}(\mathfrak{A})$ is the double cover of $\mathbb{P}^n$ branched along the hyperplane arrangement $\mathfrak{A} \in \mathfrak{M}_{AR}$. It is easy to see the canonical resolution gives rise to a simultaneous resolution $\tilde{\mathcal{X}}_{AR} \to \mathcal{X}_{AR}$, and the family $\tilde{f} : \tilde{\mathcal{X}}_{AR} \to \mathfrak{M}_{AR}$ is a smooth family of CY manifolds.

2.2. The Kummer cover

Let $a = (a_{ij})$ denote an $n \times n$-matrix associated with a hyperplane arrangement $\mathfrak{A}$ as described above. We let Y denote the complete intersection of the $n+1$ hypersurfaces in $\mathbb{P}^{2n+1}$ defined by the $n+1$ equations:

$$y_{n+1}^2 - (y_0^2 + \cdots + y_n^2) = 0;$$
$$y_{n+i+1}^2 - (y_0^2 + a_{1i}y_1^2 + \cdots + a_{ni}y_n^2) = 0, \quad 1 \le i \le n.$$

Here $[y_0 : \cdots : y_{2n+1}]$ are the homogeneous coordinates on $\mathbb{P}^{2n+1}$. In case $\mathfrak{A}$ is in general position, the space Y is smooth (see Proposition 3.1.2 in [37]).

Let $N = \bigoplus_{j=0}^{2n+1} \mathbb{F}_2$, where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Consider the following group

$$N_1 := Ker(N \to \mathbb{F}_2)$$
$$(a_j) \mapsto \sum_{j=0}^{2n+1} a_j.$$

We define a natural action of N on Y. $\forall g = (a_0, \ldots, a_{2n+1}) \in N$, the action of g on Y is induced by

$$g \cdot y_j := (-1)^{a_j} y_j, \quad \forall 0 \le j \le 2n+1.$$

Then for a given arrangement $\mathfrak{A}$ in general position, we have the following relations between the double cover X and the Kummer cover Y:

Proposition 2.3.

(1) *The map $\pi_1 : Y \to \mathbb{P}^n$, $[y_0 : \cdots : y_{2n+1}] \mapsto [y_0^2 : \cdots : y_n^2]$ defines a cover of degree 2^{2n+1}.*
(2) *$X \simeq Y/N_1$.*
(3) *There exists a natural isomorphism of rational Hodge structures $H^n(X, \mathbb{Q}) \simeq H^n(Y, \mathbb{Q})^{N_1}$, where $H^n(Y, \mathbb{Q})^{N_1}$ denotes the subspace of invariants under N_1. In particular, the natural mixed Hodge structure on $H^n(X, \mathbb{Q})$ is in fact a pure one.*
(4) *Via the identification $H^n(X, \mathbb{Q}) = H^n(Y, \mathbb{Q})^{N_1}$, we have $H^n(X, \mathbb{Q})_{(1)} \subset H^n(Y, \mathbb{Q})_{pr}$.*

Proof. For (1), (2), (3), see Lemma 2.4, Proposition 2.5 and Proposition 2.6 in [17].

(4) Since the isomorphism $X \simeq Y/N_1$ is compatible with the $\mathbb{F}_2$-action, we know $H^{n-2}(X, \mathbb{Q})_{(1)} = H^{n-2}(Y, \mathbb{Q})_{(1)}^{N_1}$, where the $\mathbb{F}_2$-action on Y/N_1 is induced by the identification $\mathbb{F}_2 = N/N_1$. Since Y is a complete intersection in $\mathbb{P}^{2n+1}$, Lefschetz Hyperplane Theorem implies that $H^{n-2}(Y, \mathbb{Q}) = H^{n-2}(\mathbb{P}^{2n+1}, \mathbb{Q})$, hence $H^{n-2}(X, \mathbb{Q})_{(1)} = H^{n-2}(Y, \mathbb{Q})_{(1)}^{N_1} = 0$. Since the Lefschetz operator

$$H^{n-2}(X,\mathbb{Q}) \xrightarrow{L} H^n(X,\mathbb{Q})$$

preserves the $\mathbb{F}_2$-action, we get $L(H^{n-2}(X,\mathbb{Q})) \cap H^n(X,\mathbb{Q})_{(1)} = 0$. This shows $H^n(X,\mathbb{Q})_{(1)} \subset H^n(Y,\mathbb{Q})_{pr}$. □

2.3. Hyperelliptic locus

There is an interesting locus in $\mathfrak{M}_{AR}$ where the Hodge structure of X comes from a hyperelliptic curve.

Note that there exists a natural Galois covering with Galois group S_n, the permutation group of n letters:

$$\gamma : \left(\mathbb{P}^1\right)^n \to Sym^n\left(\mathbb{P}^1\right) = \mathbb{P}^n.$$

Here the identification attaches to a divisor of degree n the ray of its equation in $H^0(\mathbb{P}^1,\mathcal{O}(n))$.

Lemma 2.4. *Let $(p_1,\dots,p_{2n+2})$ be a collection of $2n+2$ distinct points on $\mathbb{P}^1$, and put $H_i = \gamma(\{p_i\}\times\mathbb{P}^1\times\cdots\times\mathbb{P}^1)$. Then $(H_1,\dots,H_{2n+2})$ is an arrangement of hyperplanes in general position.*

Proof. The divisors of degree n in $\mathbb{P}^1$ containing a given point form a hyperplane and, as a divisor of degree n cannot contain $n+1$ distinct points, no $n+1$ hyperplanes in the arrangement do meet. □

Let C be the double cover of $\mathbb{P}^1$ branched at $p_1,\dots,p_{2n+2}$, and let X_C be the double cover of $\mathbb{P}^n$ branched along $H_1,\dots,H_{2n+2}$. Then the double covering structures induce natural actions of the cyclic group $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ on C and X.

The group $\mathbb{F}_2^n$ and the permutation group S_n act naturally on the product C^n. These actions induce an action of the semi-direct product $\mathbb{F}_2^n \rtimes S_n$ on C^n. Let N' be the kernel of the summation homomorphism:

$$\mathbb{F}_2^n \to \mathbb{F}_2$$
$$(a_i) \mapsto \sum_{i=1}^{n} a_i.$$

Then we have:

Lemma 2.5. *There exists a natural isomorphism: $X_C \simeq C^n/N_2$, where $N_2 := N' \rtimes S_n$.*

Proof. Let $p : C \to \mathbb{P}^1$ be the covering map. The n-fold product

$$h : C^n \xrightarrow{p^n} \left(\mathbb{P}^1\right)^n \xrightarrow{\gamma} \mathbb{P}^n$$

is a Galois cover with Galois group $\mathbb{F}_2^n \rtimes S_n$. Similar to [17, Lemma 2.8], one checks that the natural map $C^n/N_2 \to \mathbb{P}^n$ induced by h is a double cover and its branch locus is exactly $H_1 + \cdots + H_{2n+2}$. As the Picard group of a projective space has no torsion, one concludes that C^n/N_2 is isomorphic to the double cover X_C of $\mathbb{P}^n$ branched along $\sum_i H_i$. □

The following lemma is well known

Lemma 2.6. $\dim H^1(C, \mathbb{C})_{(0)} = 0$, $\dim H^{1,0}(C)_{(1)} = \dim H^{0,1}(C)_{(1)} = n$.

Proof. See [26, (2.7)]. □

We can get some information about the Hodge structure of X_C from C.

Proposition 2.7.

$$H^n(X_C, \mathbb{C})_{(1)} \simeq \bigwedge^n H^1(C, \mathbb{C}).$$

Proof. Lemma 2.5 gives an identification

$$H^n(X_C, \mathbb{C}) = H^n\left(C^n, \mathbb{C}\right)^{N_2}.$$

Taking into account the action of $\mathbb{F}_2$, the Künneth formula gives

$$H^n\left(C^n, \mathbb{C}\right)^{N_2}_{(1)} = \left(\bigotimes_{i=1}^n H^1(C, \mathbb{C})\right)^{N_2}.$$

Then it is easy to see the following map is an isomorphism

$$\begin{aligned} \bigwedge^n H^1(C, \mathbb{C}) &\xrightarrow{\sim} \left(\bigotimes_{i=1}^n H^1(C, \mathbb{C})\right)^{N_2} = H^n(X_C, \mathbb{C})_{(1)} \\ \alpha_1 \wedge \cdots \wedge \alpha_n &\mapsto \sum_{\nu \in S_n} (-1)^{Sign(\nu)} \alpha_{\nu(1)} \otimes \cdots \otimes \alpha_{\nu(n)} \end{aligned}$$

and this gives the desired isomorphism. □

Let $\mathfrak{M}_{hp} \subset \mathfrak{M}_{AR}$ be the moduli space of ordered distinct $2n+2$ points on $\mathbb{P}^1$ and $g : \mathcal{C} \to \mathfrak{M}_{hp}$ be the natural universal family of hyperelliptic curves. Recall that $f : \mathcal{X}_{AR} \to \mathfrak{M}_{AR}$ is the natural family of double covers of $\mathbb{P}^n$ branched along $2n+2$ hyperplanes in general position and $\tilde{f} : \tilde{\mathcal{X}}_{AR} \to \mathfrak{M}_{AR}$ is the simultaneous crepant resolution of f. Consider the $\mathbb{Q}$-VHS (rational variation of Hodge structures) $\mathbb{V} := R^n f_* \mathbb{Q}$. Since $\mathbb{F}_2$ acts naturally on $\mathbb{V}$, we have a decomposition of $\mathbb{Q}$-VHS: $\mathbb{V} = \mathbb{V}_{(0)} \oplus \mathbb{V}_{(1)}$. Similarly, let $\tilde{\mathbb{V}}_{AR} := (R^n \tilde{f}_* \mathbb{Q})_{pr}$ be the $\mathbb{Q}$-PVHS of primitive cohomologies. Then we have

Proposition 2.8. *There are isomorphisms of* $\mathbb{Q}$*-PVHS:*

(1) $\tilde{\mathbb{V}}_{AR} \simeq \mathbb{V}_{(1)}$, *whose Hodge numbers are:*

$$h^{p,n-p} = \binom{n}{p}^2, \quad \forall 0 \le p \le n.$$

(2) $\mathbb{V}_{(1)}|_{\mathfrak{M}_{hp}} \simeq \bigwedge^n \mathbb{V}_C$, *where* $\mathbb{V}_C := R^1 g_* \mathbb{Q}$ *is the weight one* $\mathbb{Q}$*-PVHS associated to* $g : \mathcal{C} \to \mathfrak{M}_{hp}$.

Proof. (1) By Proposition 2.3, we can see that the simultaneous crepant resolution gives a natural morphism $\mathbb{V}_{(1)} \to \tilde{\mathbb{V}}_{AR}$ of $\mathbb{Q}$-PVHS. Theorem 5.41 in [30] shows that this morphism is injective. It is easy to see the isomorphism in Proposition 2.7 preserves the Hodge filtrations. Then Proposition 2.2 and Lemma 2.6 show that $\tilde{\mathbb{V}}_{AR}$ and $\mathbb{V}_{(1)}$ have the same Hodge numbers. So the natural morphism $\mathbb{V}_{(1)} \to \tilde{\mathbb{V}}_{AR}$ is an isomorphism of $\mathbb{Q}$-PVHS.

(2) follows directly from Proposition 2.7. □

3. Type A canonical variation and characteristic subvariety

In order to compare two $\mathbb{C}$-PVHSs over S, besides using the obvious Hodge numbers as invariants, we can also use another important series of invariants: characteristic subvarieties, which are contained in the projectivized tangent bundle $\mathbb{P}(\mathcal{T}_S)$. The basic theory of characteristic subvarieties is developed in [33]. We recall the definition.

Definition 3.1. Let $\mathbb{W}$ be a $\mathbb{C}$-PVHS of weight n over S and (E, θ) the associated Higgs bundle. For every q with $1 \le q \le n$, the q-th iterated Higgs field

$$E^{n,0} \xrightarrow{\theta^{n,0}} E^{n-1,1} \otimes \Omega_S \xrightarrow{\theta^{n-1,1}} \cdots \xrightarrow{\theta^{n-q+1,q-1}} E^{n-q,q} \otimes S^q \Omega_S$$

defines a morphism

$$\theta^q : Sym^q(T_S) \to Hom\big(E^{n,0}, E^{n-q,q}\big).$$

Then $\forall s \in S$, the q-th characteristic subvariety of $\mathbb{W}$ at s is

$$C_{q,s} = \big\{[v] \in \mathbb{P}(T_{S,s}) \;\big|\; v^{q+1} \in \ker\big(\theta^{q+1}\big)\big\}.$$

Moreover, we call $\mathbb{W}$ of Calabi–Yau type (CY-type) if rank $E^{n,0} = 1$ and $\theta^{n,0} : T_S \to Hom(E^{n,0}, E^{n-1,1})$ is an isomorphism at the generic point.

Remark 3.2. Our definition of characteristic variety is slightly different with that in [33]. Here we only need the reduced subvariety. See Lemma 4.3 in [17].

As an example, we can compute the first characteristic subvariety of the type A canonical $\mathbb{C}$-PVHS explicitly. We recall the following description of the canonical $\mathbb{C}$-PVHS $\mathbb{V}_{can}$ over $D^I_{n,n}$ from [33].

Let $V = \mathbb{C}^{2n}$ be a complex vector space equipped with a Hermitian symmetric bilinear form h of signature (n,n). Then $D^I_{n,n}$ parameterizes the dimension n complex subspaces $U \subset V$ such that

$$h|_U : U \times U \to \mathbb{C}$$

is positive definite. This forms the tautological subbundle $S \subset V \times D^I_{n,n}$ of rank n and denote by Q the tautological quotient bundle of rank n. We have the natural isomorphism of holomorphic vector bundles

$$T_{D^I_{n,n}} \simeq Hom(S, Q).$$

The standard representation V of $SU(n,n)$ gives rise to a weight one $\mathbb{C}$-PVHS $\mathbb{W}$ over $D^I_{n,n}$, and its associated Higgs bundle

$$F = F^{1,0} \oplus F^{0,1}, \qquad \eta = \eta^{1,0} \oplus \eta^{0,1}$$

is determined by

$$F^{1,0} = S, \qquad F^{0,1} = Q, \qquad \eta^{0,1} = 0,$$

and $\eta^{1,0}$ is defined by the above isomorphism. The canonical $\mathbb{C}$-PVHS is

$$\mathbb{V}_{can} = \bigwedge^n \mathbb{W}$$

and its associated system of Higgs bundle (E_{can}, θ_{can}) is then

$$(E_{can}, \theta_{can}) = \bigwedge^n (F, \eta).$$

Proposition 3.3.

(1) *The Hodge numbers of the type A canonical $\mathbb{C}$-PVHS $\mathbb{V}_{can}$ over $D^I_{n,n}$ are*

$$h^{p,n-p} = \binom{n}{p}^2, \quad \forall 0 \le p \le n.$$

(2) *The first characteristic subvarieties of the type A canonical $\mathbb{C}$-PVHS $\mathbb{V}_{can}$ over $D^I_{n,n}$ are*

$$C_{1,s} \simeq \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}, \quad \forall s \in D^I_{n,n}.$$

Proof. From the description of $\mathbb{V}_{can}$ above, $\forall 0 \le p \le n$, $E_{can}^{p,n-p}$ can be identified with the subspace of $\bigwedge^n F$ linearly spanned by the following set:

$$\{e_1 \wedge \cdots \wedge e_p \wedge \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_{n-p} \mid e_i \in F^{1,0},\ 1 \le i \le p;\ \tilde{e}_j \in F^{0,1},\ 1 \le j \le n-p\}.$$

From this we get (1). Moreover, with this identification, $\forall v \in T_{D^I_{n,n}}$, if $e_1, \dots, e_n \in F^{1,0}$, then

$$\theta^{n,0}_{can,v}(e_1 \wedge \cdots \wedge e_n) = \sum_{i=1}^{n} (-1)^{i-1} \eta^{1,0}_v(e_i) \wedge e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_n,$$

where $\hat{e_i}$ means deleting the term e_i.

Similarly, if $e_1, \dots, e_{n-1} \in F^{1,0}$, $\tilde{e}_1 \in F^{0,1}$, then

$$\theta^{n-1,1}_{can,v}(e_1 \wedge \cdots \wedge e_{n-1} \wedge \tilde{e}_1) = \sum_{i=1}^{n-1} (-1)^{i-1} \eta^{1,0}_v(e_i) \wedge e_1 \wedge \cdots \wedge \hat{e_i} \wedge \cdots \wedge e_{n-1} \wedge \tilde{e}_1.$$

Since $\eta^{1,0}$ is defined by the identification $T_{D^I_{n,n}} = Hom(F^{1,0}, F^{0,1})$, we can see that $\forall s \in D^I_{n,n}$,

$$\begin{aligned} C_{1,s} &= \{v \in T_{D^I_{n,n},s} \mid \theta^{n-1,1}_{can,v} \circ \theta^{n,0}_{can,v} = 0\} \\ &\simeq \{A \in M(n \times n, \mathbb{C}) \mid \operatorname{rank}(A) = 1\} \\ &\simeq \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}. \end{aligned}$$

This proves (2). □

4. Characteristic subvariety and nonfactorization

4.1. Nonfactorization of the period map

There is an upper bound of the dimension of the first characteristic variety of the $\mathbb{Q}$-PVHS $\mathbb{V}_{(1)}$.

Proposition 4.1. *If $n \ge 3$, then for a generic $a \in \mathfrak{M}_{AR}$, the first characteristic variety of $\mathbb{V}_{(1)}$ has dimension* $\dim C_{1,a} \le 2$.

Proof. This theorem follows from Proposition 7.16, whose proof we postpone to the Section 7.2. □

As over a coarse moduli space $\mathfrak{M}$ of a polarized algebraic variety usually does not exist a universal family, we use the following weaker notion. We say that a proper smooth morphism $f : \mathcal{X} \to S$ over a smooth connected base is a good family for $\mathfrak{M}$, if the moduli map $S \to \mathfrak{M}$ is dominant and generically finite. With this definition, our main theorem is

Theorem 4.2. *If $n \geq 3$, then:*

(1) $\tilde{\mathcal{X}}_{AR} \xrightarrow{\tilde{f}} \mathfrak{M}_{AR}$ *is a good family for the coarse moduli space* $\mathcal{M}_{n,2n+2}$*;*
(2) *Let $f : \mathcal{X} \to S$ be a good family of $\mathcal{M}_{n,2n+2}$ and $\mathbb{W} = (R^n f_* \mathbb{Q})_{pr}$ be the associated weight n $\mathbb{Q}$-PVHS. Then $\mathbb{W}$ does not factor through the $\mathbb{C}$-PVHS $\mathbb{V}_{can}$ over the type A symmetric domain $D^I_{n,n}$.*

Proof. (1) follows directly from Lemma 2.1.

(2) Assume the contrary. By Proposition 3.3, for any $s \in S$ away from the ramification locus of the moduli map $S \to \mathcal{M}_{n,2n+2}$, the first characteristic variety $C_{1,s}$ is isomorphic to $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. In particular, the dimension of $C_{1,s}$ has dimension $2n - 2 \geq 4$. On the other hand, by Proposition 2.8, Proposition 4.1 and (1), for a generic $s \in \mathcal{X}$, the dimension of the first characteristic variety $C_{1,s}$ has dimension ≤ 2. This gives a contradiction. □

5. Zariski density of monodromy group

In this section, we will use the notation as given in [16] for representations of Lie algebras. We are mainly concerned with representations of $\mathfrak{sp}_{2n}\mathbb{C}$. Following the notation in Section 16.1 in [16], it is well known that the weight lattice of $\mathfrak{sp}_{2n}\mathbb{C}$ is the lattice of integral linear combinations of $L_1, \ldots, L_n \in \mathfrak{h}^*$, where $\mathfrak{h} \subset \mathfrak{sp}_{2n}\mathbb{C}$ is the Cartan subalgebra as defined in Section 16.1 in [16], and after fixing a positive direction, the corresponding closed Weyl chamber is

$$\{a_1 L_1 + a_2 L_2 + \cdots + a_n L_n\colon\ a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}.$$

We have the following

Lemma 5.1. *Suppose W is a nontrivial finite dimensional complex representation of $\mathfrak{sp}_{2n}\mathbb{C}$ and Q is an $\mathfrak{sp}_{2n}\mathbb{C}$-invariant bilinear form on W. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of an integer d with $k \leq \dim W - 1$, then there exist k linearly independent weight vectors $v_1, \ldots, v_k \in W$ such that if we define $v_\lambda \in W^{\otimes d}$ as follows:*

$$\underbrace{v_1 \otimes \cdots \otimes v_1}_{\lambda_1} \otimes \cdots \otimes \underbrace{v_k \otimes \cdots \otimes v_k}_{\lambda_k}$$

then $v_\lambda \cdot c_\lambda \in \mathbb{S}_\lambda W$ is a nonzero weight vector, and the weight $a_1 L_1 + \cdots + a_n L_n$ of $v_\lambda \cdot c_\lambda$ satisfies $a_1 \geq \lambda_1$. Here c_λ is the Young symmetrizer. Moreover, if $k \leq [\frac{\dim W}{2}]$, then the weight vectors $v_1, \ldots, v_k \in W$ above can be chosen such that they satisfy the additional condition: $\forall 1 \leq i, j \leq k$, $Q(v_i, v_j) = 0$.

Proof. From the definition of the Young symmetrizer, $v_\lambda \cdot c_\lambda \in \mathbb{S}_\lambda W$ is nonzero as long as $v_1, \dots, v_k \in W$ are linearly independent. Suppose

$$W = \bigoplus_{\alpha \in \mathfrak{h}^*} W_\alpha$$

is the weight space decomposition of W. It follows from the basic representation theory of $\mathfrak{sp}_{2n}\mathbb{C}$ that $\dim W_\alpha = \dim W_{-\alpha}$, $\forall \alpha \in \mathfrak{h}^*$. So we can write

$$W = \bigoplus_{i=1}^{l} (W_{\alpha_i} \oplus W_{-\alpha_i}) \oplus U$$

such that

(1) U is a trivial representation of $\mathfrak{sp}_{2n}\mathbb{C}$;
(2) $\forall 1 \le i \le l$, $W_{\alpha_i} \ne 0$;
(3) if $\alpha_i = a_{i1}L_1 + \cdots + a_{in}L_n$, then there exists an integer k_i between 1 and n, with $a_{ik_i} > 0$ and $a_{i1} = a_{i2} = \cdots = a_{i,k_i-1} = 0$.

Since Q is $\mathfrak{sp}_{2n}\mathbb{C}$-invariant, the spaces U and $\bigoplus_{i=1}^{l}(W_{\alpha_i} \oplus W_{-\alpha_i})$ are orthogonal under Q, and $Q|_{\bigoplus_{i=1}^{l} W_{\alpha_i}} = 0$. It is easy to see that if $v_i \in W_{\beta_i}$, $i = 1, \dots, k$, then $v_\lambda \cdot c_\lambda$ is a weight vector of $\mathbb{S}_\lambda W$ with weight $\sum_{i=1}^{k} \beta_i$. From these and since at least one a_{i1} is a positive integer, it is not difficult to see that we can choose linearly independent weight vectors $v_1, \dots, v_k \in W$ such that the weight of $v_\lambda \cdot c_\lambda$ satisfies the required condition. Moreover, if $k \le [\frac{\dim W}{2}]$, since $Q|_{\bigoplus_{i=1}^{l} W_{\alpha_i}} = 0$ and $U \perp \bigoplus_{i=1}^{l} W_{\alpha_i}$ under Q, we can choose weight vectors $v_1, \dots, v_k$ from the spaces U and $\bigoplus_{i=1}^{l} W_{\alpha_i}$ such that they satisfy the additional condition: $\forall 1 \le i, j \le k$, $Q(v_i, v_j) = 0$. □

Recall the definition of the complex Lie algebra $\mathfrak{g}_n$ in Section 1.

Proposition 5.2. *The n-th wedge product $V = \bigwedge^n \mathbb{C}^{2n}$ of the standard representation of $\mathfrak{sp}_{2n}\mathbb{C}$ induces an embedding $\mathfrak{sp}_{2n}\mathbb{C} \hookrightarrow \mathfrak{g}_n$. Suppose $\mathfrak{g}$ is a complex semi-simple Lie algebra lying between $\mathfrak{sp}_{2n}\mathbb{C}$ and $\mathfrak{g}_n$ such that the induced representation of $\mathfrak{g}$ on V is irreducible, then $\mathfrak{g}$ is one of the following:*

(1) $\mathfrak{g}_n$,
(2) $\mathfrak{sl}_{2n}\mathbb{C}$, *in which case the induced representation of $\mathfrak{g}$ on V is isomorphic to the n-th wedge product of the standard representation on $\mathbb{C}^{2n}$.*

Proof. We first show that $\mathfrak{g}$ is simple. Since $\mathfrak{g}$ is semi-simple, we can write $\mathfrak{g} = \bigoplus_{i=1}^{m} \mathfrak{g}_i$ into a direct sum of simple Lie algebras. By Schur's lemma and since V is an irreducible $\mathfrak{g}$-module, we have the tensor decomposition of $V = \bigotimes_{i=1}^{m} V_i$, where each V_i is an irreducible $\mathfrak{g}_i$-module. Since $\mathfrak{sp}_{2n}\mathbb{C}$ is simple, $\forall 1 \le i \le m$, the composition

$$\mathfrak{sp}_{2n}\mathbb{C} \hookrightarrow \mathfrak{g} = \bigoplus_{i=1}^{m} \mathfrak{g}_i \to \mathfrak{g}_i$$

is either an embedding or a zero map. So $\bigwedge^n \mathbb{C}^{2n} = V = \bigotimes_{i=1}^m V_i$ is also a tensor decomposition of the $\mathfrak{sp}_{2n}\mathbb{C}$-module $\bigwedge^n \mathbb{C}^{2n}$. A direct computation of highest weights shows that $m = 1$, which implies $\mathfrak{g}$ is simple. So the only possibilities of $\mathfrak{g}$ are exceptional Lie algebras, of type A, type B, type C or type D.

Exceptional cases: Since V is an irreducible $\mathfrak{g}$-module, by checking the dimensions of irreducible representations, we can exclude the exceptional Lie algebra cases.

Case A: Suppose $\mathfrak{g} \simeq \mathfrak{sl}_m\mathbb{C}$ is a simple Lie algebra of type A, then by Weyl's construction, there exists a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of a positive integer d with $k \le m-1$, such that $V \simeq \mathbb{S}_\lambda W$ as $\mathfrak{sl}_m\mathbb{C}$-modules, where $W = \mathbb{C}^m$ is the standard representation of $\mathfrak{sl}_m\mathbb{C}$ (cf. Proposition 15.15 in [16]). By Lemma 5.1, we can always choose k linearly independent vectors $v_1, \dots, v_k \in W$ such that $v_\lambda \cdot c_\lambda \in \mathbb{S}_\lambda W$ is a nonzero weight vector of $\mathfrak{sp}_{2n}\mathbb{C}$ and its weight $a_1L_1 + \dots + a_nL_n$ satisfies $a_1 \ge \lambda_1$. Since $V \simeq \mathbb{S}_\lambda W$ as $\mathfrak{sp}_{2n}\mathbb{C}$-modules and the highest weight of V is $L_1 + \dots + L_n$, we deduce that $\lambda_1 = 1$. So the partition $\lambda = (1, \dots, 1)$ and $\mathbb{S}_\lambda W = \bigwedge^d W$. Then it is easy to see that the only possibility is $d = n$, and $m = 2n$. This gives case (2).

Case B: Suppose $\mathfrak{g} \simeq \mathfrak{so}_{2m+1}\mathbb{C}$ is a simple Lie algebra of type B. Following Section 18.1 in [16], we choose the basis $L'_1, \dots, L'_m$ of the dual space of a Cartan subalgebra (note we add a '′' to distinguish with the weights of $\mathfrak{sp}_{2n}\mathbb{C}$). We have two cases to discuss.

B1: If the highest weight of the irreducible $\mathfrak{so}_{2m+1}\mathbb{C}$-representation V is $a_1L'_1 + \dots + a_mL'_m$, with each a_i an integer, then by Weyl's construction, there exists a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of a positive integer d with $k \le m$, such that $V \simeq \mathbb{S}_{[\lambda]}W$ as $\mathfrak{so}_{2m+1}\mathbb{C}$-modules. Here $W = \mathbb{C}^{2m+1}$ is the standard representation of $\mathfrak{so}_{2m+1}\mathbb{C}$ (cf. Theorem 19.22 in [16]). By Lemma 5.1, we can choose k linearly independent vectors $v_1, \dots, v_k \in W$ such that $v_\lambda \cdot c_\lambda \in \mathbb{S}_{[\lambda]}W$ is a nonzero weight vector of $\mathfrak{sp}_{2n}\mathbb{C}$ and its weight $b_1L_1 + \dots + b_nL_n$ satisfies $b_1 \ge \lambda_1$. Since $V \simeq \mathbb{S}_\lambda W$ as $\mathfrak{sp}_{2n}\mathbb{C}$-modules and the highest weight of V is $L_1 + \dots + L_n$, we deduce that $\lambda_1 = 1$. So the partition $\lambda = (1, \dots, 1)$ and $V = \mathbb{S}_{[(1,\dots,1)]}W$. Then it is easy to see that the only possibility is $d = 1$, $2m + 1 = \binom{2n}{n}$, and $\mathfrak{g} = \mathfrak{g}_n$. This gives case (1).

B2: If the highest weight of the irreducible $\mathfrak{so}_{2m+1}\mathbb{C}$-representation V is $a_1L'_1 + \dots + a_mL'_m$, with each a_i a nonzero half integer, then consider the representation Sym^2V, and let V_1 be the irreducible $\mathfrak{so}_{2m+1}\mathbb{C}$-submodule of Sym^2V with highest weight $2a_1L'_1 + \dots + 2a_mL'_m$. By Weyl's construction, there exists a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of a positive integer d with $k \le m$, such that $V_1 \simeq \mathbb{S}_{[\lambda]}W$ as $\mathfrak{so}_{2m+1}\mathbb{C}$-modules. Here $W = \mathbb{C}^{2m+1}$ is the standard representation of $\mathfrak{so}_{2m+1}\mathbb{C}$. Then arguing in the same way as the **B1** case, we find $\lambda_1 \le 2$. Since as a representation of $\mathfrak{so}_{2m+1}\mathbb{C}$, the highest weight of $\mathbb{S}_{[\lambda]}W$ is $\lambda_1L'_1 + \dots + \lambda_kL'_k$, we see that $2a_1 = \lambda_1 \le 2$. Hence $a_1 = \frac{1}{2}$ and the irreducible $\mathfrak{so}_{2m+1}\mathbb{C}$-representation V is the fundamental spin representation. Then the dimension $\dim V = \binom{2n}{n}$ must be a power of 2. It is elementary to see that this can never happen if $n \ge 2$. So we can exclude this case.

Case C: Suppose $\mathfrak{g} \simeq \mathfrak{sp}_{2m}\mathbb{C}$ is of type C. By Weyl's construction, there exists a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of a positive integer d with $k \le m$, such that $V \simeq \mathbb{S}_{\langle\lambda\rangle}W$ as $\mathfrak{sp}_{2m}\mathbb{C}$-modules. Here $W = \mathbb{C}^{2m}$ is the standard representation of $\mathfrak{sp}_{2m}\mathbb{C}$ (cf. Theorem 17.11 in [16]). By Lemma 5.1 and arguing in the same way as the **B1** case, we find $\lambda_1 = 1$ and $V = \mathbb{S}_{\langle(1,\dots,1)\rangle}W$ as $\mathfrak{sp}_{2m}\mathbb{C}$-modules. Then it is easy to see that the only possibility is $d = 1$, $2m = \binom{2n}{n}$ and $\mathfrak{g} = \mathfrak{g}_n$. This gives case (1).

Case D: Suppose $\mathfrak{g} \simeq \mathfrak{so}_{2m}\mathbb{C}$ is of type D. Following Section 18.1 in [16], we choose the basis $L'_1, \dots, L'_m$ of the dual space of a Cartan subalgebra (note we add a $'$ to distinguish with the weights of $\mathfrak{sp}_{2n}\mathbb{C}$). We have three cases to discuss.

D1: If the highest weight of the irreducible $\mathfrak{so}_{2m}\mathbb{C}$-representation V is $a_1L'_1 + \dots + a_{m-1}L'_{m-1}$, with each a_i an integer, then by Weyl's construction, there exists a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of a positive integer d with $k \le m-1$, such that $V \simeq \mathbb{S}_{[\lambda]}W$ as $\mathfrak{so}_{2m}\mathbb{C}$-modules. Here $W = \mathbb{C}^{2m}$ is the standard representation of $\mathfrak{so}_{2m}\mathbb{C}$ (cf. Theorem 19.22 in [16]). By Lemma 5.1 and arguing in the same way as the **B1** case, we find $\lambda_1 = 1$ and $V = \mathbb{S}_{[(1,\dots,1)]}W$ as $\mathfrak{so}_{2m}\mathbb{C}$-modules. Then it is easy to see that the only possibility is $d = 1$, $2m = \binom{2n}{n}$ and $\mathfrak{g} = \mathfrak{g}_n$. This gives case (1).

D2: Suppose the highest weight of the irreducible $\mathfrak{so}_{2m}\mathbb{C}$-representation V is $\lambda_1L'_1 + \dots + \lambda_mL'_m$, with each λ_i an integer and $\lambda_m \ne 0$. Let $\lambda = (\lambda_1, \dots, \lambda_{m-1}, |\lambda_m|)$ be the partition of $d = \sum_{i=1}^{m-1} \lambda_i + |\lambda_m|$. The standard representation $W = \mathbb{C}^{2m}$ of $\mathfrak{so}_{2m}\mathbb{C}$ can also be viewed as a representation of the complex Lie group $SO_{2m}\mathbb{C}$. By Theorem 19.22 in [16], $\mathbb{S}_{[\lambda]}W$ is an irreducible representation of the complex Lie group $O_{2m}\mathbb{C}$ and as a representation of $SO_{2m}\mathbb{C}$, we have $\mathbb{S}_{[\lambda]}W = V \oplus V'$, where V' is conjugate to V. In particular, for any $\sigma \in O_{2m}\mathbb{C}$, if $\det\sigma = -1$, then $V' = \sigma V$. As a representation of $\mathfrak{sp}_{2n}\mathbb{C}$, consider the weight space decomposition of W:

$$W = \bigoplus_{\alpha\in\mathfrak{h}^*} W_\alpha.$$

Since W is a nontrivial representation of $\mathfrak{sp}_{2n}\mathbb{C}$, we can choose a weight $\alpha_1 = a_2L_2 + \dots + a_nL_n$ with $W_{\alpha_1} \ne 0$. Let $W_1 = W_{\alpha_1} \oplus W_{-\alpha_1}$, and let W_2 be the direct sum of other nonzero weight spaces of W. Then $W = W_1 \oplus W_2$ and $W_1 \perp W_2$ under the standard $\mathfrak{so}_{2m}\mathbb{C}$-invariant symmetric form Q, since Q is also invariant under $\mathfrak{sp}_{2n}\mathbb{C}$ and the sum of weights of any two nonzero weight vectors from W_1 and W_2 respectively is not zero. Since Q is non-degenerate, we can choose a basis $e_1, \dots, e_l$ of W_{α_1} and a basis $e'_1, \dots, e'_l$ of $W_{-\alpha_1}$, such that

$$Q(e_i, e_i) = Q(e'_i, e'_i) = 0, \quad \forall 1 \le i \le l,$$

and $\forall 1 \le i, j \le l$,

$$Q(e_i, e'_j) = \begin{cases} 1, & i = j; \\ 0, & i \ne j. \end{cases}$$

Define a linear transformation σ of W by $\sigma|_{W_2} = id$, $\sigma(e_i) = e_i$, $\sigma(e'_i) = e'_i$, $\forall 1 \leq i \leq l-1$, and σ interchanges e_l and e'_l. Then obviously $\sigma \in O_{2m}\mathbb{C}$ and $\det \sigma = -1$. By the proof of Lemma 5.1 and the definition of σ, we can find m linearly independent weight vectors $v_1, \ldots, v_m = e_l$ of $\mathfrak{sp}_{2n}\mathbb{C}$ such that

(1) $\sigma(v_i) = v_i$, $\forall 1 \leq i \leq m-1$;
(2) both $v_\lambda \cdot c_\lambda \in \mathbb{S}_{[\lambda]}W$ and $\sigma(v_\lambda \cdot c_\lambda) \in \mathbb{S}_{[\lambda]}W$ are nonzero;
(3) the weight of $v_\lambda \cdot c_\lambda$ is $b_1L_1 + \cdots + b_nL_n$, with $b_1 \geq \lambda_1$.

By the construction, we see the weight of $\sigma(v_\lambda \cdot c_\lambda)$ is $b'_1L_1 + \cdots + b'_nL_n$ with $b'_1 = b_1 \geq \lambda_1$. From this, we deduce that in V, there always exists a nonzero weight vector of $\mathfrak{sp}_{2n}\mathbb{C}$ with weight $c_1L_1 + \cdots + c_nL_n$, $c_1 \geq \lambda_1$. Then arguing as before, we get $\lambda_1 = 1$, hence $\lambda_1 = \cdots = \lambda_{m-1} = 1$, $\lambda_m = \pm 1$. Then it is not difficult to see that this cannot happen. So this case is excluded.

D3: If the highest weight of the irreducible $\mathfrak{so}_{2m}\mathbb{C}$-representation V is $a_1L'_1 + \cdots + a_mL'_m$, with each a_i a nonzero half integer, then consider the representation Sym^2V, and let V_1 be the irreducible $\mathfrak{so}_{2m}\mathbb{C}$-submodule of Sym^2V with highest weight $2a_1L'_1 + \cdots + 2a_mL'_m$. By the discussion of case **D2**, if we define the partition $\lambda = (2a_1, \ldots, 2a_{m-1}, 2|a_m|)$, then as $\mathfrak{so}_{2m}\mathbb{C}$-modules, V_1 is a direct summand of $\mathbb{S}_{[\lambda]}W$, where $W = \mathbb{C}^{2m}$ is the standard representation of $\mathfrak{so}_{2m}\mathbb{C}$, and there exists a nonzero weight vector of $\mathfrak{sp}_{2n}\mathbb{C}$ in V_1 with weight $c_1L_1 + \cdots + c_nL_n$, $c_1 \geq 2a_1$. Since as an $\mathfrak{sp}_{2n}\mathbb{C}$-module, V_1 is a direct summand of $Sym^2(\bigwedge^n \mathbb{C}^{2n})$, we find easily that $2a_1 \leq 2$. So $a_1 = \frac{1}{2}$ and V is a fundamental spin representation of $\mathfrak{so}_{2m}\mathbb{C}$. Then the dimension of V is a power of 2. On the other hand, $\dim V = \binom{2n}{n}$, which can never be a power of 2 if $n \geq 2$. So we can exclude this case. □

Proposition 5.3. *Let $\mathbb{V}$ be an absolutely irreducible $\mathbb{C}$-PVHS over a quasi-projective variety S with quasi-unipotent local monodromy around each component of $D = \bar{S} - S$ and $\mathbb{W}$ be a rank $2n$ local system of complex vector spaces over S. Suppose $\mathbb{V} \simeq \bigwedge^n \mathbb{W}$ as local systems. Then $\mathbb{W}$ admits a $\mathbb{C}$-PVHS structure such that the induced $\mathbb{C}$-PVHS on the wedge product $\bigwedge^n \mathbb{W}$ coincides with the given $\mathbb{C}$-PVHS on $\mathbb{V}$.*

Proof. Assume first $D = \emptyset$ to illustrate the idea. By the result of Simpson (cf. [35]), a complex local system admits a $\mathbb{C}$-PVHS structure if and only if it is fixed by the $\mathbb{C}^*$-action on the moduli space of semisimple representations of π_1. Now the wedge n product of $\mathbb{C}^{2n}$ induces a homomorphism $GL(2n) \xrightarrow{\rho_{\wedge^n}} GL(\binom{2n}{n})$, which has a finite kernel. This homomorphism induces the morphism

$$\phi_{\wedge^n} : \mathfrak{M}\big(\pi_1(S), GL(2n)\big)^{ss} \to \mathfrak{M}\left(\pi_1(S), GL\left(\binom{2n}{n}\right)\right)^{ss}$$

between the corresponding moduli spaces of semi-simple representations. By Corollary 9.18 in [36], the morphism $\phi_{\wedge^n}$ is finite. Note $\mathbb{C}^*$ acts on both moduli spaces continuously via the Hermitian Yang–Mills metric on the corresponding polystable Higgs bundles, and this action is compatible with $\phi_{\wedge^n}$. Thus since $[\mathbb{V} = \bigwedge^n \mathbb{W}]$ is fixed by the $\mathbb{C}^*$-action, it follows that $[\mathbb{W}]$ itself is fixed by the $\mathbb{C}^*$-action. Thus $\mathbb{W}$ admits a $\mathbb{C}$-PVHS structure such that it induces a $\mathbb{C}$-PVHS structure on $\bigwedge^n \mathbb{W}$. By Deligne's uniqueness theorem of $\mathbb{C}$-PVHS structures on an irreducible local system, it coincides with the given one on $\mathbb{V}$.

Consider the general case. First we show $\mathbb{W}$ has also quasi-unipotent local monodromy. Let $\gamma \in \pi_1(S)$ be a loop around a component of $D = \bar{S} - S$ and T be the corresponding local monodromy of $\mathbb{W}$. Then $\bigwedge^n T$ is quasi-unipotent. Since $T = T_s T_u$, where T_s (T_u) is the semisimple (unipotent) part of T, we can assume the eigenvalues of $\bigwedge^n T$ are all one. Let $\lambda_1, \ldots, \lambda_{2n}$ be the eigenvalues of T. Then $\{\lambda_{i_1} \cdots \lambda_{i_n},\ 1 \le i_1 < \cdots < i_n \le 2n\}$ are all one. It implies that $\lambda_1 = \cdots = \lambda_{2n} = \pm 1$. Thus T is quasi-unipotent. After a finite base change, we assume that the local monodromies are unipotent. By the result of Jost–Zuo [21], there exists a harmonic metric on the flat bundle $\mathbb{W}$ with finite energy which makes $\mathbb{W}$ into a Higgs bundle (F, η) on S. T. Mochizuki [25] has further analyzed the singularity of this harmonic metric and in particular shown that (F, η) admits a logarithmic extension $(\bar{F}, \bar{\eta})$ with logarithmic poles of Higgs field along D. By the uniqueness of such harmonic metrics, the induced metric $\bigwedge^n \mathbb{W}$ coincides with the Hodge metric given by the $\mathbb{C}$-PVHS $\mathbb{V}$.

Let $C \subset \bar{S}$ be a general complete intersection curve of a very ample divisor of $\bar{S}$. Set $C_0 = C - C \cap D$. Taking the restrictions, we obtain $[\mathbb{W}|_{C_0}] \in \mathfrak{M}(\pi_1(C_0), Gl(n))^{ss}$ such that $[\bigwedge^n \mathbb{W}|_{C_0}] \in \mathfrak{M}(\pi_1(C_0), Gl(\binom{2n}{n}))^{ss}$. By Simpson [34], there exists Hermitian–Yang–Mills metrics on polystable Higgs bundles on C with logarithmic poles of Higgs field along $C \cap D$. The $\mathbb{C}^*$-action can be defined on both spaces of semisimple representations on C_0 via a Hermitian–Yang–Mills metric on $(\bar{F}, t\bar{\eta})$, $t \in \mathbb{C}^*$. By the same arguments as above, we show that the restriction of $(\bar{F}, \bar{\eta})$ to C_0 is a fixed point of the $\mathbb{C}^*$-action. If we choose C_0 sufficiently ample, then $(\bar{F}, \bar{\eta})$ is also a fixed point of the $\mathbb{C}^*$-action. Again by Simpson [35], $\mathbb{W}$ admits a $\mathbb{C}$-PVHS structure. This concludes the proof. □

Now we consider the monodromy representation associated to $\tilde{\mathbb{V}}_{AR}$:

$$\rho : \pi_1(\mathfrak{M}_{AR}, s) \to Aut(V, Q).$$

Here and in the following part of this section we keep the notation as in Section 1.

Theorem 5.4. *If $n \ge 3$, then $Mon^0 = Aut^0(V, Q)$. That is, the monodromy group of $\tilde{\mathbb{V}}_{AR}$ is Zariski dense in $Aut^0(V, Q)$.*

Proof. We first show the monodromy representation is absolutely irreducible. For otherwise there would exist local systems $\mathbb{V}_1$, $\mathbb{V}_2$ of complex linear spaces over $\mathfrak{M}_{AR}$, such that $\tilde{\mathbb{V}}_{AR} \otimes \mathbb{C} = \mathbb{V}_1 \oplus \mathbb{V}_2$. Then by a result of P. Deligne (cf. [9]), there exist $\mathbb{C}$-PVHS structures on $\mathbb{V}_1$ and $\mathbb{V}_2$ such that $\tilde{\mathbb{V}}_{AR} \otimes \mathbb{C} = \mathbb{V}_1 \oplus \mathbb{V}_2$ as $\mathbb{C}$-PVHS. So the Higgs bundle

$(E, \theta) = (\bigoplus E^{p,q}, \bigoplus \theta^{p,q})$ associated to $\tilde{\mathbb{V}}_{AR} \otimes \mathbb{C}$ admits a direct sum decomposition. On the other hand, Proposition 7.11 and Proposition 7.13 show that $\forall 1 \leq q \leq n$, the Higgs map

$$Sym^q \mathcal{T}_{\mathfrak{M}_{AR}} \xrightarrow{\theta^q} Hom\big(E^{n,0}, E^{n-q,q}\big) = E^{n-q,q}$$

is surjective. This is a contradiction with the decomposition of (E, θ). So we get the monodromy representation is absolutely irreducible.

Consider the universal family of hyperelliptic curves $g : \mathcal{C} \to \mathfrak{M}_{hp}$. By Proposition 2.8, we have an inclusion $\mathfrak{M}_{hp} \subset \mathfrak{M}_{AR}$ and $\tilde{\mathbb{V}}_{AR}|_{\mathfrak{M}_{hp}} = \bigwedge^n \mathbb{V}_C$, where $\mathbb{V}_C$ is the weight one $\mathbb{Q}$-PVHS associated to g. Suppose the base point $s \in \mathfrak{M}_{hp}$. Denote the monodromy representation of g by

$$\tau : \pi_1(\mathfrak{M}_{hp}, s) \to Sp(2n, \mathbb{Q}).$$

Then we have a commutative diagram

$$\begin{array}{ccc} \pi_1(\mathfrak{M}_{hp}, s) & \xrightarrow{\tau} & Sp(2n, \mathbb{Q}) \\ \downarrow & & \downarrow \rho_{\wedge^n} \\ \pi_1(\mathfrak{M}_{AR}, s) & \xrightarrow{\rho} & Aut(V, Q) \end{array}$$

where $\rho_{\wedge^n}$ is the homomorphism induced by the n-th wedge product of the standard representation of $Sp(2n, \mathbb{Q})$.

By Theorem 1 of [1], $\tau(\pi_1(\mathfrak{M}_{hp}, s))$ is Zariski dense in $Sp(2n, \mathbb{Q})$. So we get the commutative diagram

$$\begin{array}{ccccc} Sp(2n, \mathbb{Q}) & & \xrightarrow{\rho_{\wedge^n}} & & Aut(V, Q) \\ & \searrow & & \nearrow & \\ & & Mon & & \end{array}$$

Note the complexification

$$Aut^0(V, Q)_{\mathbb{C}} = \begin{cases} Sp(\binom{2n}{n}, \mathbb{C}), & n \text{ odd;} \\ SO(\binom{2n}{n}, \mathbb{C}), & n \text{ even.} \end{cases}$$

By a result of Deligne (cf. Corollary 4.2.9 in [7]), Mon is semi-simple. Then apply Proposition 5.2 to the Lie algebra version of the commutative diagram above, we get either $Mon^0 = Aut^0(V, Q)$ or (after a possible finite étale base change) there exists a local system $\mathbb{W}$ over $\mathfrak{M}_{AR}$ such that $\tilde{\mathbb{V}}_{AR} \otimes \mathbb{C} = \bigwedge^n \mathbb{W}$. In the latter case, Proposition 5.3 implies that there exists a $\mathbb{C}$-PVHS structure on $\mathbb{W}$ such that $\tilde{\mathbb{V}}_{AR} \otimes \mathbb{C} = \bigwedge^n \mathbb{W}$ as

$\mathbb{C}$-PVHS over $\mathfrak{M}_{AR}$. This would imply $\tilde{\mathbb{V}}_{AR}$ factors through the $\mathbb{C}$-PVHS $\mathbb{V}_{can}$ over the type A symmetric domain $D^I_{n,n}$. But this can not happen by Theorem 4.2. So we get $Mon^0 = Aut^0(V, Q)$. □

Corollary 5.5. *If $n \geq 3$, then the special Mumford–Tate group of a general member X in $\mathcal{M}_{n,2n+2}$ is $Aut^0(V, Q)$.*

Proof. Consider the good family $\tilde{f} : \tilde{\mathcal{X}}_{AR} \to \mathfrak{M}_{AR}$ and the associated weight n $\mathbb{Q}$-PVHS $\tilde{\mathbb{V}}_{AR}$. By Deligne and Schoen (see for example Lemma 2.4 in [40]), the identity component of the $\mathbb{Q}$-Zariski closure of the monodromy group is a normal subgroup of the special Mumford–Tate group $Hg(\tilde{\mathbb{V}}_{AR})$ of $\tilde{\mathbb{V}}_{AR}$, which is equal to the special Mumford–Tate group of a general closed fiber of $\tilde{f}$. We can easily deduce from Theorem 5.4 that $Mon^0 = Hg(\tilde{\mathbb{V}}_{AR}) = Aut^0(V, Q)$. Then the corollary follows since the moduli map of $\tilde{f}$ is dominant. □

Corollary 5.6. *Let $f : \mathcal{X} \to S$ be a good family of $\mathcal{M}_{n,2n+2}$ and $\mathbb{W}$ be the associated weight n $\mathbb{Q}$-PVHS. Let $s \in S$ be a base point and let*

$$\rho : \pi_1(S, s) \to Aut(V, Q)$$

be the monodromy representation associated to $\mathbb{W}$, where $V = \mathbb{W}_s$ and Q is the bilinear form on V induced by the cup product. If $n \geq 3$, then the image of ρ is Zariski dense in $Aut^0(V, Q)$.

Proof. By Corollary 5.5 and the results of Deligne and Schoen we used above, the identity component of the $\mathbb{Q}$-Zariski closure of the monodromy group is a normal subgroup of $Aut^0(V, Q)$. Then the corollary follows from the fact that $Aut^0(V, Q)$ is an almost simple algebraic group. □

6. Gross's geometric realization problem

Motivated by Theorem 4.2, we consider a particular subclass of moduli spaces of hyperplane arrangements in projective spaces, namely those related to the CY varieties, and ask whether their (sub) VHSs will realize the canonical PVHSs over type A bounded symmetric domains. More precisely, the question of Gross in this setting is described as follows.

Question 6.1. (See B. Gross [20].) Let $D = G_{\mathbb{R}}/K$ be an irreducible type A bounded symmetric domain and $\mathbb{V}_{can}$ be the canonical $\mathbb{C}$-PVHS of CY type over D. Does there exist a good family of CY manifolds $f : X \to S$ which is obtained from a crepant resolution of cyclic covers of $\mathbb{P}^n$ branched along m hyperplanes in general position, where $S = D/\Gamma$ with $\Gamma \subset G_{\mathbb{Q}}$ an arithmetic subgroup, such that $\mathbb{V}_{can}$ is the pull back to D of any sub $\mathbb{C}$-PVHS of the $\mathbb{Q}$-PVHS $\mathbb{V}_X$ attached to f?

Gross stated his question only for the tube domain case. However, it is equally interesting to consider other cases, e.g. the complex ball. Then one has to extend the construction of Gross of $\mathbb{V}_{can}$ (in this case and only in this case it is an $\mathbb{R}$-PVHS) to the remaining cases. This was done in [33]. A recent work of Friedmann–Laza [15] showed that $\mathbb{C}$-PVHS of CY types over bounded symmetric domains come basically from the $\mathbb{V}_{can}$ of Gross and Sheng–Zuo. Note that, in the tube domain case, the affirmative answer to the above question is in fact a weaker reformulation of Dolgachev's conjecture. What we intend to do in this section is to make a definite and complete answer to the geometric realization problem for type A domains with moduli spaces of CY manifolds coming hyperplane arrangements as potential candidate in mind. In fact, we will prove results analogous to Theorem 4.2 and Theorem 5.4. Let us fix the following notation throughout this section.

- m, n, k, r are positive integers such that $m = kr$ and $n = m - k - 1$.
- $\zeta = \exp(\frac{2\pi i}{r})$ is a primitive r-root of unit.

We call an ordered arrangement $\mathfrak{A} = (H_1, \ldots, H_m)$ of m hyperplanes in $\mathbb{P}^n$ in general position if no $n + 1$ of the hyperplanes intersect in a point. The hyperplanes of the arrangement $\mathfrak{A}$ determine a divisor $H = \sum_{i=1}^{m} H_i$ on $\mathbb{P}^n$. As the same as the $r = 2$ case, this divisor determines a unique r-fold cyclic cover $\pi : X \to \mathbb{P}^n$ that ramifies over H and the canonical line bundle of X is trivial. In the same way as Section 2.2, we can construct the Kummer cover of X which is smooth projective, so the Hodge structure on $H^n(X, \mathbb{Q})$ is a weight n pure $\mathbb{Q}$-Hodge structure. In our earlier work [32], we constructed a crepant resolution $\tilde{X}$ of X. Thus the projective variety $\tilde{X}$ is a smooth CY manifold.

Lemma 6.2. *The crepant resolution $\psi : \tilde{X} \to X$ induces isomorphisms:*

(1) $H^{p,q}(X, \mathbb{C}) \xrightarrow{\sim} H^{p,q}(\tilde{X}, \mathbb{C})$, $\forall p + q = n$, $p \neq q$.
(2) $\psi^* : H^n(X, \mathbb{C})_{(i)} \xrightarrow{\sim} H^n(\tilde{X}, \mathbb{C})_{(i)}$, $\forall 1 \leq i \leq r - 1$.

Proof. (1) is just Proposition 2.8 in [32].

(2) can be proved in the same way as Proposition 2.8 in [32], by replacing everything by its i-eigenspace, and noting that by induction, at every blow-up step, $\forall 1 \leq i \leq r - 1$, $H^n(E)_{(i)} = 0$, where E is the exceptional divisor. □

Let $\mathfrak{M}_{AR}$ denote the coarse moduli space of ordered arrangements of m hyperplanes in $\mathbb{P}^n$ in general position, and let $\mathcal{M}_{n,m}$ denote the coarse moduli space of $\tilde{X}$. In the same way as the $r = 2$ case, $\mathfrak{M}_{AR}$ can be realized as an open subvariety of the affine space $\mathbb{C}^{n(k-1)}$ and it admits a natural family $f : \mathcal{X}_{AR} \to \mathfrak{M}_{AR}$, where each fiber $f^{-1}(\mathfrak{A})$ is the r-fold cyclic cover of $\mathbb{P}^n$ branched along the hyperplane arrangement $\mathfrak{A}$. It is easy to see the crepant resolution in [32] gives a simultaneous crepant resolution $\pi : \tilde{\mathcal{X}}_{AR} \to \mathcal{X}_{AR}$ for the family f. We denote this smooth projective family of CY manifolds by $\tilde{f}:\tilde{\mathcal{X}}_{AR} \to \mathfrak{M}_{AR}$.

Let $\mathfrak{M}_C$ be the moduli space of ordered distinct m points on $\mathbb{P}^1$ and $g:\mathcal{C} \to \mathfrak{M}_C$ be the universal family of r-fold cyclic covers of $\mathbb{P}^1$ branched at m distinct points.

We consider the $\mathbb{Q}$-VHS attached to the three families f, $\tilde{f}$, g:

$$\mathbb{V} := R^n f_* \mathbb{Q}, \qquad \tilde{\mathbb{V}} := \left(R^n \tilde{f}_* \mathbb{Q}\right)_{pr}, \qquad \mathbb{V}_C := R^1 g_* \mathbb{Q}.$$

Since $\mathbb{Z}/r\mathbb{Z}$ acts naturally on the three families, we have a decomposition of the three $\mathbb{Q}$-VHS into eigen–sub $\mathbb{C}$-VHS:

$$\mathbb{V} \otimes \mathbb{C} = \bigoplus_{i=0}^{r-1} \mathbb{V}_{(i)}, \qquad \tilde{\mathbb{V}} \otimes \mathbb{C} = \bigoplus_{i=0}^{r-1} \tilde{\mathbb{V}}_{(i)}, \qquad \mathbb{V}_C \otimes \mathbb{C} = \bigoplus_{i=0}^{r-1} \mathbb{V}_{C(i)}.$$

Proposition 6.3. *$\forall 1 \le i \le r-1$, we have:*

(1) *the crepant resolution induces an isomorphism of $\mathbb{C}$-PVHS: $\tilde{\mathbb{V}}_{(i)} \simeq \mathbb{V}_{(i)}$;*
(2) *there is an embedding $\mathfrak{M}_C \hookrightarrow \mathfrak{M}_{AR}$ such that $\mathbb{V}_{(i)}|_{\mathfrak{M}_C} \simeq \bigwedge^n \mathbb{V}_{C(i)}$;*
(3) *as a $\mathbb{C}$-PVHS of weight n, the Hodge numbers of $\mathbb{V}_{(1)}$ are:*

$$h^{n-q,q} = \begin{cases} \binom{n}{q}\binom{k-1}{q}, & 0 \le q \le k-1; \\ 0, & k \le q \le n. \end{cases}$$

Proof. (1) follows from Lemma 6.2.

The proof of (2) and (3) is similar to that of Proposition 2.8. □

Analogous to the $r = 2$ case, for any $2 \le k \le n+1$, on the type A symmetric domain $D^I_{n,k-1} = \frac{SU(n,k-1)}{S(U(n)\times U(k-1))}$, there is a canonical $\mathbb{C}$-PVHS $\mathbb{V}_{can}$, which has the same Hodge numbers as $\mathbb{V}_{(1)}$. For details of the construction, one can see [33]. From the construction, one can deduce in the same way as Proposition 3.3 that

Proposition 6.4. *The first characteristic variety of $\mathbb{V}_{can}$ is $C_{1,s} \simeq \mathbb{P}^{n-1} \times \mathbb{P}^{k-2}$, $\forall s \in D^I_{n,k-1}$.*

In order to calculate the characteristic varieties of $\mathbb{V}$, we use the same method as the $r = 2$ case and reduce the situation to a calculation in a Jacobian ring. The construction and properties of Jacobian rings are similar to the $r = 2$ case. So we only summarize and state the results we need.

For each parameter $a \in \mathbb{C}^{n(k-1)}$, there is a bi-graded $\mathbb{C}$-algebra $R = \bigoplus_{p,q\ge 0} R_{(p,q)}$ and the group $N = \bigoplus_{j=0}^{m-1} \mathbb{Z}/r\mathbb{Z}$ acts on R, preserving the grading. Consider the summation homomorphism $\bigoplus_{j=0}^{m-1} \mathbb{Z}/r\mathbb{Z} \xrightarrow{\sum} \mathbb{Z}/r\mathbb{Z}$ and let N_1 be the kernel of this homomorphism. Define $R^{N_1} = \bigoplus_{p,q\ge 0} R^{N_1}_{(p,q)}$ to be the N_1-invariant part of R, then the cyclic group

$\mathbb{Z}/r\mathbb{Z} = N/N_1 = \langle\sigma\rangle$ acts on R^{N_1}. Let $R^{N_1}_{(p,q)(i)} = \{\alpha \in R^{N_1}_{(p,q)} \mid \sigma(\alpha) = \zeta^i\alpha\}$ be the i-th eigenspace of $R^{N_1}_{(p,q)}$ under the action of $\mathbb{Z}/r\mathbb{Z}$. Then analogous to Proposition 7.11, we have

Proposition 6.5.

(1) $\forall 1 \le i \le r-1$, $\forall 0 \le q \le n$, $H^{n-q,q}(X)_{(i)} \simeq R^{N_1}_{(q,qr)(i-1)}$, *where* X *is the cyclic cover of* $\mathbb{P}^n$ *corresponding to the parameter* $a \in \mathfrak{M}_{AR} \subset \mathbb{C}^{n(k-1)}$. *In particular,* $H^{n-q,q}(X)_{(1)} \simeq R^{N_1}_{(q,qr)(0)} = R^N_{(q,qr)}$;

(2) $\forall 1 \le i \le r-1$, $\forall 0 \le q \le n$, *we have a commutative diagram*

$$\begin{array}{ccc} T_{\mathfrak{M}_{AR},a} \otimes H^{n-q,q}(X)_{(i)} & \xrightarrow{\theta^{n-q,q}} & H^{n-q-1,q+1}(X)_{(i)} \\ \downarrow \simeq & & \downarrow \simeq \\ R^N_{(1,r)} \otimes R^{N_1}_{(q,qr)(i-1)} & \longrightarrow & R^{N_1}_{(q+1,qr+r)(i-1)} \end{array}$$

Here the lower horizontal arrow is the ring multiplication map.

For a parameter $a \in \mathbb{C}^{n(k-1)}$, we define a subvariety in the projective space $\mathbb{P}(R^N_{(1,r)})$ as follows

$$C'_{1,a} := \{[\alpha] \in \mathbb{P}(R^N_{(1,r)}) \mid \alpha^2 = 0 \in R^N_{(2,2r)}\}.$$

We have the following upper bound of the dimension of $C'_{1,a}$.

Proposition 6.6. *If* $n \ge 3$, $k \ge 3$, *then for generic* $a \in \mathbb{C}^{n(k-1)}$, $\dim C'_{1,a} \le 2$.

For the proof of this proposition, one can follow the proof of Proposition 7.16 without any difficulty.

The following theorem is a generalization of Theorem 4.2.

Theorem 6.7. *If* $n \ge 3$, $k \ge 3$, *then:*

(1) $\tilde{\mathcal{X}}_{AR} \xrightarrow{\tilde{f}} \mathfrak{M}_{AR}$ *is a good family for the coarse moduli space* $\mathcal{M}_{n,m}$;

(2) *Let* $f : \mathcal{X} \to S$ *be a good family of* $\mathcal{M}_{n,m}$ *and* $\mathbb{W} = (R^n f_*\mathbb{Q})_{pr}$ *be the associated weight* n $\mathbb{Q}$*-PVHS. Then any sub* $\mathbb{C}$*-VHS* $\tilde{\mathbb{W}}$ *of Calabi–Yau type in* $\mathbb{W} \otimes \mathbb{C}$ *does not factor through the* $\mathbb{C}$*-PVHS* $\mathbb{V}_{can}$ *over the type A symmetric domain* $D^I_{n,k-1}$. *In particular, the first eigenspace* $\tilde{\mathbb{V}}_{(1)}$ *of* $\tilde{\mathbb{V}}$ *associated to the family* $\tilde{f}$ *in (1) does not factor through* $\mathbb{V}_{can}$ *over* $D^I_{n,k-1}$.

Proof. (1) For each fiber $\tilde{X}$ of the family $\tilde{f}$, by Lemma 6.2 and Proposition 6.5, we can identify $H^{n-1,1}(\tilde{X}, \mathbb{C})$ with the Jacobian ring $R^N_{(1,r)}$ and the Higgs map

$$T_{\mathfrak{M}_{AR},s} \otimes H^{n,0}(\tilde{X}, \mathbb{C}) \xrightarrow{\theta^{n,0}} H^{n-1,1}(\tilde{X}, \mathbb{C})$$

can be identified with the multiplication map

$$R^N_{(1,r)} \otimes R^N_{(0,0)} \to R^N_{(1,r)}$$

which is obviously an isomorphism. By the local Torelli theorem for CY manifolds, we get the Kodaira–Spencer map of $\tilde{f}$ is an isomorphism at each point $s \in \mathfrak{M}_{AR}$. This shows $\tilde{f}$ is a good family.

(2) It suffices to prove the statement for the good family $\tilde{\mathcal{X}}_{AR} \xrightarrow{\tilde{f}} \mathfrak{M}_{AR}$. Recall that for this good family, there is a decomposition of the associated $\mathbb{C}$-VHS: $\tilde{\mathbb{V}} \otimes \mathbb{C} = \bigoplus_{i=0}^{r-1} \tilde{\mathbb{V}}_{(i)}$. Suppose we have a sub $\mathbb{C}$-VHS $\tilde{\mathbb{W}}$ of $\tilde{\mathbb{V}}$, a nonempty (analytically) open subset $U \subset S$ and a holomorphic map $j : U \to D^I_{n,k-1}$, such that $\tilde{\mathbb{W}}$ is of Calabi–Yau type, and $\tilde{\mathbb{W}}|_U \simeq j^*\mathbb{V}_{can}$ as $\mathbb{C}$-VHS. Since both $\tilde{\mathbb{W}}$ and $\mathbb{V}_{can}$ are of Calabi–Yau type, j is a local isomorphism, so we can assume j is an open embedding of complex manifolds. Let $(\tilde{E}, \tilde{\theta})$ and (E_{can}, θ_{can}) be the Higgs bundles corresponding to $\tilde{\mathbb{W}}$ and $\mathbb{V}_{can}$ respectively, then under the embedding j, $(\tilde{E}, \tilde{\theta})|_U = (E_{can}, \theta_{can})|_U$. It can be seen easily from the description of (E_{can}, θ_{can}) (cf. Section 4.1 in [33]) that $\forall s \in D^I_{n,k-1}$, $\forall q \geq 0$, the map $\theta^q : Sym^q T_{D^I_{n,k-1},s} \otimes E^{n,0}_{can} \to E^{n-q,q}_{can}$ is surjective. Moreover, by Proposition 6.4, the first characteristic variety of $\mathbb{V}_{can}$ at each point of $D^I_{n,k-1}$ is isomorphic to $\mathbb{P}^{n-1} \times \mathbb{P}^{k-2}$. So the Higgs bundle $\tilde{E}$ also satisfies the following two properties:

(1) $\forall s \in U$, $\forall q \geq 0$, the map $\theta^q : Sym^q T_{U,s} \otimes \tilde{E}^{n,0} \to \tilde{E}^{n-q,q}$ is surjective.
(2) $\forall s \in U$, the first characteristic variety of $\tilde{\mathbb{W}}$ at s is isomorphic to $\mathbb{P}^{n-1} \times \mathbb{P}^{k-2}$.

Since the $\mathbb{Z}/r\mathbb{Z}$-invariant part $\tilde{\mathbb{V}}_{(0)}$ of $\tilde{\mathbb{V}}$ is obviously a constant $\mathbb{C}$-VHS, (1) implies that we must have $\tilde{\mathbb{W}} \subset \bigoplus_{i=1}^{r-1} \tilde{\mathbb{V}}_{(i)}$. Then by Proposition 6.5 and taking into account the Hodge numbers of $\mathbb{V}_{can}$, we can translate the properties of $\tilde{E}$ above to the following properties of the Jacobian ring R: there exists a nonempty open subset U of the parameter space $\mathbb{C}^{n(k-1)}$, and for any parameter $a \in U$, there is an element $\beta \in R^{N_1}$, such that:

(1)′ $\forall 0 \leq q \leq k-1$, the dimension of the linear space $J_q := \{\beta \cdot \gamma \in R^{N_1} \mid \gamma \in Sym^q R^N_{(1,r)}\}$ is $\binom{n}{q}\binom{k-1}{q}$;
(2)′ the variety $\tilde{C}_{1,a} := \{\alpha \in \mathbb{P}(R^N_{(1,r)}) \mid \beta \cdot \alpha^2 = 0\}$ is isomorphic to $\mathbb{P}^{n-1} \times \mathbb{P}^{k-2}$.

It is easy to see from the definition of Jacobian ring that

$$J_q = \{\beta \cdot \gamma \in R^{N_1} \mid \gamma \in Sym^q R^N_{(1,r)}\} = \{\beta \cdot \gamma \in R^{N_1} \mid \gamma \in R^N_{(q,qr)}\}.$$

Note also by Proposition 6.5, the dimension of the linear space $R^N_{(q,qr)}$ is $\binom{n}{q}\binom{k-1}{q}$, equal to that of J_q. So we get the map of multiplication by β:

$$R^N_{(q,qr)} \xrightarrow{\cdot\beta} J_q$$

is a linear isomorphism. From this by taking $q = 2$ we deduce that

$$\tilde{C}_{1,a} = \{\alpha \in \mathbb{P}(R^N_{(1,r)}) \mid \beta \cdot \alpha^2 = 0\} = \{\alpha \in \mathbb{P}(R^N_{(1,r)}) \mid \alpha^2 = 0\} = C'_{1,a}.$$

Then $(2)'$ implies the dimension of $C'_{1,a}$ is $n + k - 3 \geq 3$, and this contradicts with Proposition 6.6. So we finally get that any sub $\mathbb{C}$-VHS $\mathbb{W}$ of Calabi–Yau type in $\mathbb{V}$ does not factor through $\mathbb{V}_{can}$. □

The above theorem leaves tiny possibilities for a positive answer of Gross's question, since for simple reasons, it is easy to exclude any other type A domain but $D^I_{n,m-n-2}$ for $\mathcal{M}_{n,m}$. Indeed, we can list each of the remaining cases as follows:

$\mathcal{M}_{1,4}$: This is the starting point.

$\mathcal{M}_{2,6}$: This is a four dimensional family of K3 surfaces with generic Picard number 16. A detailed study of this family was done in Matsumoto–Sasaki–Yoshida [24]. The connection of the weight two Hodge structure of such a K3 surface with the weight one Hodge structure of an abelian variety in view of Kuga–Satake construction was geometrically realized by K. Paranjape [29].

$\mathcal{M}_{3,6}, \mathcal{M}_{5,8}, \mathcal{M}_{9,12}$: These are only cases with $n \geq 3$ and $m - n = 3$ which can realize the problem of Gross (the partial compactification issue however remains to be done). Our earlier work [32] studied the series $\mathcal{M}_{n,n+3}, n \geq 3$, and related the Hodge structure with that in Deligne–Mostow [10]. Besides [10], the classification was obtained thanks to the works of Mostow [27,28] on discrete subgroups of the automorphism group of a complex ball.

Now we study the monodromy representation of the family $\tilde{\mathcal{X}}_{AR} \xrightarrow{\tilde{f}} \mathfrak{M}_{AR}$. Suppose $r \geq 3$, then there is a weight n $\mathbb{R}$-VHS $\tilde{\mathbb{V}}_{\mathbb{R},(1)}$ such that

$$\tilde{\mathbb{V}}_{\mathbb{R},(1)} \otimes \mathbb{C} = \tilde{\mathbb{V}}_{(1)} \oplus \tilde{\mathbb{V}}_{(r-1)}$$

and the polarization on $\tilde{\mathbb{V}}$ induces a parallel hermitian form h on $\tilde{\mathbb{V}}_{(1)}$ of signature (p, q). Here by Proposition 6.3

$$p = \sum_{i=0}^{[\frac{k-1}{2}]} \binom{n}{2i}\binom{k-1}{2i}, \qquad q = \sum_{i=0}^{[\frac{k}{2}]-1} \binom{n}{2i+1}\binom{k-1}{2i+1}.$$

Take a base point $s \in \mathfrak{M}_{AR}$ and consider the real monodromy representation

$$\rho : \pi_1(\mathfrak{M}_{AR}, s) \to GL(V)$$

where V is the fiber of $\tilde{\mathbb{V}}_{\mathbb{R},(1)}$ over s. Let $Mon_{\mathbb{R}}$ be the Zariski closure of $\rho(\pi_1(\mathfrak{M}_{AR}, s))$ in the real algebraic group $GL(V)$. The following theorem is parallel to Theorem 5.4.

Theorem 6.8. *Suppose $r \geq 3$ and $k \geq 3$, then the identity component $Mon^0_{\mathbb{R}}$ of $Mon_{\mathbb{R}}$ is isomorphic to $SU(p, q)$.*

Before starting the proof of this theorem, we state two propositions parallel to Proposition 5.2 and Proposition 5.3.

Proposition 6.9. *The n-th wedge product $V = \bigwedge^n \mathbb{C}^{n+k-1}$ of the standard representation of $\mathfrak{sl}_{n+k-1}\mathbb{C}$ induces an embedding $\mathfrak{sl}_{n+k-1}\mathbb{C} \hookrightarrow \mathfrak{sl}_{p+q}$. Suppose $k \geq 3$ and $\mathfrak{g}$ is a complex semi-simple Lie algebra lying between $\mathfrak{sl}_{n+k-1}\mathbb{C}$ and $\mathfrak{sl}_{p+q}$ such that the induced representation of $\mathfrak{g}$ on V is irreducible, then $\mathfrak{g}$ is one of the following:*

(1) $\mathfrak{sl}_{p+q}$,
(2) $\mathfrak{sl}_{n+k-1}\mathbb{C}$, *in which case the induced representation of $\mathfrak{g}$ on V is isomorphic to the n-th wedge product of the standard representation on $\mathbb{C}^{n+k-1}$.*

Proof. The proof of Proposition 5.2 goes through without difficulty. □

Proposition 6.10. *Let $\mathbb{V}$ be a $\mathbb{C}$-PVHS over a quasi-projective variety S with quasi-unipotent local monodromy around each component of $D = \bar{S} - S$ and $\mathbb{W}$ be a rank $n+k-1$ local system of complex vector spaces over S. Suppose $\mathbb{V} \simeq \bigwedge^n \mathbb{W}$ as local systems and $k \geq 3$. Then $\mathbb{W}$ admits a $\mathbb{C}$-PVHS structure such that the induced $\mathbb{C}$-PVHS on the wedge product $\bigwedge^n \mathbb{W}$ coincides with the given $\mathbb{C}$-PVHS on $\mathbb{V}$.*

Proof. The proof of Proposition 5.3 goes through without difficulty. □

Now we can proceed to the proof of Theorem 6.8.

Proof of Theorem 6.8. Consider the family of curves $g : \mathcal{C} \to \mathfrak{M}_C$ and assume the base point $s \in \mathfrak{M}_C \hookrightarrow \mathfrak{M}_{AR}$. Let C and X be the fibers over s of the families $\mathcal{C} \xrightarrow{g} \mathfrak{M}_C$ and $\mathcal{X}_{AR} \xrightarrow{f} \mathfrak{M}_{AR}$ respectively. By Proposition 6.3,

$$H^n(X, \mathbb{C})_{(1)} = \bigwedge\nolimits^n H^1(C, \mathbb{C})_{(1)}.$$

The embedding $\mathbb{R} \hookrightarrow \mathbb{C}$ allows to consider $H^1(C, \mathbb{C})_{(1)}$ and $H^n(X, \mathbb{C})_{(1)}$ as $\mathbb{R}$-vector spaces. Consider the monodromy representation of the family $\mathcal{X}_{AR} \xrightarrow{f} \mathfrak{M}_{AR}$:

$$\tau : \pi_1(\mathfrak{M}_{AR}, s) \to GL_{\mathbb{R}}\big(H^n(X, \mathbb{C})_{(1)}\big).$$

Since $\tilde{\mathbb{V}}_{(1)} = \mathbb{V}_{(1)}$, and $\tilde{\mathbb{V}}_{\mathbb{R},(1)} \simeq \tilde{\mathbb{V}}_{(1)}$ as $\mathbb{R}$-local systems, we can identify $Mon_{\mathbb{R}}$ with the Zariski closure of $\tau(\pi_1(\mathfrak{M}_{AR}, s))$ in $GL_{\mathbb{R}}(H^n(X, \mathbb{C})_{(1)})$. By Theorem 5.1.1 in [31], for the family $g : \mathcal{C} \to \mathfrak{M}_C$, the identity component of the Zariski closure of the monodromy representation

$$\pi_1(\mathfrak{M}_C, s) \to GL_{\mathbb{R}}\big(H^1(C, \mathbb{C})_{(1)}\big)$$

is $SU(n, k-1)$. So similar to the proof of Theorem 5.4 we have a commutative diagram

$$\begin{array}{ccc} SU(n,k-1) & \xrightarrow{\rho_{\wedge^n}} & Aut(H^n(X,\mathbb{C})_{(1)},h) = U(p,q) \\ & \searrow \quad \nearrow & \\ & Mon^0_{\mathbb{R}} & \end{array}$$

where the homomorphism $\rho_{\wedge^n}$ is induced by the n-th wedge product of the standard representation of $SU(n,k-1)$. Note that there exists a parallel $\mathbb{Z}[\zeta]$-lattice $H^n(X,\mathbb{Z}[\zeta])_{(1)}$ inside $H^n(X,\mathbb{C})_{(1)}$. From this one can deduce that $Mon^0_{\mathbb{R}} \subset SU(p,q)$. Arguing in the same way as the proof of Theorem 5.4, we can show the complex representation of $Mon^0_{\mathbb{R}}$ on $H^n(X,\mathbb{C})_{(1)}$ is irreducible. Since $Mon_{\mathbb{R}}$ is semi-simple by a result of Deligne (cf. Corollary 4.2.9 in [7]), by taking the complexification and using Proposition 6.9, Proposition 6.10 in the same way as Theorem 5.4, we get $Mon^0_{\mathbb{R}} = SU(p,q)$. □

7. Two calculations

In this section, we do two concrete calculations, one is for the primitive Hodge numbers of the CY manifold $\tilde{X}$, and the other is for the dimension of the characteristic variety $C_{1,a}$.

7.1. Calculation of Hodge numbers

Let us keep the notation as in Section 2.1. We will calculate the primitive Hodge numbers of $\tilde{X}$ and complete the proof of Proposition 2.2. Let $\sigma : \tilde{\mathbb{P}}^n \to \mathbb{P}^n$ be the composite of all blow-ups and $\tilde{H}$ the strict transform of H. Let $\tilde{\mathcal{L}}$ be a line bundle on $\tilde{X}$ such that $\tilde{\mathcal{L}}^2 = \mathcal{O}_{\tilde{X}}(\tilde{H})$. As

$$\tilde{\pi}_*\Omega^p_{\tilde{X}} = \Omega^p_{\tilde{\mathbb{P}}^n} \oplus \Omega^p_{\tilde{\mathbb{P}}^n}(\log \tilde{H}) \otimes \tilde{\mathcal{L}}^{-1},$$

it follows that

$$H^{p,q}_{prim}(\tilde{X}) = H^q\big(\tilde{\mathbb{P}}^n, \Omega^p_{\tilde{\mathbb{P}}^n}(\log \tilde{H}) \otimes \tilde{\mathcal{L}}^{-1}\big).$$

Claim 7.1. *For all $k \neq q$, $H^k(\Omega^p_{\tilde{\mathbb{P}}^n}(\log \tilde{H}) \otimes \tilde{\mathcal{L}}^{-1}) = 0$. Therefore,*

$$\chi\big(\Omega^p_{\tilde{\mathbb{P}}^n}(\log \tilde{H}) \otimes \tilde{\mathcal{L}}^{-1}\big) = (-1)^q \dim H^q\big(\Omega^p_{\tilde{\mathbb{P}}^n}(\log \tilde{H}) \otimes \tilde{\mathcal{L}}^{-1}\big).$$

Proof. This is a direct application of the vanishing result [14, Proposition 6.1]. □

Next, we show that the Euler characteristic keep unchanged under resolution. Namely, we have the following

Claim 7.2. *Put $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(n+1)$. It holds that*

$$\chi\big(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(\log H) \otimes \mathcal{L}^{-1}\big) = \chi\big(\tilde{\mathbb{P}}^n, \Omega^p_{\tilde{\mathbb{P}}^n}(\log \tilde{H}) \otimes \tilde{\mathcal{L}}^{-1}\big).$$

The claim follows from a general consideration, which we postpone after stating the last

Claim 7.3. *One has the following formula:*

$$\chi\big(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(\log H)\otimes\mathcal{L}^{-1}\big) = (-1)^q\binom{n}{p}^2.$$

It is clear that the above claims imply Proposition 2.2.

Proof of Claim 7.2. Let X be an n-dimensional smooth projective variety, $D_X\subset X$ an SNCD(simple normal crossing divisor). Let $Z\subset D_X$ be a smooth irreducible component of the singularities of D_X. Let $\sigma: Y\xrightarrow{\sigma} X$ be the blow-up of X along Z and E be the exceptional divisor. Put $D_Y := \sigma^*D_X - 2E$.

Proposition 7.4. *Notation as above. Let $\mathcal{L}_X$ be an ample invertible sheaf on X. Put $\mathcal{L}_Y := \sigma^*\mathcal{L}_X - E$. Then it holds that*

$$\chi\big(X, \Omega^p_X(\log D_X)\otimes\mathcal{L}_X^{-1}\big) = \chi\big(Y, \Omega^p_Y(\log D_Y)\otimes\mathcal{L}_Y^{-1}\big).$$

Proof. For $p\geq 1$, the residue exact sequence reads (cf. [14, Properties 2.3]):

$$0\to\Omega^p_Y(\log D_Y)\to\sigma^*\Omega^p_X(\log D_X) = \Omega^p_X\big(\log\sigma^*D_X\big)\xrightarrow{\text{res}}\Omega^{p-1}_E(\log D_Y\cdot E)\to 0.$$

We shall tensor the above short exact sequence with $\sigma^*\mathcal{L}_X^{-1}\otimes\mathcal{O}_Y(E)$ and take the Euler characteristic of the resulting exact sequence. First, applying the Leray spectral sequence to the morphism σ, it follows that

$$\chi\big(\sigma^*\Omega^p_X(\log D_X)\otimes\sigma^*\mathcal{L}_X^{-1}\otimes\mathcal{O}_Y(E)\big) = \chi\big(\Omega^p_X(\log D_X)\otimes\mathcal{L}_X^{-1}\big).$$

Furthermore, as $\mathcal{L}_X$ is ample, one has

$$\Omega^{p-1}_E(\log D_Y\cdot E)\otimes\sigma^*\mathcal{L}_X^{-1}\otimes\mathcal{O}_E(E) = \Omega^{p-1}_E(\log D_Y\cdot E)\otimes\mathcal{O}_E(-1).$$

Thus we get

$$\chi\big(\Omega^p_X(\log D_X)\otimes\mathcal{L}_X^{-1}\big) = \chi\big(\Omega^p_Y(\log D_Y)\otimes\mathcal{L}_Y^{-1}\big) + \chi\big(\Omega^{p-1}_E(\log D_Y\cdot E)(-1)\big).$$

It is to show $\chi(\Omega^p_E(\log D_Y\cdot E)(-1)) = 0$ for all $p\geq 0$. We proceed by induction on p. Write $D = D_Y\cdot E = S_1 + S_2 + F_1 + \cdots + F_k$, where S_i, $i = 1, 2$, are two sections of $\sigma: E\to Z$ and $F_i\xrightarrow{\sigma} H_i\subset Z$ are $\mathbb{P}^1$-bundles over hypersurfaces in Z for $i = 1,\dots,k$. Then the residue sequence reads:

$$0 \to \Omega_E \to \Omega_E(\log D) \xrightarrow{\text{res}} \bigoplus_{i=1}^{2} \mathcal{O}_{S_i} \oplus \bigoplus_{j=1}^{k} \mathcal{O}_{F_j} \to 0.$$

Write $\mathcal{N} := \mathcal{N}_{Z/X} = \mathcal{O}_Z(D_1) \oplus \mathcal{O}_Z(D_2)$. Then $E = \mathbf{Proj}(\mathcal{N}^*)$. For $i = 1, 2$, we assume the quotient invertible sheaf $\mathcal{O}_Z(-D_i)$ of $\mathcal{N}^*$ gives the section map $s_i : Z \to E$ and the image of s_i is just S_i. Then

$$\begin{aligned} \mathcal{O}_{S_i} \otimes \mathcal{O}_E(-1) &= s_{i*}\big(\mathcal{O}_Z \otimes s_{i*}\mathcal{O}_E(-1)\big) \\ &= s_{i*}\big(\mathcal{O}_Z \otimes \mathcal{O}_Z(D_i)\big) \\ &= s_{i*}\big(\mathcal{O}_Z(D_i)\big). \end{aligned}$$

Thus

$$\begin{aligned} \chi\left(E, \bigoplus_{i=1}^{2} \mathcal{O}_{S_i} \otimes \mathcal{O}_E(-1)\right) &= \sum_{i=1}^{2} \chi\big(E, s_{i*}\big(\mathcal{O}_Z(D_i)\big)\big) \\ &= \sum_{i=1}^{2} \chi\big(Z, \mathcal{O}_Z(D_i)\big) \\ &= \chi(Z, \mathcal{N}). \end{aligned}$$

And obviously, $\mathcal{O}_{F_i} \otimes \mathcal{O}_E(-1) = \mathcal{O}_{F_i}(-1)$. Now we shall compute $\chi(\Omega_E \otimes \mathcal{O}_E(-1))$. Recall the following exact sequence:

$$0 \to \sigma^* \Omega_Z \to \Omega_E \to \Omega_{E/Z} \to 0.$$

Tensoring with $\mathcal{O}_E(-1)$ and taking the Euler characteristic, we obtain

$$\chi\big(\Omega_E \otimes \mathcal{O}_E(-1)\big) = \chi\big(\sigma^* \Omega_Z \otimes \mathcal{O}_E(-1)\big) + \chi\big(\Omega_{E/Z} \otimes \mathcal{O}_E(-1)\big).$$

For $\mathrm{R}^i\sigma_*(\mathcal{O}_E(-1)) = 0$ for all i, $\chi(\sigma^*\Omega_Z \otimes \mathcal{O}_E(-1)) = 0$. And since

$$\mathrm{R}^i\sigma_*\big(\Omega_{E/Z} \otimes \mathcal{O}_E(-1)\big) = \begin{cases} 0 & i = 0, \\ \mathcal{N}_{Z/X} & i = 1, \end{cases}$$

one computes that

$$\begin{aligned} \chi\big(\Omega_{E/Z} \otimes \mathcal{O}_E(-1)\big) &= \sum_{i,j} (-1)^{i+j} \dim H^i\big(Z, \mathrm{R}^j\sigma_*\Omega_{E/Z} \otimes \mathcal{O}_E(-1)\big) \\ &= -\sum_i (-1)^i \dim H^i(Z, \mathcal{N}) \\ &= -\chi(Z, \mathcal{N}). \end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\chi\big(\Omega_E(\log D)\otimes\mathcal{O}_E(-1)\big) &= \chi\big(\Omega_E\otimes\mathcal{O}_E(-1)\big)+\sum_{i=1}^{2}\chi\big(\mathcal{O}_{S_i}\otimes\mathcal{O}_E(-1)\big)\\ &\quad+\sum_{j=1}^{k}\chi\big(\mathcal{O}_{F_i}\otimes\mathcal{O}_E(-1)\big)\\ &= -\chi(Z,\mathcal{N})+\chi(Z,\mathcal{N})\\ &= 0.\end{aligned}$$

This shows the $p=1$ case. Consider the residue sequence along F_k,

$$0\to\Omega_E^p(\log D-F_k)\to\Omega_E^p(\log D)\to\Omega_{F_k}^{p-1}\big(\log(D-F_k)\cdot F_k\big)\to 0.$$

By the induction hypothesis, we have that

$$\chi\big(\Omega_E^p(\log D)\otimes\mathcal{O}_E(-1)\big)=\chi\big(\Omega_E^p(\log D-F_k)\otimes\mathcal{O}_E(-1)\big),$$

and then

$$\chi\big(\Omega_E^p(\log D)\otimes\mathcal{O}_E(-1)\big)=\chi\big(\Omega_E^p(S_1+S_2)\otimes\mathcal{O}_E(-1)\big).$$

Consider the following three short exact sequences:

$$\begin{gathered}0\to\Omega_E^p(\log S_1)\to\Omega_E^p(\log S_1+S_2)\to\Omega_{S_2}^{p-1}\to 0,\\ 0\to\Omega_E^p\to\Omega_E^p(\log S_1)\to\Omega_{S_1}^{p-1}\to 0,\\ 0\to\sigma^*\Omega_Z^p\to\Omega_E^p\to\sigma^*\Omega_Z^{p-1}\otimes\Omega_{E/Z}\to 0.\end{gathered}$$

From the last sequence, it follows that

$$\begin{aligned}\chi\big(\Omega_E^p\otimes\mathcal{O}_E(-1)\big) &= \chi\big(\sigma^*\Omega_Z^{p-1}\otimes\Omega_{E/Z}\otimes\mathcal{O}_E(-1)\big)\\ &= \sum_{i,j}(-1)^{i+j}\dim H^i\big(Z,\Omega_Z^{p-1}\otimes\mathrm{R}^j\sigma_*\big(\Omega_{E/Z}\otimes\mathcal{O}_E(-1)\big)\big)\\ &= -\sum_i(-1)^i\dim H^i\big(Z,\Omega_Z^{p-1}\otimes\mathcal{N}\big)\\ &= -\chi\big(Z,\Omega_Z^{p-1}\otimes\mathcal{N}\big).\end{aligned}$$

Moreover, we compute that

$$\begin{aligned}\sum_{i=1}^{2}\chi\big(S_i,\Omega_{S_i}^{p-1}\otimes\mathcal{O}_E(-1)\big)&=\sum_{i=1}^{2}\chi\big(Z,\Omega_Z^{p-1}\otimes s_i^*\mathcal{O}_E(-1)\big)\\&=\chi\left(Z,\Omega_Z^{p-1}\otimes\left(\bigoplus_{i=1}^{2}s_i^*\mathcal{O}_E(-1)\right)\right)\\&=\chi\big(Z,\Omega_Z^{p-1}\otimes\mathcal{N}\big).\end{aligned}$$

Therefore, we get finally that

$$\begin{aligned}\chi\big(\Omega_E^p(\log D)\otimes\mathcal{O}_E(-1)\big)&=\sum_{i=1}^{2}\chi\big(S_i,\Omega_{S_i}^{p-1}\otimes\mathcal{O}_E(-1)\big)+\chi\big(\Omega_E^p\otimes\mathcal{O}_E(-1)\big)\\&=\chi\big(Z,\Omega_Z^{p-1}\otimes\mathcal{N}\big)-\chi\big(Z,\Omega_Z^{p-1}\otimes\mathcal{N}\big)\\&=0.\end{aligned}$$

This completes the proof. □

Proof of Claim 7.3. Put $\mathcal{E}:=\bigoplus^{2n+2}\mathcal{O}_{\mathbb{P}^n}(1)$. Let $\mathbb{P}=\mathbf{Proj}(S(\mathcal{E}))\xrightarrow{p}\mathbb{P}^n$ be the associated projective vector bundle together with the invertible sheaf $\mathcal{M}:=\mathcal{O}(1)$. The sheaf of differential operators $\Sigma_{\mathcal{M}}$ of $\mathcal{M}$ with order ≤ 1 is defined by the following exact sequence:

$$0\to\mathcal{O}_{\mathbb{P}}\to\Sigma_{\mathcal{M}}\xrightarrow{q}T_{\mathbb{P}}\to 0$$

with the extension class $-c_1(\mathcal{M})\in\mathrm{Ext}^1(T_{\mathbb{P}},\mathcal{O}_{\mathbb{P}})\simeq H^1(\mathbb{P},\Omega^1_{\mathbb{P}})$. For $i\in\{1,\dots,2n+2\}$, we let $\lambda_i\in H^0(\mathbb{P},\mathcal{M}\otimes p^*\mathcal{O}_{\mathbb{P}^n}(-1))$ such that $\mathrm{P}_i=\{\lambda_i=0\}$ is the divisor of $\mathbb{P}$ whose fiber under the projection p is the i-th coordinate hyperplane of the fiber of $\mathbb{P}$ under p. Put $\mathcal{P}=\sum_{i=1}^{2n+2}\mathrm{P}_i$.

Suppose H is defined by the equation $\prod_{i=1}^{2n+2}F_i=0$ in $\mathbb{P}^n$. Then we associate to H the section $\sigma=\sum_{i=1}^{2n+2}F_i\cdot\lambda_i\in H^0(\mathbb{P},\mathcal{M})$. Put $\mathcal{Z}=\{\sigma=0\}$. Since H is normal crossing, $\mathcal{Z}$ is smooth in $\mathbb{P}$. Note that the section $\sigma\in H^0(\mathbb{P},\mathcal{M})$ defines the evaluation map $j(\sigma):\Sigma_{\mathcal{M}}\to\mathcal{M}$.

Lemma 7.5. *$j(\sigma)$ is surjective with kernel equal to $T_{\mathbb{P}}(-\log\mathcal{Z})$. That is, the following exact sequence holds:*

$$0\to T_{\mathbb{P}}(-\log\mathcal{Z})\to\Sigma_{\mathcal{M}}\xrightarrow{j(\sigma)}\mathcal{M}\to 0.$$

Proof. Since $\mathcal{Z}$ is smooth, the system of equations $\{\frac{\partial}{\partial x_1}(\mathcal{M}_x)=\cdots=\frac{\partial}{\partial\lambda_{2n+2}}(\mathcal{M}_x)=0\}$ has no common solutions in $\mathbb{P}$. This means $j(\sigma)$ is surjective. The local sections of $T_{\mathbb{P}}(-\log\mathcal{Z})$ are the first order differential operators preserving $\mathcal{Z}$. Then for each open subset $U\subset\mathbb{P}$, one has

$$T_{\mathbb{P}}(-\log \mathcal{Z})(U) = \{P \in \Sigma_{\mathcal{M}}(U),\ P(\sigma) = \sigma\}$$
$$\simeq \{P \in \Sigma_{\mathcal{M}}(U),\ P(\sigma) = 0\}.$$

Hence $\ker(\sigma) \simeq T_{\mathbb{P}}(-\log \mathcal{Z})$. □

Now define $\Sigma_{\mathcal{M}}(-\log \mathcal{P}) = q^{-1}(T_{\mathbb{P}}(-\log \mathcal{P})) \subset \Sigma_{\mathcal{M}}$.

Lemma 7.6. *The following two exact sequences are exact:*

$$0 \to T_{\mathbb{P}}(-\log \mathcal{P} + \mathcal{Z}) \to \Sigma_{\mathcal{M}}(-\log \mathcal{P}) \to \mathcal{M} \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}}^{\oplus 2n+2} \to \Sigma_{\mathcal{M}}(-\log \mathcal{P}) \to p^* T_{\mathbb{P}^n} \to 0.$$

Proof. The proof for the first sequence is the same as the one in Lemma 7.5. The second sequence follows from the defining sequence of $\Sigma_{\mathcal{M}}$ and the Euler sequences. □

Define $\Sigma_{\mathcal{E}}^0 = p_* \Sigma_{\mathcal{M}}(-\log \mathcal{P})$. Note that $\mathrm{R}^1 p_* T_{\mathbb{P}}(-\log \mathcal{P} + \mathcal{Z}) = \mathrm{R}^1 p_* \mathcal{O}_{\mathbb{P}} = 0$. Then p_* of two short exact sequences in Lemma 7.6 gives the following two short exact sequences of $\Sigma_{\mathcal{E}}^0$:

Corollary 7.7. *The following two sequences of vector bundles are exact:*

$$0 \to T_{\mathbb{P}^n}(-\log H) \to \Sigma_{\mathcal{E}}^0 \to \mathcal{E} \to 0,$$
$$0 \to \mathcal{O}_{\mathbb{P}^n}^{\oplus 2n+2} \to \Sigma_{\mathcal{E}}^0 \to T_{\mathbb{P}^n} \to 0.$$

Recall the following

Lemma 7.8. *Let X be a compact complex manifold. Given a short exact sequence of vector bundles on X:*

$$0 \to E \to F \to G \to 0,$$

one has then a long exact sequence of the form:

$$0 \to S^k E \to S^{k-1} E \otimes F \to \cdots \to \bigwedge^k F \to \bigwedge^k G \to 0.$$

Applying the previous lemma to the dual of the first short exact sequence in Corollary 7.7, one obtains the following long exact sequence:

$$0 \to \Omega_{\mathbb{P}^n}^{n-p}(\log H)(-n-1) \to \bigwedge^p \Sigma_{\mathcal{E}}^0 \to \cdots \to S^p \mathcal{E} \to 0.$$

Lemma 7.9. *$H^i(\mathbb{P}^n, S^{p-k}\mathcal{E} \otimes \bigwedge^k \Sigma_{\mathcal{E}}^0) = 0$, for each $i > 0$ and $0 \le k \le p$.*

Proof. It suffices to show $H^i(\mathbb{P}^n, \bigwedge^k \Sigma^0_{\mathcal{E}}) = 0$. The sheaf $\bigwedge^k \Sigma^0_{\mathcal{E}}$ has a filtration from the second exact sequence in Corollary 7.7,

$$\bigwedge^k \Sigma^0_{\mathcal{E}} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^k \supset \mathcal{F}^{k+1} = 0,$$

with for each $0 \le \nu \le k$,

$$\begin{aligned}\mathrm{Gr}^\nu = \mathcal{F}^\nu/\mathcal{F}^{\nu+1} &= \bigwedge^\nu \Big(\bigoplus^{2n+2} \mathcal{O}_{\mathbb{P}^n} \otimes \bigwedge^{k-\nu} T_{\mathbb{P}^n}\Big) \\ &= \bigoplus^{\binom{2n+2}{\nu}} \bigwedge^{k-\nu} T_{\mathbb{P}^n} \\ &= \bigoplus^{\binom{2n+2}{\nu}} \Omega^{n+\nu-k}_{\mathbb{P}^n}(n+1).\end{aligned}$$

By the Bott vanishing theorem, $H^i(\mathbb{P}^n, \Omega^{n+\nu-k}_{\mathbb{P}^n}(n+1)) = 0$, which implies $H^i(\mathbb{P}^n, \mathcal{F}^\nu) = 0$. In particular, $H^i(\mathbb{P}^n, \bigwedge^k \Sigma^0_{\mathcal{E}}) = 0$. □

Proposition 7.10. $\chi(\mathbb{P}^n, \Omega^{n-p}_{\mathbb{P}^n}(\log H)(-n-1)) = (-1)^p \binom{n}{p}^2$.

Proof. Taking the Euler characteristic of the long exact sequence below Corollary 7.7, we have

$$\chi\big(\Omega^{n-p}_{\mathbb{P}^n}(\log H)(-n-1)\big) = \sum_{k=0}^{p} (-1)^k \chi\Big(S^{p-k}\mathcal{E} \otimes \bigwedge^k \Sigma^0_{\mathcal{E}}\Big).$$

Again by the filtration of the sheaf $\bigwedge^k \Sigma^0_{\mathcal{E}}$ in the proof of Lemma 7.9, one computes that

$$\begin{aligned}\chi\Big(\bigwedge^k \Sigma^0_{\mathcal{E}}\Big) &= \sum_{i=0}^{k} \chi(\mathrm{Gr}_i) \\ &= \sum_{i=0}^{k}\sum_{j=0}^{i} (-1)^{i+j} \binom{2n+2}{j}\binom{n+1}{k-i} \chi\big(\mathcal{O}_{\mathbb{P}^n}(k-i)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\chi\big(\Omega^{n-p}_{\mathbb{P}^n}(\log B)(-n-1)\big) \\ &= \sum_{i=0}^{p}\sum_{j=0}^{p-i}\sum_{k=0}^{p-i-j} (-1)^{i+k} \binom{2n+1+i}{i}\binom{n+1}{j}\binom{n+i+j}{i+j}\binom{2n+2}{p-i-j-k} \\ &= \sum_{i=0}^{p}\sum_{j=0}^{p-i} (-1)^i \binom{2n+1+i}{i}\binom{n+1}{j}\binom{n+i+j}{i+j}\binom{2n+1}{p-i-j}\end{aligned}$$

$$= \sum_{l=0}^{p} \sum_{i=0}^{l} (-1)^i \binom{2n+1+i}{i} \binom{n+1}{l-i} \binom{n+l}{n} \binom{2n+1}{p-l}$$
$$= \sum_{l=0}^{p} (-1)^l \binom{n+l}{l}^2 \binom{2n+1}{p-l}$$
$$= (-1)^p \binom{n}{p}^2. \qquad \square$$

7.2. Calculations in Jacobian ring

In this subsection we will prove the upper bound of the dimension of the characteristic variety $C_{1,a}$ claimed in Proposition 4.1. Recall the definitions of X and Y from Section 2.2. We want to compute the Hodge structure and the Higgs maps on X. Since the Hodge structure on X is determined by that on Y, we first analyze the Hodge structure on Y. In order to do that, we use the tool of Jacobian ring. It is constructed as follows. In the polynomial ring $\mathbb{C}[\mu_0, \ldots, \mu_n, y_0, \ldots, y_{2n+1}]$, consider the polynomial

$$F = \mu_0 F_0 + \cdots + \mu_n F_n$$

where

$$F_0 := y_{n+1}^2 - \left(y_0^2 + y_1^2 + \cdots + y_n^2\right),$$
$$F_i := y_{n+i+1}^2 - \left(y_0^2 + a_{1i} y_1^2 + \cdots + a_{ni} y_n^2\right), \quad 1 \le i \le n.$$

Let $J = \langle \frac{\partial F}{\partial \mu_i}, \frac{\partial F}{\partial y_j} \mid 0 \le i \le n,\ 0 \le j \le 2n+1 \rangle$ be the ideal of $\mathbb{C}[\mu_0, \ldots, \mu_n, y_0, \ldots, y_{2n+1}]$ generated by the partial derivatives of F. Define the Jacobian ring to be

$$R := \mathbb{C}[\mu_0, \ldots, \mu_n, y_0, \ldots, y_{2n+1}]/J.$$

There is a natural bigrading on the polynomial ring $\mathbb{C}[\mu_0, \ldots, \mu_n, y_0, \ldots, y_{2n+1}]$, that is: the (p,q)-part $\mathbb{C}[\mu_0, \ldots, \mu_n, y_0, \ldots, y_{2n+1}]_{(p,q)}$ is linearly spanned by the monomials $\prod_{i=0}^{n} \mu_i^{a_i} \prod_{j=0}^{2n+1} y_j^{b_j}$ with $\sum_{i=0}^{n} a_i = p$, $\sum_{j=0}^{2n+1} b_j = q$. Since the ideal J is a homogeneous ideal, there is a natural induced bigrading of $R = \mathbb{C}[\mu_0, \ldots, \mu_n, y_0, \ldots, y_{2n+1}]/J$, written as $R = \bigoplus_{p,q \ge 0} R_{(p,q)}$.

The group $N = \bigoplus_{j=0}^{2n+1} \mathbb{F}_2$ acts on R through $y_0, \ldots, y_{2n+1}$. Explicitly, $\forall g = (a_j) \in N$, we define the action of g on R by

$$g \cdot y_j := (-1)^{a_j} y_j, \quad \forall 0 \le j \le 2n+1,$$
$$g \cdot \mu_i := \mu_i, \quad \forall 0 \le i \le n.$$

It is obvious that the action of N on R preserves the bigrading. Let $R_{(p,q)}^N$ be the N-invariant part of $R_{(p,q)}$, then we have the decomposition of the N-invariant subring: $R^N = \bigoplus_{p,q \ge 0} R_{(p,q)}^N$.

Recall $\mathbb{V}_{(1)}$ is the (-1)-eigen $\mathbb{Q}$-PVHS associated to the family $\mathcal{X}_{AR} \to \mathfrak{M}_{AR}$. The following proposition gives an identification of the Higgs map associated to $\mathbb{V}_{(1)}$ and the Jacobian ring multiplication.

Proposition 7.11.

(1) *$\forall 0 \le q \le n$, $H^{n-q,q}(X)_{(1)} \simeq R^N_{(q,2q)}$;*
(2) *$\forall 0 \le q \le n$, we have a commutative diagram*

$$\begin{array}{ccc} T_{\mathfrak{M}_{AR},a} \otimes H^{n-q,q}(X)_{(1)} & \xrightarrow{\theta^{n-q,q}} & H^{n-q-1,q+1}(X)_{(1)} \\ \downarrow \simeq & & \downarrow \simeq \\ R^N_{(1,2)} \otimes R^N_{(q,2q)} & \longrightarrow & R^N_{(q+1,2q+2)} \end{array}$$

Here X is the fiber over $a \in \mathfrak{M}_{AR}$, and the lower horizontal arrow is the ring multiplication map.

Proof. (1) Let $N_1 = Ker(\bigoplus_{j=0}^{2n+1} \mathbb{F}_2 \xrightarrow{\sum} \mathbb{F}_2)$ be the subgroup of $N = \bigoplus_{j=0}^{2n+1} \mathbb{F}_2$. By Corollary 2.5 in [38] and its proof, we can see that $\forall 0 \le q \le n$, $H^{n-q,q}(Y)^{N_1}_{(1)} \simeq R^{N_1}_{(q,2q)(0)} = R^N_{(q,2q)}$. By Proposition 2.3, $\forall 0 \le q \le n$, $H^{n-q,q}(X)_{(1)} \simeq H^{n-q,q}(Y)^{N_1}_{(1)}$, these two isomorphisms together give (1).

(2) follows from (1) and Proposition 2.6 in [38]. □

Next we will present $\mathbb{C}$-bases of $R^N_{(1,2)}$ and $R^N_{(2,4)}$. Note that the ideal J is generated by the following elements:

$$\begin{aligned} \frac{\partial F}{\partial \mu_0} &= y_{n+1}^2 - (y_0^2 + y_1^2 + \cdots + y_n^2); \\ \frac{\partial F}{\partial \mu_i} &= y_{n+i+1}^2 - (y_0^2 + a_{1i} y_1^2 + \cdots + a_{ni} y_n^2), \quad 1 \le i \le n; \\ -\frac{\partial F}{2\partial y_0} &= y_0(\mu_0 + \mu_1 + \cdots + \mu_n); \\ -\frac{\partial F}{2\partial y_i} &= y_i(\mu_0 + a_{i1}\mu_1 + \cdots + a_{in}\mu_n), \quad 1 \le i \le n; \\ \frac{\partial F}{2\partial y_{n+i+1}} &= \mu_i y_{n+i+1}, \quad 0 \le i \le n. \end{aligned}$$

By the relations above, we can see easily that

$$R^N_{(1,2)} = \mathbb{C}\langle \mu_i y_j^2 \mid 0 \le i, j \le n \rangle,$$

where $\mathbb{C}\langle \mu_i y_j^2 \mid 0 \le i, j \le n\rangle$ means the linear subspace of R spanned by elements in the set $\{\mu_i y_j^2 \mid 0 \le i, j \le n\}$. Similarly, one finds that

$$R^N_{(2,4)} = \mathbb{C}\big\langle \mu_i y_j^2 \mu_p y_q^2 \bigm| 0 \le i, j, p, q \le n\big\rangle.$$

In order to obtain bases from $\{\mu_i y_j^2 \mid 0 \le i, j \le n\}$ and $\{\mu_i y_j^2 \mu_p y_q^2 \mid 0 \le i, j, p, q \le n\}$, we study the relations in R.

In order to write the relations more symmetrically, we define

$$a_{i0} = a_{0j} = 1, \quad \forall 0 \le i, j \le n.$$

Then we find easily that the following relations hold in R:

$$\begin{cases} \displaystyle\sum_{j=0}^{n} a_{ji}\mu_i y_j^2 = 0, \quad \forall 0 \le i \le n; \\ \displaystyle\sum_{i=0}^{n} a_{ji}\mu_i y_j^2 = 0, \quad \forall 0 \le j \le n. \end{cases} \tag{7.11.1}$$

Note that all the discussion above depends on the parameter $a := (a_{ij}) \in M(n\times n, \mathbb{C})$. From these basic relations (7.11.1), we can get some other useful relations.

Lemma 7.12. *For a generic parameter $a \in M(n\times n, \mathbb{C})$, the following relations hold in R:*

(R1) $(\sum_{j=0}^{n} a_{ji}\mu_i y_j^2)\mu_p y_q^2 = 0$, $\forall 0 \le i, p, q \le n$.
(R2) $(\sum_{i=0}^{n} a_{ji}\mu_i y_j^2)\mu_p y_q^2 = 0$, $\forall 0 \le p, q, j \le n$.
(R3) $\mu_p y_q^2 \mu_i y_q^2 = \sum_{j=1, j\ne q}^{n} \frac{a_{0p}a_{ji} - a_{0i}a_{jp}}{a_{0i}a_{qp} - a_{0p}a_{qi}} \mu_p y_q^2 \mu_i y_j^2$, $\forall 0 \le p \ne i \le n$, $\forall 1 \le q \le n$.
(R4) $\mu_p y_j^2 \mu_p y_q^2 = \sum_{i=1, i\ne p}^{n} \frac{a_{q0}a_{ji} - a_{j0}a_{qi}}{a_{j0}a_{qp} - a_{q0}a_{jp}} \mu_i y_j^2 \mu_p y_q^2$, $\forall 1 \le p \le n$, $\forall 0 \le j \ne q \le n$.

Proof. The relations (R1) and (R2) follow obviously from the basic relations (7.11.1).

Given $0 \le p \ne i \le n$, $1 \le q \le n$, in order to prove (R3), we consider the element $\mu_p \mu_i y_0^2 y_q^2$. By relations (R1), we have

$$a_{0p}\mu_p\mu_i y_0^2 y_q^2 = a_{0p}\mu_p y_0^2 \mu_i y_q^2 = -\sum_{j=1}^{n} a_{jp}\mu_p y_j^2 \mu_i y_q^2,$$

$$a_{0i}\mu_p\mu_i y_0^2 y_q^2 = a_{0i}\mu_i y_0^2 \mu_p y_q^2 = -\sum_{j=1}^{n} a_{ji}\mu_i y_j^2 \mu_p y_q^2.$$

Let a_{0i} times the first identity and a_{0p} times the second identity, then we get

$$\sum_{j=1}^{n} a_{0i}a_{jp}\mu_p y_j^2 \mu_i y_q^2 = \sum_{j=1}^{n} a_{0p}a_{ji}\mu_i y_j^2 \mu_p y_q^2.$$

From this equality we get

$$\begin{aligned}(a_{0i}a_{qp} - a_{0p}a_{qi})\mu_p y_q^2 \mu_i y_q^2 &= a_{0i}a_{qp}\mu_p y_q^2 \mu_i y_q^2 - a_{0p}a_{qi}\mu_i y_q^2 \mu_p y_q^2 \\ &= \sum_{j=1, j\neq q}^{n} a_{0p}a_{ji}\mu_i y_j^2 \mu_p y_q^2 - \sum_{j=1, j\neq q}^{n} a_{0i}a_{jp}\mu_p y_j^2 \mu_i y_q^2 \\ &= \sum_{j=1, j\neq q}^{n} (a_{0p}a_{ji} - a_{0i}a_{jp})\mu_p y_q^2 \mu_i y_j^2.\end{aligned}$$

Since for a generic parameter $a \in \mathbb{C}^{n^2}$, $a_{0i}a_{qp} - a_{0p}a_{qi} = a_{qp} - a_{qi} \neq 0$, we get (R3). The relations (R4) can be proved similarly. □

Now we can determine $\mathbb{C}$-bases of $R^N_{(1,2)}$ and $R^N_{(2,4)}$.

Proposition 7.13.

(1) *For any parameter $a \in \mathbb{C}^{n^2}$, $R^N_{(1,2)}$ has a $\mathbb{C}$-basis $\{\mu_i y_j^2 \mid 1 \leq i, j \leq n\}$.*
(2) *For a generic parameter $a \in \mathbb{C}^{n^2}$, $R^N_{(2,4)}$ has a $\mathbb{C}$-basis $\{\mu_i y_j^2 \mu_p y_q^2 \mid 1 \leq i < p \leq n,\ 1 \leq j < q \leq n\}$.*
(3) *For any parameter $a \in \mathbb{C}^{n^2}$, $\forall q \geq 1$, the multiplication map $Sym^q R^N_{(1,2)} \to R^N_{(q,2q)}$ is surjective.*

Proof. (1): By the basic relations (7.11.1), we have

$$R^N_{(1,2)} = \mathbb{C}\big\langle \mu_i y_j^2 \bigm| 1 \leq i, j \leq n \big\rangle.$$

Proposition 2.8 and Proposition 7.11 imply that the dimension of the $\mathbb{C}$-linear space $R^N_{(1,2)}$ is n^2. So we get that $\{\mu_i y_j^2 \mid 1 \leq i, j \leq n\}$ is a $\mathbb{C}$-basis of $R^N_{(1,2)}$. This proves (1).

(2): Similarly as in (1), the dimension of the $\mathbb{C}$-linear space $R^N_{(2,4)}$ is $\binom{n}{2}^2$, and $R^N_{(2,4)}$ is linearly spanned by $\{\mu_i y_j^2 \mu_p y_q^2 \mid 0 \leq i, j, p, q \leq n\}$, so it suffices to show that $\forall 0 \leq i, j, p, q \leq n$,

$$\mu_i y_j^2 \mu_p y_q^2 \in \mathbb{C}\big\langle \mu_i y_j^2 \mu_p y_q^2 \bigm| 1 \leq i < p \leq n,\ 1 \leq j < q \leq n \big\rangle.$$

This can be proved using the relations (R1)–(R4) in Lemma 7.12. Next we show $\mu_0 y_0^2 \mu_0 y_0^2 \in \mathbb{C}\langle \mu_i y_j^2 \mu_p y_q^2 \mid 1 \leq i < p \leq n,\ 1 \leq j < q \leq n\rangle$ by the following steps to illustrate the ideas.

Step 1: By relations (R1), $\mu_0 y_0^2 \mu_0 y_0^2 \in \mathbb{C}\langle \mu_0 y_0^2 \mu_0 y_q^2 \mid 1 \leq q \leq n\rangle$.

Step 2: $\forall 1 \leq q \leq n$, by relations (R2), $\mu_0 y_0^2 \mu_0 y_q^2 \in \mathbb{C}\langle \mu_0 y_0^2 \mu_p y_q^2 \mid 1 \leq p \leq n\rangle$.

Step 3: $\forall 1 \leq p, q \leq n$, by relations (R2), $\mu_0 y_0^2 \mu_p y_q^2 \in \mathbb{C}\langle \mu_i y_0^2 \mu_p y_q^2 \mid 1 \leq i \leq n\rangle$.

Step 4: $\forall 1 \leq i, p, q \leq n$, if $i = p$, by relations (R4), $\mu_i y_0^2 \mu_p y_q^2 \in \mathbb{C}\langle \mu_{i_1} y_0^2 \mu_p y_q^2 \mid 1 \leq i_1 \leq n,\ i_1 \neq p\rangle$.

Step 5: $\forall 1 \le q \le n$, $\forall 1 \le i \ne p \le n$, by relations (R1), $\mu_i y_0^2 \mu_p y_q^2 \in \mathbb{C}\langle \mu_i y_j^2 \mu_p y_q^2 \mid 1 \le j \le n \rangle$.

Step 6: $\forall 1 \le q \le n$, $\forall 1 \le i \ne p \le n$, by relations (R3), $\mu_i y_q^2 \mu_p y_q^2 \in \mathbb{C}\langle \mu_i y_j^2 \mu_p y_q^2 \mid 1 \le j \le n,\ j \ne q \rangle$.

After these six steps, we have shown $\mu_0 y_0^2 \mu_0 y_0^2 \in \mathbb{C}\langle \mu_i y_j^2 \mu_p y_q^2 \mid 1 \le i < p \le n,\ 1 \le j < q \le n \rangle$. Other cases can be treated similarly.

(3) follows directly from the definition of the Jacobian ring. □

Given $\alpha \in R^N_{(1,2)}$, we can expand it under the basis above:

$$\alpha = \sum_{1 \le i,j \le n} \lambda_{ij} \mu_i y_j^2,$$

with $(\lambda_{ij}) \in M(n \times n, \mathbb{C})$.

Since $\{\mu_i y_j^2 \mu_p y_q^2 \mid 1 \le i < p \le n,\ 1 \le j < q \le n\}$ is a basis of $R^N_{(2,4)}$, we have the expression

$$\alpha^2 = \sum_{1 \le i < p \le n, 1 \le j < q \le n} f_{ijpq} \mu_i y_j^2 \mu_p y_q^2.$$

Obviously each f_{ijpq} is a homogeneous quadratic polynomial of λ_{ij} $(1 \le i, j \le n)$, with coefficients being rational functions of the parameters a_{ij} $(1 \le i, j \le n)$. As for the information of these f_{ijpq}, we have the following proposition.

Proposition 7.14. *The following statements hold:*

(1) $\forall 1 \le i < p \le n$, $\forall 1 \le j < q \le n$, *we have*

$$\begin{aligned} f_{ijpq} = {}& c^{ijij}_{ijpq} \lambda_{ij}^2 + c^{iqiq}_{ijpq} \lambda_{iq}^2 + c^{pjpj}_{ijpq} \lambda_{pj}^2 + c^{pqpq}_{ijpq} \lambda_{pq}^2 \\ & + c^{ijiq}_{ijpq} \lambda_{ij} \lambda_{iq} + c^{pjpq}_{ijpq} \lambda_{pj} \lambda_{pq} + c^{ijpj}_{ijpq} \lambda_{ij} \lambda_{pj} + c^{iqpq}_{ijpq} \lambda_{iq} \lambda_{pq} \\ & + c^{ijpq}_{ijpq} \lambda_{ij} \lambda_{pq} + c^{iqpj}_{ijpq} \lambda_{iq} \lambda_{pj}, \end{aligned}$$

where each of the ten coefficients $c^{ijij}_{ijpq}, \ldots, c^{iqpj}_{ijpq}$ *is a nonzero rational function of* $a_{ji}, a_{qi}, a_{jp}, a_{qp}$.

(2) *Notation as in (1).* $\forall 2 \le j < q \le n$, *let* R_{jq} *be the following resultant:*

$$R_{jq} := \det \begin{pmatrix} c^{1q1q}_{112q} & 0 & c^{1q1q}_{1j2q} & 0 \\ c^{1q2q}_{112q} & c^{1q1q}_{112q} & c^{1q2q}_{1j2q} & c^{1q1q}_{1j2q} \\ c^{2q2q}_{112q} & c^{1q2q}_{112q} & c^{2q2q}_{1j2q} & c^{1q2q}_{1j2q} \\ 0 & c^{2q2q}_{112q} & 0 & c^{2q2q}_{1j2q} \end{pmatrix}.$$

Then R_{jq} *is a nonzero rational function of* a_{ij} $(1 \le i, j \le n)$.

(3) *Notation as in (1).* $\forall 2 \le i < p \le n$, *let* Q_{ip} *be the following resultant:*

$$Q_{ip} := \det \begin{pmatrix} c_{11p2}^{p1p1} & 0 & c_{i1p2}^{p1p1} & 0 \\ c_{11p2}^{p1p2} & c_{11p2}^{p1p1} & c_{i1p2}^{p1p2} & c_{i1p2}^{p1p1} \\ c_{11p2}^{p2p2} & c_{11p2}^{p1p2} & c_{i1p2}^{p2p2} & c_{i1p2}^{p1p2} \\ 0 & c_{11p2}^{p2p2} & 0 & c_{i1p2}^{p2p2} \end{pmatrix}.$$

Then Q_{ip} *is a nonzero rational function of* a_{ij} $(1 \le i, j \le n)$.

Proof. (1): Examining the proof of Proposition 7.13, one can get the following fact:

$\forall 1 \le i_1, i_2, j_1, j_2 \le n$, if $\{(i_1, j_1), (i_2, j_2)\} \nsubseteq \{(i, j), (i, q), (p, j), (p, q)\}$, then when we express $\mu_{i_1} y_{j_1}^2 \mu_{i_2} y_{j_2}^2$ as a linear combination of the basis $\{\mu_i y_j^2 \mu_p y_q^2 \mid 1 \le i < p \le n,\ 1 \le j < q \le n\}$, the coefficient before $\mu_i y_j^2 \mu_p y_q^2$ is zero.

Using this observation, we find f_{ijpq} has the required expression. That each of the ten coefficients $c_{ijpq}^{ijij}, \dots, c_{ijpq}^{iqpj}$ is a nonzero rational function of $a_{ji}, a_{qi}, a_{jp}, a_{qp}$ follows from an explicit computation. Explicitly, we have:

$$c_{ijpq}^{ijpq} = 2; \qquad c_{ijpq}^{iqpj} = 2;$$

$$c_{ijpq}^{ijiq} = \frac{2(a_{jp} - a_{qp})}{a_{qi} - a_{ji}}; \qquad c_{ijpq}^{pjpq} = \frac{2(a_{ji} - a_{qi})}{a_{qp} - a_{jp}};$$

$$c_{ijpq}^{ijpj} = \frac{2(a_{qi} - a_{qp})}{a_{jp} - a_{ji}}; \qquad c_{ijpq}^{iqpq} = \frac{2(a_{ji} - a_{jp})}{a_{qp} - a_{qi}};$$

$$c_{ijpq}^{ijij} = \frac{1}{a_{ji}} \cdot \frac{a_{jp} - 1}{a_{ji} - 1} \cdot \left(\frac{a_{jp}(a_{qp} - a_{qi})}{a_{jp} - a_{ji}} - a_{qp} \right) + \frac{a_{qi}}{a_{ji}} \cdot \frac{a_{qp} - a_{jp}}{a_{qi} - a_{ji}};$$

$$c_{ijpq}^{iqiq} = \frac{1}{a_{qi}} \cdot \frac{a_{qp} - 1}{a_{qi} - 1} \cdot \left(\frac{a_{qp}(a_{jp} - a_{ji})}{a_{qp} - a_{qi}} - a_{jp} \right) + \frac{a_{ji}}{a_{qi}} \cdot \frac{a_{jp} - a_{qp}}{a_{ji} - a_{qi}};$$

$$c_{ijpq}^{pjpj} = \frac{1}{a_{jp}} \cdot \frac{a_{ji} - 1}{a_{jp} - 1} \cdot \left(\frac{a_{ji}(a_{qi} - a_{qp})}{a_{ji} - a_{jp}} - a_{qi} \right) + \frac{a_{qp}}{a_{jp}} \cdot \frac{a_{qi} - a_{ji}}{a_{qp} - a_{jp}};$$

$$c_{ijpq}^{pqpq} = \frac{1}{a_{qp}} \cdot \frac{a_{qi} - 1}{a_{qp} - 1} \cdot \left(\frac{a_{qi}(a_{ji} - a_{jp})}{a_{qi} - a_{qp}} - a_{ji} \right) + \frac{a_{jp}}{a_{qp}} \cdot \frac{a_{ji} - a_{qi}}{a_{jp} - a_{qp}}.$$

(2) and (3): From the above explicit computation of the coefficients $c_{ijpq}^{ijij}, \dots, c_{ijpq}^{iqpj}$, we can see that R_{jq} and Q_{ip} are nonzero rational functions of a_{ij} $(1 \le i, j \le n)$. □

Now we can complete the proof of Proposition 4.1. Recall $C_{1,a}$ is the first characteristic variety of $\mathbb{V}_{(1)}$ at $a \in \mathfrak{M}_{AR}$. By Proposition 7.11, it is easy to see that

$$C_{1,a} \simeq C'_{1,a} := \left\{ [\alpha] \in \mathbb{P}\big(R_{(1,2)}^N\big) \;\middle|\; \alpha^2 = 0 \in R_{(2,4)}^N \right\},$$

where $\mathbb{P}(R^N_{(1,2)})$ is the projectification of the $\mathbb{C}$-linear space $R^N_{(1,2)}$, and $[\alpha]$ means the class represented by an element $\alpha \in R^N_{(1,2)}$. For each $a \in \mathbb{C}^{n^2}$, $C'_{1,a}$ is a closed subvariety in the projective space $\mathbb{P}(R^N_{(1,2)})$.

For later use, we state the following elementary lemma.

Lemma 7.15. *Let $V = \mathbb{C}^p$ be an affine space with coordinates $x_1, \ldots, x_p$. Then the following hold:*

(1) *Given*

$$\begin{array}{ccc} V_1 & \subset & V \\ \cup & & \cup \\ X_1 & & X \end{array}$$

where
- *V_1 is the subspace of V defined by $x_1 = 0$;*
- *X_1 is the subvariety of V_1 defined by the simultaneous vanishing of the polynomials $f_j(x_2, \ldots, x_p)$, $2 \leq j \leq q$;*
- *X is the subvariety of V defined by the simultaneous vanishing of the polynomials $f_1(x_1, \ldots, x_p)$, $f_j(x_2, \ldots, x_p)$, $2 \leq j \leq q$, where $f_1(x_1, \ldots, x_p) = ax_1^2 + g_1(x_2, \ldots, x_p)x_1 + h_1(x_2, \ldots, x_p)$, with $0 \neq a \in \mathbb{C}$.*

Then we have $\dim X \leq \dim X_1$.

(2) *Given*

$$\begin{array}{ccc} V_2 & \subset & V \\ \cup & & \cup \\ X_2 & & X \end{array}$$

where
- *V_2 is the subspace of V defined by $x_1 = x_2 = 0$;*
- *X_2 is the subvariety of V_2 defined by the simultaneous vanishing of the polynomials $f_j(x_3, \ldots, x_p)$, $3 \leq j \leq q$;*
- *X is the subvariety of V defined by the simultaneous vanishing of the polynomials $f_1(x_1, \ldots, x_p)$, $f_2(x_1, \ldots, x_p)$, $f_j(x_3, \ldots, x_p)$, $3 \leq j \leq q$, and for $i = 1, 2$, $f_i(x_1, \ldots, x_p) = a_i x_1^2 + b_i x_1 x_2 + c_i x_2^2 + x_1 g_i(x_3, \ldots, x_p) + x_2 h_i(x_3, \ldots, x_p) + r_i(x_3, \ldots, x_p)$, with $a_i, b_i, c_i \in \mathbb{C}$ such that the following resultant is not zero:*

$$\det \begin{pmatrix} a_1 & 0 & a_2 & 0 \\ b_1 & a_1 & b_2 & a_2 \\ c_1 & b_1 & c_2 & b_2 \\ 0 & c_1 & 0 & c_2 \end{pmatrix} \neq 0.$$

Then we have $\dim X \leq \dim X_2$.

Proof. The proof is direct. Since in each case we can consider the natural projections: $\pi_i : V \to V_i$ $(i = 1, 2)$ and the conditions guarantee that $\forall x \in V_i$, the dimension of $X \cap \pi^{-1}(x)$ is either empty or a zero dimensional variety. □

Proposition 4.1 follows directly from

Proposition 7.16. *If $n \geq 2$, then for generic $a \in \mathbb{C}^{n^2}$, we have* $\dim C'_{1,a} \leq 2$.

Proof. By Proposition 7.14, we can choose a generic parameter $a = (a_{ij}) \in \mathbb{C}^{n^2}$, such that Proposition 7.13 holds, and $\forall 1 \leq i < p \leq n$, $\forall 1 \leq j < q \leq n$, each of the rational functions c^{pqpq}_{ijpq}, R_{jq}, Q_{ip} takes nonzero value at the point a. We only need to show that at this point a, $\dim C'_{1,a} \leq 2$. In the following, we fix this parameter a.

Under the basis $\mu_j y_j^2$ $(1 \leq i, j \leq n)$, we identify $R^N_{(1,2)}$ with $\mathbb{C}^{n^2}$, and we view (λ_{ij}) $(1 \leq i, j \leq n)$ as the coordinates on the affine space $\mathbb{C}^{n^2}$.

It is obvious that the cone in $R^N_{(1,2)} = \mathbb{C}^{n^2}$ corresponding to $C'_{1,a}$ is the variety $\tilde{X} \subset \mathbb{C}^{n^2}$ defined by the simultaneously vanishing of the $\binom{n}{2}^2$ homogeneous quadratic polynomials f_{ijpq} $(1 \leq i < p \leq n,\ 1 \leq j < q \leq n)$. Define $X \subset \mathbb{C}^{n^2}$ by the simultaneously vanishing of the following $n^2 - 3$ homogeneous quadratic polynomials:

$$f_{i,j,i+1,j+1} \quad (1 \leq i, j \leq n-1);$$
$$f_{1,1,2,q} \quad (3 \leq q \leq n);$$
$$f_{1,1,p,2} \quad (3 \leq p \leq n).$$

Since $\dim C'_{1,a} = \dim \tilde{X} - 1$ and $\tilde{X} \subset X$, it suffices to show $\dim X \leq 3$.

In order to prove $\dim X \leq 3$ using Lemma 7.15, we give a filtration of $\mathbb{C}^{n^2}$ by affine spaces and define a subvariety in each of these affine spaces, i.e. we want to get the following diagram:

$$\begin{array}{ccccccccc} V_1 & \subset & V_2 & \subset & \cdots & \subset & V_t & = & \mathbb{C}^{n^2} \\ \cup & & \cup & & & & \cup & & \cup \\ X_1 & & X_2 & & \cdots & & X_t & = & X \end{array}$$

with $t = (n-1)^2$.

Recall (λ_{ij}) $(1 \leq i, j \leq n)$ are coordinates on $\mathbb{C}^{n^2}$. First give a filtration of the index set $S := \{(i, j) \mid 1 \leq i, j \leq n\}$ as follows

$$S_1 \subset S_2 \subset \cdots \subset S_t = S,$$

where we define inductively

- $\forall 1 \leq p \leq n-1$, $S_p := \{(i, j) \in S \mid 1 \leq i \leq 2,\ 1 \leq j \leq p+1\}$;
- $\forall k \geq 1$, $\forall k(n-1) + 1 \leq p \leq (k+1)(n-1)$, $S_p := S_{k(n-1)} \cup \{(k+2, j) \mid 1 \leq j \leq p - k(n-1) + 1\}$.

Now $\forall 1 \leq p \leq t$, define the affine space

$$V_p := \{(\lambda_{ij}) \in \mathbb{C}^{n^2} \mid \lambda_{ij} = 0, \ \forall (i,j) \notin S_p\}.$$

Then $\forall 1 \leq p \leq t$, we define $X_p \subset V_p$ by the simultaneous vanishing of the polynomials in $\mathcal{F}_p$, where $\mathcal{F}_p$ is the set of polynomials defined inductively as follows:

- $\mathcal{F}_1 := \{f_{1122}\}$;
- $\forall 2 \leq p \leq n-1$, $\mathcal{F}_p := \mathcal{F}_{p-1} \cup \{f_{1,1,2,p+1}, f_{1,p,2,p+1}\}$;
- $\forall k \geq 1$, $\forall k(n-1)+1 \leq p \leq (k+1)(n-1)$, $\mathcal{F}_p := \mathcal{F}_{p-1} \cup \{f_{1,1,k+2,2}, f_{k+1,p-k(n-1),k+2,p-k(n-1)+1}\}$.

By Proposition 7.14, each f_{ijpq} is a polynomial of the four variables $\lambda_{ij}, \lambda_{iq}, \lambda_{pi}, \lambda_{pq}$, so $X_p \subset V_p$ is well defined. According to our choice of the parameter a, $\forall 1 \leq i < p \leq k-1$, $\forall 1 \leq j < q \leq n$, each of the rational functions c_{ijpq}^{pqpq}, R_{jq}, Q_{ip} takes nonzero value at a. Then a direct verification shows that $\forall 1 \leq p \leq t-1$, the diagram

$$\begin{array}{ccc} V_{p-1} & \subset & V_p \\ \cup & & \cup \\ X_{p-1} & & X_p \end{array}$$

satisfies the conditions in (1) or (2) of Lemma 7.15, hence we get $\dim X = \dim X_t \leq \dim X_{t-1} \leq \cdots \leq \dim X_1$. By definition, one can see that X_1 is a hypersurface in $\mathbb{C}^4$ defined by a nonzero polynomial, so $\dim X_1 \leq 3$, and finally we get $\dim X \leq 3$. □

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