Kähler-Einstein metrics, G-invariants and uniformization

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1 Introduction

Given a complex curve C which is uniformized by the unit disk, there are two structures induced by the uniformization, one is the Kähler-Einstein metric on C, the other is the projective structure. In Poincaré's famous work on uniformization [13],*Les fonctions fuchsiennes et L'équation* $\Delta u = e^u$, he established an explict relations between these two structures. In this note, we first recover Poincaré's work via the canonical system of Hodge bundle defined over C, which should be well-known to experts. This can recover the well-known projective invariants in dimension 1, the Schwarzian derivatives, and gives a geometric explanation of the associated Fuchsian type equations governing the uniformization map. The theory of uniformization for higher dimensional algebraic manifolds was pioneered by the third author [19], where he proved that an algebraic manifold is a ball quotient if and only if certain Chern numbers equality holds. And later the theory was generalized to algebraic geometric characterization of locally Hermitian symmetric space of non-compact type depending on existence of certain holomorphic tensor fields, see [20],[3]. The idea of using analytic methods to construct uniformization was adapted by the third author with Uhlenbeck [18] by proving that a holomorphic bundle over a compact Kähler manifold admits a Hermitian-Yang-Mills connection iff it is polystable. Such a statement was also proved by Donaldson for algebraic surfaces and later projective manifolds. This DUY theorem between polystable holomorphic bundles and Hermitian-Yang-Mills connection immediately led to uniformization theorem for characterizing projective flat bundles by Chern numbers equalities. (It is a generalization of the work of Narasimhan and Seshadri [12] for algebraic curves). When there is an extra endomorphism called Higgs field acting on the bundle , DUY theorem was observed to be true by Simpson in [15] using the destabilizing sheaf construction from Uhlenbeck-Yau [18]. Hitchin in [9] found a more general notion of Higgs field and proved the similar results Riemann surfaces.

Depending on these ideas, we used the construction of uniformization system of Hodge bundles for higher dimensional complex ball quotients and find the corresponding relations between Kähler-Einstein metrics and projective invariants, generalizing the work of Poincaré.

Another motivation of our work is to find generalizations to Schwarzian derivatives. The classical Schwarzian derivative is used to study projective structures on curves. Following Thurston's terminology (for example, see section 3 of [17]), projective structures on curves are (X, G)-structure for (X, G) being $(\mathbb{C}P^1, PSL(2, \mathbb{C}))$. In [11], Molzon and Tamanoi found a complete generating set of $(\mathbb{C}P^n, PSL(n+1, \mathbb{C}))$-invariants. The relations between the projective invariants via KE metrics and the projective invariants in [11] is discussed in the last part of section 3. They can be both viewed as holomorphic connection matrix under different choices of frames for the system of Hodge bundles. Using this interpretation of Molzon and Tamanoi's work, we can generalize Schwarzian derivatives to G-invariants for any semisimple Lie group G and Hermitian symmetric space X. The Hodge bundle associated with G is defined by Gross in [8] for tube domains and generalized by the second author and Zuo to general Hermitian symmetric domains in [14]. There are two essential ingredients used here from Gross and Sheng-Zuo's work, one is that the system of Hodge bundles are of Calabi-Yau type and the other is a local Torelli-type theorem, see section 5 of [8].

2 Systems of Hodge bundles over hyperbolic curve

The relation between hyperbolic structure and projective structure was first observed by Poincaré in [13], page 229. It is an explicit formula relating metric tensor and Schwarzian derivative, see Theorem 2.2. The formula can be derived by a canonical system of Higgs bundle by Hitchin[9]. This should be already well-known to experts. For reader's convenience, we include the construction in this section. Note that a sign change of the Hodge metric also yields relations between KE metric with positive Ricci curvature and projective structures. This comes from considering the extension of Hodge bundles to the compact dual.

We recall the construction of system of Hodge bundles and Hermitian-Yang-Mills connections from [9],[16].

A system of Hodge bundles (E, θ) over complex manifold X is a pair of graded holomorphic vector bundle $E = \oplus_{i+j=k} E^{i,j}$ with a Higgs field, i.e. a holomorphic map $\theta\colon E^{i,j} \to E^{i-1,j+1} \otimes \Omega^1_X$, such that $\theta \wedge \theta = 0$. A metric h on E is a Hermitian bilinear form on E such that $E^{i,j}$ are othogonal to each other and $(\sqrt{-1})^{i-j} h$ defines a positive definite Hermitian metric on E. Given a metric h, there is a bundle map $\bar{\theta}_h\colon E^{i,j} \to E^{i+1,j-1} \otimes \overline{\Omega^1_X}$. Let D_h be the Chern connection with respect to h. Then $D = D_h + \theta + \bar{\theta}_h$ is a connection on E compatible with metric h. As explained in [16], the definition of system of Hodge bundles is motivated by variation of Hodge structure. The connection D is expected to recover the corresponding Gauss-Manin connection. When the connection D is flat, we can get a complex variation of Hodge structure on X. The connection D is called Hermitian-Yang-Mills if the curvature of D satisfies the equation,

$$\Lambda F_D = \lambda Id_E. \tag{1}$$

Using the arguments from Donaldson [5] and Uhlenbeck-Yau [18], Simpson proved that the existence of Hermitian-Yang-Mills connection is equivalent to the corresponding notion of slope polystability. When X is compact and the first and second Chern classes of E vanish, HYM equation (1) implies D being flat. In this case, the classifying space is a period domain. The classifying map defines a holomorphic map from the universal cover $\tilde{X}$ to the classifying space which is equivariant under monodromy representation.

There is a canonical construction of system of Hodge bundles over any complex manifold, i.e.

$$E^{1,0} = \mathcal{O}_X, E^{0,1} = T_X \tag{2}$$

with $\theta\colon \mathcal{O}_X \to T_X \otimes \Omega_X \cong \mathrm{End}(T_X)$ corresponding to the identity map on T_X. When the connection D is projectively flat, the classifying space is the complex hyperbolic ball. The classifying map gives us the uniformization. So what we will do is to write down the holomorphic differential equations which governs the classifying map. The flat sections of $E \otimes_{Hol(X)} C^\infty(X)$ give rise to the new holomorphic structure on the C^∞ bundle. Call it H, and equip it with $D = D^{1,0}$, the holomorphic flat connection.

We first focus on the case when $\dim X = 1$. Since the following calculations are all local, we can assume the canonical bundle K_X has a square root (or globally this involves a choice of $Spin^c$ structure on X). In order to get flat connections instead of projectively flat connections, we twist the system of Hodge bundles (2) by $K_X^{1/2}$ to make $c_1(E) = 0$. The metric h_1 on $\mathcal{O}_X$ is the trivial metric such that $|1|_{h_1} = 1$. Fix a local coordinate $z\colon U \to \mathbb{C}$, the Hermitian metric g on T_X is $|\frac{\partial}{\partial z}|_g = e^u$. The metric on E is denoted by $h = (-h_1, g)$. Since we need the calculations from [9], Example 1.5, we include the proof of the following proposition.

Proposition 2.1 (Hitchin [9])**.** The connection D being flat is equivalent to metric g on tangent bundle of X having the constant sectional curvature -4.

Proof. We choose local basis for $K_X^{1/2} \oplus K_X^{-1/2}$

$$s_1 = \sqrt{dz}, s_2 = \frac{1}{\sqrt{dz}}. \tag{3}$$

Under the metric h,

$$\langle s_1, s_1\rangle = e^{-u}, \langle s_2, s_2\rangle = -e^u, \langle s_1, s_2\rangle = 0. \tag{4}$$

We write $D_h(s_1, s_2) = (s_1, s_2)A, D(s_1, s_2) = (s_1, s_2)B$, where

$$A = \begin{pmatrix} -\partial u & \\ & \partial u \end{pmatrix}, B = \begin{pmatrix} -\partial u & e^u d\overline{z} \\ dz & \partial u \end{pmatrix}. \tag{5}$$

So the curvature for connection D is

$$dB + B \wedge B = \begin{pmatrix} \partial\overline{\partial}u - e^{2u} dz \wedge d\overline{z} & \\ & e^{2u} dz \wedge d\overline{z} - \partial\overline{\partial}u \end{pmatrix}. \tag{6}$$

So the vanish of the above curvature form is equivalent to that g has constant sectional curvature -4. □

Now we assume the curvature for the connection D vanishes. The classifying map to the unit disk $f\colon \tilde{X} \to B_1$ is governed by the following holomorphic differential equation.

Theorem 2.2 (Poincaré [13])**.** *Under the notations above, denote $S(u) = 2(u_z^2 - u_{zz})$. Then the uniformization map $f(z)$ up to a Möbius transformation is the quotient of two linearly independent solutions of the following differential equation of holomorphic functions*

$$y''(z) + \frac{1}{2} S(u) y(z) = 0. \tag{7}$$

Moreover, $S(u)$ is the Schwarzian derivative of f.

Proof. The parallel sections with respect to D will induce the classifying map f. In order to get holomorphic sections, we choose a gauge transformation

$$G = \begin{pmatrix} 1 & -u_z \\ & 1 \end{pmatrix}. \tag{8}$$

i.e. choosing a new frame $(S_1, S_2) = (s_1, s_2)G$. The connection matrix under the frame (S_1, S_2) is

$$\tilde{B} = G^{-1}BG + G^{-1}dG = \begin{pmatrix} 0 & ((u_z)^2 - u_{zz})dz \\ dz & 0 \end{pmatrix} \tag{9}$$

with $(0,1)$-part vanishes. Let $f_1(z)S_1 + f_2(z)S_2$ be any parallel section. Then $(f_1(z), f_2(z))$ are holomorphic and

$$d\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \tilde{B}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0 \tag{10}$$

which gives us the holomorphic ordinary differential equation

$$(f_2)'' + (u_{zz} - u_z^2)f_2 = 0 \tag{11}$$

Denote two linearly independent solutions to (11) by $(f_1^\alpha, f_2^\alpha), (f_1^\beta, f_2^\beta)$. Then two linearly independent parallel sections are $X_1 = f_1^\alpha S_1 + f_2^\alpha S_2$ and $X_2 = f_1^\beta S_1 + f_2^\beta S_2$. The generating section for $K_X^{-1/2}$ is $s_1 = (X_1, X_2) \cdot C(f_2^\beta, -f_2^\alpha)^T$. On each point $x \in X$, the fiber of $K_X^{-1/2}$ is a one dimensional subspace of E. If we trivialize E by the parallell sections X_1, X_2, the homogenous coordinate corresponding to $K_X^{1/2}$ in $P(E)$ is given by $[f_2^\beta, -f_2^\alpha]$. This proves that the uniformization map f is the quotient of two solutions of (7).

It is straightforward to show that $S(u)$ is holomorphic from the Kähler-Einstein equation for metric g. The metric g is the pull back of the Poincaré on B_1 under map f, i.e.

$$u = \frac{1}{2}\log\left(\frac{4|f'|^2}{(1-|f|^2)^2}\right) \tag{12}$$

Direct calculation shows that

$$2(u_{zz} - u_z u_z) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Hence the Schwarzian derivative is

$$S(f) = 2(u_{zz} - u_z^2) \tag{13}$$

□

Remark 2.3. Since $S(f)$ is constructed from the Poincaré metric on the unit disk, it is invariant under the isometry action of $U(1,1)$. Moreover, the differential of S with respect to f is complex linear. So $S(f)$ is also invariant under the complexification of $U(1,1)$. This provides a proof for the well-known fact that $S(f)$ is invariant under projective transformations.

Remark 2.4. When the metric on E is changed to positive definite $h = (h_1, g)$, i.e. the metric for local sections in (4) is changed to

$$\langle s_1, s_1\rangle = e^{-u}, \langle s_2, s_2\rangle = e^{u}, \langle s_1, s_2\rangle = 0. \tag{14}$$

Then the Hermitian-Yang-Mills equation implies that g has constant sectional curvature 4, i.e. $\Delta u = -e^{2u}$. Similar gauge choice as (8) yields the relation between u and developing map f to $\mathbb{P}^1$

$$S(f) = 2(u_{zz} - u_z^2). \tag{15}$$

This should also be well-known to experts. For example, the above relation is used in Chang-Shou Lin and his collaborators in studying Mean field equation with prescribed singularity, see [10].

3 Higher dimensional ball quotients

We generalize the results from the previous section to higher dimensional ball quotients. The crucial step is still to find the gauge transformation (8). Let X be a complex manifold of dimension n. Choose E to be the system of Hodge bundles defined in (2). The metric on E is still denoted by $h = (-h_1, g)$. The next proposition says that the Hermitian-Yang-Mills metric over the bundle E is exactly the homogeneous Kähler-Einstein metric on the classifying space.

Proposition 3.1. Under the notations above, the connection D is projectively flat if and only if g is Kähler with constant bisectional curvature -1.

Proof. Since the calculation is local, we can twist E by $K_X^{\frac{1}{n+1}}$. Choose a local coordinate $(z^1, z^2, \cdots, z^n)$. Under the basis $s_0 = (dz^1 \wedge \cdots \wedge dz^n)^{\frac{1}{n+1}}, s_1 = \frac{\partial}{\partial z^1} \otimes s_0, \cdots, s_n = \frac{\partial}{\partial z^n} \otimes s_0$, the matrices for bundle maps are

$$\theta = \begin{pmatrix} 0 & 0 \\ d\hat{z} = (dz^1 \dots dz^n)^T & 0 \end{pmatrix}$$

$$\bar{\theta}_h = -\overline{h^{-1}\theta^T h} = \begin{pmatrix} 0 & d\bar{\hat{z}}^T(g^T) \\ 0 & 0 \end{pmatrix}$$

The connection matrix is given by

$$B = \begin{pmatrix} -\frac{1}{n+1}trA_g & d\bar{\hat{z}}^T(g^T) \\ d\hat{z} & A_g - \frac{1}{n+1}trA_g Id \end{pmatrix}$$

So the curvature is given by

$$F_B = \begin{pmatrix} -\frac{1}{n+1}trF_g + d\bar{\hat{z}}^T g^T \wedge d\hat{z} & d\bar{\hat{z}}^T(g^T)A_g \\ A_g d\hat{z} & F_g - \frac{1}{n+1}trF_g Id + d\hat{z} \wedge d\bar{\hat{z}}^T(g^T) \end{pmatrix} \tag{16}$$

The off-diagonal term being zero is equivalent to g being Kähler. The first diagonal term being zero is equivalent to g being Kähler-Einstein with Einstein constant $-(n+1)$.

The last diagonal block being zero is equivalent to

$$(R_{j\ k\bar{l}}^{\ i} - \frac{1}{n+1}\delta_{ij}R_{m\ k\bar{l}}^{\ m})dz^k \wedge dz^{\bar{l}} + g_{j\bar{l}}dz^i \wedge dz^{\bar{l}} = 0 \tag{17}$$

i.e.

$$R_{j\ k\bar{l}}^{\ i} = -(\delta_{ij}g_{k\bar{l}} + \delta_{ik}g_{j\bar{l}})$$

The proposition is proved. □

Following the philosophy in the curve case, we wish to find a holomorphic system of linear differential equations governing the trivializing sections of the bundle E, allowing us to construct the explicit map to the uniformization domain. To do so, we find a gauge in terms of the metric tensor to eliminate the $(0,1)$ components of the connection matrix.

Let us assume the form of the group element producing the gauge transformation is

$$G = \begin{pmatrix} 1 & a \\ 0 & Id \end{pmatrix}$$

The connection matrix under the new basis is

$$\tilde{B} = G^{-1}BG + G^{-1}dG = \begin{pmatrix} -\frac{1}{n+1}trA_g - ad\hat{z} & -ad\hat{z}a - aA_g + d\bar{\hat{z}}^T(g^T) + da \\ d\hat{z} & d\hat{z}A_g - \frac{1}{n+1}trA_g Id + d\hat{z}a \end{pmatrix} \tag{18}$$

The components $a = (a_1, \cdots, a_n)$ can be chosen as

$$a_i = (-\Gamma^{j_i}_{ij_i} + \frac{1}{n+1}\delta_{ij_i}\sum_m \Gamma^m_{mj_i}) \tag{19}$$

Here j_i can be any number in $\{1, \cdots, n\}$. From the equation (17), we have

$$\partial_{\bar{k}} a_i = -g_{i\bar{k}} \tag{20}$$

This implies that $\tilde{B}$ only has $(1,0)$ forms. So we have the following analogous results to the previous section.

Theorem 3.2. *Under the choice of a in (19), the connection matrix $\tilde{B}$ (18) is holomorphic. Consider the holomorphic differential equation*

$$\partial f - \tilde{B}^T f = 0 \tag{21}$$

The uniformization map up to a projective transformation is given by $f = [f^0_0 \colon f^0_1 \cdots f^0_n]$, where $f_i = (f^0_i, \cdots, f^n_i)^T$ are linear independent solutions for (21). Moreover set $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \log(1 - |w_1|^2 - \cdots - |w_n|^2)$ for $w = (w_1, \cdots, w_n)$ being any holomorphic and nondegenerate map of $(z_1, \cdots, z_n)$. Then the coefficients in matrix $\tilde{B}$ are rational functions of the derivatives of w_i and are projective invariants for the map w. Here projective invariants are holomorphic functions depending on w and invariant when w is composed with a projective transformation.

Proof. The constant bisectional curvature condition (17) shows that $\bar{\partial} B = 0$. Any parallel basis $(X_0, \cdots, X_n) = (s_0, \cdots s_n)GF$ of E satisfies $\bar{\partial} F = 0$ and

$$\partial F + \tilde{B} F = 0.$$

Under the trivialization of these basis, the coordinate for s_0 is given by the first row of $(F^{-1})^T$. On the other hand, $(F^{-1})^T$ must satisfy the equation

$$\partial (F^{-1})^T - \tilde{B}^T (F^{-1})^T = 0.$$

So the uniformization map f is determined by equation (21) in the stated way.

Notice that $\tilde{B}$ only involves rational functions of the differentials of g and is holomorphic. So the coefficients in $\tilde{B}$ are rational functions of derivatives of w_i. They are all invariant under the action of $U(1, n)$. Now we claim it's also invariant under the complexification of $U(1,n)$, i.e. $GL(n+1,\mathbb{C})$. For any Y in the Lie algebra of $U(1,n)$, the matrix $\tilde{B}(\exp(tY)w)$ does not depend on t, i.e.

$$d\tilde{B}(L_Y w) = 0.$$

From the structure of $\tilde{B}$, it's differential is complex linear. So

$$d\tilde{B}(L_{\sqrt{-1}Y} w) = 0.$$

This implies $\tilde{B}$ is invariant under the infinitesimal transformations of $GL(n+1,\mathbb{C})$. So they are projective invariants of w. □

Remark 3.3. The gauge transformation depends on the choice of j_i. In fact the difference between any two choices of a^i also gives us projective invariants. The proof follows from the same argument in Theorem 3.2, i.e. the differences are holomorphic due to (20) and invariant under group $U(1,n)$ action. In fact, these are well-known projective invariants involving the second-order derivatives. For example, they can be found in [21], Chapter 8, up to a change of basis. In other words, the projective invariants in [21] can be viewed as gauge transformations.

Remark 3.4. The same as curve case, we can assume the metric on E to be positive definite. This gives the similar relation between positive constant bisectional curvature metric and projective invariants in Theorem 3.2.

4 Generalized Schwarzian derivatives by Molzon and Tamanoi

Now we discuss the relation with the work of Molzon and Tamanoi. First we recall the definition of generalized Schwarzian derivatives in [11]. Consider a local biholomorphism $f\colon U \to \mathbb{CP}^n$ defined on some open domain $U \subset \mathbb{C}^n$. Choose a lifting map $\tilde{f}\colon U \to \mathbb{C}^{n+1}$ such that

$$\det(\tilde{f}, \partial\tilde{f}/\partial z_1, \cdots, \partial\tilde{f}/\partial z_n) = 1 \tag{22}$$

Then $M(f) = (\tilde{f}, \partial\tilde{f}/\partial z_1, \cdots, \partial\tilde{f}/\partial z_n)$ defines a frame of the flat $\mathbb{C}^n$ bundle over U. The projective invariants are defined to be the connection matrix under this basis, i.e. $D = M^{-1}dM$. The choice of this frame satisfies the following relation:

$$M(Tf) = TM(f) \tag{23}$$

for any $T \in PSL(n+1, \mathbb{C})$. The choice of $\tilde{f}$ is fixed by (22) up to a $(n+1)$-th root of unity, which does not change D.

The following proposition tells us that the relation (23) guarantees that the connection matrix D is projective invariant.

Proposition 4.1. For any $M(f)$ that depends holomorphically on f satisfying (23), the connection matrix $D(f)$ is projective invariant, i.e. $D(Tf) = D(f)$. Moreover, if M, M' are two choices of frames satisfying (23) and related by a gauge transformation matrix G, i.e.$M = M'G$, then G is also projective invariant.

Proof. By direct calculation. □

Consider the Euler sequence of $\mathbb{CP}^n$

$$0 \to K_{\mathbb{CP}^n}^{\frac{1}{n+1}} \to \mathbb{C}^{n+1} \to K_{\mathbb{CP}^n}^{\frac{1}{n+1}} \otimes T_{\mathbb{CP}^n} \to 0 \tag{24}$$

Choose a metric of signature $(1, n)$ on $\mathbb{C}^{n+1}$, we have a splitting of the sequence (24) on complex hyperbolic ball B^n which does not preserve holomorphic structure. Moreover, for any discrete subgroup Γ of $SU(1, n)$, the splitting sequence descends to $X = \Gamma \backslash B^n$ since it's equivariant under $SU(1, n)$.

Consider any local chart of X, or equivalently the uniformization map f. The coordinate system gives a canonical choice of frame of $E = K_X^{\frac{1}{n+1}} \oplus K_X^{\frac{1}{n+1}} \otimes T_X$, i.e. $s_0, s_1, \cdots s_n$ in the proof of Proposition 3.1. This gives a frame for the bundle $f^*(C^{n+1})$ using f_* and the splitting for (24), denoted by $\tilde{M}$. This frame satisfies the relation (23) for any $T \in SU(1, n)$, but it's not holomorphic frame. In order to get holomorphic frame, we use the gauge transformation G as (19). Then the new frame $M = \tilde{M}G$ is holomorphic. Since G only involves the complex hyperbolic metric, so M still satisfies (23) for any $T \in SU(1, n)$. Following the same argument in the proof of Theorem 3.2, it's also true for any $T \in SL(n, \mathbb{C})$. So the difference between our choice of frames from the metric and the frame in [11] is describe by Proposition 4.1. Especially the gauge transformation matrix G only involves the second derivatives of f, whose components are the projective invariants from Chapter 8 of [21].

5 System of Hodge bundles of Calabi-Yau type

The system of Hodge bundles we consider for the ball quotients is a special case of system of Hodge bundles of Calabi-Yau type. The definition of system of Hodge bundles of Calabi-Yau type is taken from [14].

Definition 5.1. A system of Hodge bundles ($E = \sum_{i+j=k, 0\le i\le k} E^{i,j}, \theta$) on X is of Calabi-Yau type if the following two conditions are satisfied.

1. rank $E^{k,0} = 1$.
2. $\theta : T_X \to Hom(E^{k,0}, E^{k-1,1})$ is an isomorphism.

In the following cases, the line bundle $E^{k,0}$ is a rational power of K_X, i.e.$E^{k,0} = K_X^{\frac{p}{q}}$ and $E^{k-1,1} = E^{k,0} \otimes T_X$. We can put metric on $E^{k,0}$ and $E^{k-1,1}$ by the metric on T_X the same as previous section. Notice that the first two off-diagonal term and first diagonal term for the curvature matrix are still the same as (16). So we have the same proposition as follows.

Proposition 5.2. If the metric h on E induces flat connection D, then the induced metric on T_X is Kähler and Kähler-Einstein with Einstein constant $-\frac{q}{p}$.

As stated in [8] [14], there is a canonical system of Hodge bundles of Calabi-Yau type over any Hermitian symmetric domain. Now we illustrate how the previous results are generalized to general locally Hermitian symmetric domain. For any Hermitian symmetric domain $D = G_{\mathbb{R}}/K$, consider the Borel embedding into its compact dual $\check{D} = G_{\mathbb{C}}/K_{\mathbb{C}}P_-$. There is a canonical variation of Hodge structure on D. The total bundle and filtrations can be extended to $\check{D}$. The Hodge metric on the tangent bundle induces the Bergman metric on the tangent bundle. The Gauss-Manin connection being zero will provide the equation for the curvatures of the Bergman metric. We hope this equation can help us find a corresponding holomorphic differential system and $G_{\mathbb{C}}$ invariants. The constructions are equivariant under G and hence descend to the quotient $X = \Gamma\backslash D$. The Hodge bundle can be reconstructed from the tangent bundle of X and its symmetric powers, with Hodge metric being recovered from the unique Kähler-Einstein metric on X when X is compact.

Example 5.3. Consider the case $D = SU(n,n)/S(U(n) \times U(n))$. Let τ be the tautological bundle on Grassmannian $G(n,2n)$.The Hodge bundle is induced from the tautological exact sequence on $G(n,2n)$.

$$0 \to \tau \to \mathbb{C}^{2n} \to F \to 0 \tag{25}$$

The Calabi-Yau type Hodge bundle is $E = \bigwedge^n \mathbb{C}^{2n}$. Since $Hom(\tau, F) \cong T_{G(n,2n)}$, we have

$$E^{n,0} = \bigwedge^n \tau \cong K_{G(n,2n)}^{\frac{1}{2n}} \tag{26}$$

$$\theta \colon T_{G(n,2n)} \cong Hom(\tau, F) \cong Hom(\bigwedge^n \tau, \bigwedge^{n-1} \tau \otimes F) \tag{27}$$

In order to generalize the Schwarzian derivative to general semi-simple Lie group, we need some constructions from the work of Gross [8] and Sheng-Zuo [14]. Let D be a Hermitian symmetric domain of non-compact type. There is a canonical choice of CY type system of Hodge bundle associated with the special node of the Dynkin diagram. It is a flat bundles E with filtration $F^n \subset \cdots \subset F^0 = E$ on D and also extended to the compact dual $\check{D}$. The construction satisfies the following three properties that we will use

- $F^n = K_H^l$ for some $l \in \mathbb{Q}$
- E is Calabi-Yau type
- E satisfies the local Torelli theorem, i.e.

$$\theta^k \colon Sym^k(T_D) \to Hom(F^n, F^{n-k}/F^{n-k-1}) \tag{28}$$

  is surjective. The kernel of θ^k is denoted by I_k.

Gross call the last property local Torelli theorem, because it means all the "periods" in the variation are generated by taking derivatives of the periods from Kodaira-Spencer map. It can also be interpreted as the oscillating filtration from the mapping into period domain coincides with the Hodge filtration.

In complex hyperbolic ball case, we know that the Hodge metric on the tangent bundle is the same as the Bergman metric. We have similar results for general Hermitian symmetric domain

Theorem 5.4. *The surjective bundle map (28) respects the metric when T_X is equipped with the Bergman metric and F^k with Hodge metric up to a constant multiple, i.e. the bundle isomorphism*

$$Sym^k(T_D)/I_k \cong Hom(F^n, F^{n-k}/F^{n-k-1}) \tag{29}$$

is an isometry up to a constant depending on k.

Proof. We only need to prove the case that G is simple. Following the notations in [22], let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G. Since we consider $D = G/K$ to be Hermitian symmetric, we have a nature way to define almost complex structure on $\mathfrak{p}$ and $\mathfrak{p} \otimes \mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. The variation of Hodge structure on D corresponds to the irreducible representation V of G with special node as highest weight. Then $V = \oplus_k V^k$ is the character decomposition of $V$with respect to the center of K, which gives the graded pieces in the Hodge decomposition. Note that V^k are irreducible representations of K. Since the bundles with metric on both sides are homogenous, this reduces to the fibers over one point. According to [4], the map over one point is given by the K-equivariant map

$$Sym^k(\mathfrak{p}^+) \to Hom(V^n, V^{n-k}) \tag{30}$$

The Bergman metric is unique K-invariant Hermitian metric on $\mathfrak{p}^+$ up to a constant multiple. So the induced metric on $Hom(V^n, V^{n-k})$ by the surjective map is also K-invariant. On the other hand, the Hodge metric on E is induced by the $k + \sqrt{-1}\mathfrak{p}$ invariant metric on V. Hence the Hodge metric on $Hom(V^n, V^{n-k})$ is also K-invariant. Since V^k is irreducible as K-representation, the two K-invariant metrics are equal up to a constant multiple. □

If we are given a locally Hermitian symmetric space $X = \Gamma\backslash D$ where Γ is a discrete subgroup of G. The characteristic variety related to I_2 can be described by the Kähler-Einstein metric from the special directions in the holomorphic sectional curvature. Hence I_2 is determined by the Kähler-Einstein metric. So is $I_k = I_2 \cdot Sym^{k-2}T_X$. Combing with the theorem above, the Calabi-Yau type variation of Hodge structure $E \cong K_X^l \otimes (\oplus_k Sym^k T_X)$ can be recovered from the Kähler-Einstein metric on X.

Remark 5.5. Assume $X = \Gamma\backslash D$ is a locally Hermitian symmetric space of tube type with dimension d and rank n. Up to a finite cover of X, the Hodge bundle E descends to X. Iterate the Higgs field θ for n times, we will get

$$\theta^n \colon E^{n,0} \to E^{0,n} \otimes S^n(\Omega_X) \tag{31}$$

This determines a non-zero section of $S^n(\Omega_X) \otimes K_X^{-\frac{n}{d}}$. The existence of holomorphic section of $S^{dn}(\Omega_X) \otimes K_X^{-n}$ is used by the third author in [20] to characterize locally Hermitian symmetric space. In [3], Catanese and di Scala proved that this condition corresponds to locally Hermitian symmetric space of tube type. The Calabi-Yau type Hodge bundle provides a natural construction of such tensors. Combining with Theorem 3.3 of [14] about the relation between θ^n and the characterization subvarieties defined by Mok, this also explains that the irreducible factorization types of θ^n determines the types of D.

6 Generalized G-invariants

Now we are dealing with the following problem. Let $\check{D}$ be a Hermitian symmetric space of compact type with $G_{\mathbb{C}}$ as the universal cover of its automorphism group. Given any local biholomorphism $f \colon U \to \check{D}$, we want to find a system of holomorphic functions in terms of derivative jets of f, denoted by $S(f)$, which is invariant under automorphism group of $\check{D}$, i.e. $S(f) = S(T \circ f)$ for any $T \in G_{\mathbb{C}}$ and determines f, up to the automorphism group of $\check{D}$. We call it a complete invariants associated with the $(G_{\mathbb{C}}, \check{D})$ structure as mentioned in the introduction.

We generalize the canonical lifting in Molzon and Tamanoi [11] to get $G_{\mathbb{C}}$-invariants using Calabi-Yau type system of Hodge bundle. From the notations in previous section, the local coordinates $(z_1, z_2, \cdots z_d)$ around $z = 0$ on U gives a section of $f^*(F^n)$, i.e. $s_0 = (dz^1 \wedge \cdots \wedge dz^d)^l$. We also view s_0 as a section of $f^*(E) = \mathbb{C}^N$, i.e. a holomorphic function from U to $\mathbb{C}^N$. In other words, $(f^{-1})^* s_0$ gives a local holomorphic section of E under the natural identification of K_D^l with F^n. Denote this function by $s_0(f)$. Since the section s_0 is defined up to a root of unity depending on the denominator of l, so the function $s_0(f)$ is only determined up to a multiple of a root of unity. This is exactly the choice of lifting $\tilde{f}$ in [11] in the projective space case, see (22). Then the partial derivatives of s_0 with respect to z of order less or equal than k gives a generating sections of $f^*(F^{n-k})$, according to local Torelli theorem.

Lemma 6.1. *Let D_z be any holomorphic partial differential operator. Then $T(D_z s_0(f)) = D_z(s_0(T \circ f))$ up to a root of unity, for any $T \in G_{\mathbb{C}}$.*

Proof. The pair $(\check{D}, E)$ is $G_{\mathbb{C}}$-equivariant. Notice that F^n is naturally isomorphic to K_D^l as $G_{\mathbb{C}}$ equivariant bundle. Here the $G_{\mathbb{C}}$ action on K_D^l is induced by $G_{\mathbb{C}}$ action on $\check{D}$ and $G_{\mathbb{C}}$ action on F^n is by the special node in the construction of E. So we have $s_0(T \circ f) = T(s_0(f))$. This implies that $D_z(s_0(T \circ f)) = D_z(Ts_0(f)) = T(D_z s_0(f))$, since T is constant with respect to T. □

When $\check{D}$ is projective space, the corresponding $I_2 = 0$. So we have a frame $M(f) = s_0, s_1, s_2 \cdots s_n$ by defining $s_i = \frac{\partial s_0(f)}{\partial z_i}$. This is exactly the choice in [11]. Then $M^{-1}dM$ gives generalized projective invariants.

When D is general Hermitian symmetric space, we have nontrivial I_2. So higher derivative of s_0 will be linear dependent. If the weight of E is n, the partial derivatives $s_{i_1 \cdots i_m} := \frac{\partial^m s_0}{\partial z_{i_1} \cdots \partial z_{i_m}}$ for $m \leq n$ generate f^*E. With a choice of sub index set $\{i_1 \cdots i_m\}_{m \leq n}$ depending on f, we have $s_{i_1 \cdots i_m}$ form a basis for $f^*(E)$ for a possibly smaller defining domain of f around $z = 0$. Here we denote the basis by $M(f)$. Then we have similar generalization of $(G_{\mathbb{C}}, \check{D})$ invariants.

Theorem 6.2. *Let $D(f) = M^{-1}dM$ be the connection matrix under the frame M. Then we have $D(Tf) = D(f)$ for any $T \in G_{\mathbb{C}}$ with the same sub index set choice for Tf. Furthermore if $D(f_1) = D(f_2)$, then there exist $T \in G_{\mathbb{C}}$ such that $f_1 = Tf_2$.*

Proof. Since I_k is homogenous under the action of $G_{\mathbb{C}}$, the same sub index set choice for Tf also gives a frame $M(Tf)$ and satisfies $M(Tf) = TM(f)$ from Lemma 6.1. So we have $D(Tf) = D(f)$.

For the second part of the theorem, consider the period map from $\check{D}$ to the compact dual $\check{\Omega}_D$ of the period domain and the further map to $\mathbb{CP}^{N-1}$ by choosing F^n in the filtration. Denote the composition of the maps f_i with the embedding $\check{D} \to \mathbb{CP}^{N-1}$ to be $\tilde{f}_i$. From the same argument in Theorem 3.2, the two maps $\tilde{f}_1$ and $\tilde{f}_2$ are equal up to a projective transformation. So we have $f_1 = gf_2$ for some $g \in SL(N, \mathbb{CP})$. Because $\check{D} \to \mathbb{CP}^{N-1}$ is equivariant under the representation of $G_{\mathbb{C}} \to SL(N, \mathbb{C})$ and f_i is local biholomorphism, so we have g in the image of $G_{\mathbb{C}}$. This proves the second part of the theorem. Actually the whole filtration on f^*E can be recovered from $\tilde{f}_i$ by taking the oscillating filtration since it coincides with Hodge filtration due to the Torelli-type property of E. □

Remark 6.3. The choice of canonical lifting depends on the isomorphism between $K_{\check{D}}^l \cong F_n$. Since any automorphism of line bundles on compact complex manifolds with base fixed is scaling, the lifting is determined up to a constant not related to f. This does not change the connection matrix $D(f)$.

Example 6.4. When $(G_{\mathbb{C}}, \check{D}) = (SL(2,\mathbb{C}), \mathbb{P}^1)$. A local biholomorphism can be written as $[1\colon f(z)]$. Let $e_1 = (1,0)^T, e_2 = (0,1)^T$ be the standard basis for $\mathbb{C}^2$. Assume $f(z_0) = 0$. Then $f_*(\frac{\partial}{\partial z}) = f'(z_0)e_1^* \otimes e_2$ at z_0. So the lifting is

$$\varphi \coloneqq s_0 = (f')^{-\frac{1}{2}}(1,0)^T \tag{32}$$

at z_0. For general value of $f_{21}(z_0)$, we can apply the inverse of

$$\begin{bmatrix} 1 & 0 \\ f(z) & 1 \end{bmatrix} \tag{33}$$

at z_0 to f and use the equivariance of lifting. Hence

$$\varphi = (f')^{-\frac{1}{2}}(1,f)^T \tag{34}$$

$$\varphi_z = -\frac{1}{2}(f')^{-\frac{3}{2}}f''(1,f)^T + (f')^{-\frac{1}{2}}(0,f')^T \tag{35}$$

$$\varphi_{zz} = (\frac{3}{4}(f')^{-\frac{5}{2}}(f'')^2 - \frac{1}{2}(f')^{\frac{3}{2}}f''')(1,f)^T \tag{36}$$

$$= -\frac{1}{2}(\frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2)\varphi. \tag{37}$$

The connection has the form

$$D(f) = \begin{bmatrix} 0 & S_1 dz \\ dz & S_2 dz \end{bmatrix} \tag{38}$$

and $S_2 = 0$ and $S_1 = -\frac{1}{2}S(f)$, which recovers the Schwarzian derivative of f.

Remark 6.5. In the case that $G_{\mathbb{C}} = SL(n+1,\mathbb{C}), \check{D} = \mathbb{P}^n$, this is exactly the invariants defined in [11]. The transformation of these projective invariants under coordinates is as follows. If $w = (w_1, \cdots, w_n)$ is another coordinate, then the section $s_0(w) = (\det(\frac{\partial w}{\partial z}))^{\frac{1}{n+1}} s_0(z)$. The change of connection matrix follows from the change of basis in s_i induced by s_0.

Remark 6.6. In the case that the Hodge bundle has weight $n = 2$, the map θ^2 gives us a section $S^2(\Omega_D) \otimes K_D^{-2l}$ according to Remark 5.5. The pull back of this section on U is invariant under action of $G_{\mathbb{C}}$, hence can be viewed as a $G_{\mathbb{C}}$ invariant. It also determines I_2 and the hence also the allowed sub index set for $m = 2$.

Remark 6.7. The obvious relation of the invariants defined above comes from the connection being horizontal and flat. But there're more constrains on the possible invariants we can get. For example, a priori we could have two invariants for $SL(2, \mathbb{C})$. The calculation shows that one of them is Schwarzian derivative without any constrain, the other one is always zero from the previous example. In general, for projective invariants by Molzon and Tamanoi, since the frame formed canonical lifting is in structure group $SL(n+1, \mathbb{C})$, the connection matrix is trace free. In [11], it's proved that these are the only constrains for possible D arises as $D(f)$, because the basis for parallel sections have constant determinant. This implies that the space of projective structures in dimension n has an affine structure. For general $G_{\mathbb{C}}$ structure, it's not clear what the complete sets of constrains are.

7 A remark on opers

There are generalizations of projective structures on curves called opers by replacing $PSL(2, \mathbb{C})$ by general simple complex Lie group G. In [6], Drinfeld-Sokolov defined the general notion of opers on curves under local coordinates and studied the relations to higher-order differential operators. When $G = PSL(2, \mathbb{C})$, this recovers the projective structure, Schwarzian derivative and the relation to Fuchsian type equation. In [1], Beilinson-Drinfeld wrote these results in a coordinate free and global manner. In [2], Indranil Biswas studied a class of equivariant immersions of universal cover of a compact Riemann surface into a projective space, which is related to the developing map of $PSL(n, \mathbb{C})$ opers. For reader's convenience, we include the explicit formulas for the relations between the developing maps and differential operators using the canonical lifting from the previous section, which is a generalization of Schwarzian derivative formula. Again, the discussion here should be well-known to experts, for example, see [6], [1], or Chapter 4 of [7]. From this point of view, the study of G-invariants in the previous sections may be regarded as generalizations of opers for higher dimensional base.

A projective structure on a Riemann surface X is given by local charts

$(U_i, \phi_i \colon U_i \to \mathbb{CP}^1)$ such that transition functions ϕ_{ij} are projective transformations in $PSL(2,\mathbb{C})$. In other words, we have a developing map $f \colon \widetilde{X} \to \mathbb{CP}^1$ with monodromy representation $\rho \colon \pi_1(X) \to PSL(2,\mathbb{C})$, such that f is equivariant under $\pi_1(X)$-action. The pull back of the Euler sequence of $\mathbb{CP}^1$ gives a flat rank-two bundle with filtration on $\widetilde{X}$ and descends to X if ρ lifts to $SL(2,\mathbb{C})$. Since the map f is nondegenerate, the Higgs field on X is nonzero. In the notion of opers, this interpretation says that the flat $G = SL(2,\mathbb{C})$-connection with a reduction to the Borel subgroup, such that the classifying map is nondegenerate. In general, opers are the above information for general simple Lie group G and replacing nondegneracy by being horizontal and strictly transverse.

When $G = SL(n,\mathbb{C})$, we can define the generalized projective structure compatible with fixed complex structure to be a horizontal and strictly transverse holomorphic map $f \colon \widetilde{X} \to G/B$ with monodromy representation $\rho \colon \pi_1(X) \to PSL(n,\mathbb{C})$, such that f is equivariant under $\pi_1(X)$-action. Here being horizontal and strictly transverse means the following: consider the tautological representation of G on $\mathbb{C}^n$ and the corresponding flat bundle with filtration. This induced a horizontal distribution on $T(G/B)$. So we require f_* lands into the horizontal distribution and no where zero when projected to each component. This is just the developing map of opers of $SL(n,\mathbb{C})$. Two generalized projective structure are equivalent iff they differ by an element in $SL(n,\mathbb{C})$. The following is the relations between the generalized projective structure with some generalized Schwarzian derivatives of f and higher order differential equations.

Theorem 7.1 (Drinfeld-Sokolov [6])**.** *For a local coordinate $z \in U \subset X$, there is a bijection between equivalent classes of generalized projective structures on U and holomorphic ordinary differential equations of the form $u^{(n)} - S_{n-1}u^{(n-2)} - \cdots - S_1 = 0$.*

Now we want to derive the theorem using the canonical lifting similar to the previous sections. Take the Borel subgroup to be upper triangular matrices and the simple roots are $\alpha_i \colon diag(a_1, \cdots, a_n) \mapsto a_i - a_{i+1}$ and denote weights $\lambda_i \colon diag(a_1, \cdots, a_n) \mapsto a_i$. The filtration on G/B is the complete flag $F^{n-1} \subset F^{n-2} \subset \cdots \subset F^0 = V = \mathbb{C}^n$. The horizontal distribution is

$$\bigoplus_{i=1}^{n-1} Hom(F^{n-i}/F^{n-i+1}, F^{n-i-1}/F^{n-i}). \tag{39}$$

with each component $F^{n-i}/F^{n-i+1} \cong L_{\lambda_i}$. Here $L_{\lambda_i} \cong G \times_B \mathbb{C}_{\lambda_i}$ is the line bundle associated to the character λ_i of B. The tangent map $Tf\colon TU \to f^*(\bigoplus_{i=1}^{n-1} Hom(F^{n-i}/F^{n-i+1}, F^{n-i-1}/F^{n-i}))$ is nowhere zero projecting to each component. So $s = \frac{\partial}{\partial z}$ under f_* induces generating sections of $s_i \in \Gamma(U, f^*L_{-\alpha_i})$. Since $F^{n-1} \cong L_{\frac{n-1}{n}\alpha_1 + \frac{n-2}{n}\alpha_2 + \cdots + \frac{1}{n}\alpha_{n-1}}$, the product of $s_i^{-\frac{n-i}{n}}$ defines a section of F^{n-1}, hence a function $\varphi_f\colon U \to \mathbb{C}^n$. This is the analogue of canonical lifting in the previous section. Since we need to take a n-th root of s, the canonical lifting is determined up to a n-th root of unity. It satisfies the equivariant property with respect to compositions with elements g in G

$$\varphi_{g\circ f} = g \cdot \varphi_f. \tag{40}$$

The sections $(\varphi, \frac{\partial}{\partial z}\varphi, \cdots, \frac{\partial^{n-1}}{\partial z^{n-1}}\varphi)$ form a basis of f^*V, denoted by M_f. For any $g \in G$, we have

$$M_{g\circ f} = g \cdot M_f. \tag{41}$$

Consider the connection matrix of M_f

$$A_f = M_f^{-1} dM_f. \tag{42}$$

It has the form

$$A_f = \begin{bmatrix} 0 & 0 & \dots & S_1 dz \\ dz & 0 & \dots & S_2 dz \\ 0 & dz & \ddots & \vdots \\ 0 & \dots & dz & S_n dz \end{bmatrix} \tag{43}$$

and satisfies the following property

$$A_{g\circ f} = A_f \tag{44}$$

for any $g \in G$ and does not depend on the root of unity factor for φ_f. So $S_1, \cdots, S_n$ are invariants under the projective transformation.

Proposition 7.2. The last invariant S_n is always 0.

We postpone the proof after the explicit formula for $n = 3$ case.

Example 7.3. The explicit formula of S_i for $n = 2$ is in Example 6.4. We include the case $n = 3$ as follows. Under a suitable basis of V, the map f locally can be written as lower triangular matrix with diagonal elements equal to 1. The subspace F^{n-i} is generated by the first i column vectors. For $n = 3$, the map is

$$\begin{bmatrix} 1 & 0 & 0 \\ f_1(z) & 1 & 0 \\ f_2(z) & f_3(z) & 1 \end{bmatrix} \tag{45}$$

Let e_i be the standard basis for $\mathbb{C}^n$. Assume $f_i(z_0) = 0$. Then $f_*(\frac{\partial}{\partial z}) = f_1' e_1^* \otimes e_2 + f_2' e_1^* \otimes e_3 + f_3' e_2^* \otimes e_3$. We have $f_2' = 0$ from f being horizontal and $\varphi = f_1'^{-\frac{2}{3}} f_3'^{-\frac{1}{3}} (1, 0, 0)^T$. Using the same argument as $n = 2$ case, we have

$$f_2' = f_1' f_3 \tag{46}$$

$$\varphi = f_1'^{-\frac{2}{3}} f_3'^{-\frac{1}{3}} (1, f_1, f_2)^T \tag{47}$$

Denote the factor $f_1'^{-\frac{2}{3}} f_3'^{-\frac{1}{3}}$ by C. A direct calculation shows that $\varphi_{zzz} = -S_1 \varphi - S_2 \varphi_z$ with

$$S_1 = \frac{C'''}{C} - \frac{C'}{C} \frac{f_1'''}{f_1'} - 3 \frac{C'' C'}{C^2} \tag{48}$$

$$S_2 = 3 \frac{C''}{C} + \frac{f_1'''}{f_1'} \tag{49}$$

Now we prove Proposition 7.2.

Proof. Write the map f as the matrix form in the previous example. The elements in the matrix is denoted by $f_{ij}(z)$. The i-th column is denoted by a_i. Because f is horizontal, we have $a_i' = f_{i+1,i} a_{i+1}$. The canonical lifting is $\varphi = C a_1$, where $C = (f_{21}')^{-\frac{n-1}{n}} (f_{32}')^{-\frac{n-2}{n}} \cdots (f_{n,n-1}')^{-\frac{1}{n}}$. We need to prove that $\varphi^{(n)} \in Span(a_1, \cdots, a_{n-1})$. So we only need to prove the last two terms in the expansion $nC' a_1^{(n-1)} + C a_1^{(n)}$ is in $Span(a_1, \cdots, a_{n-1})$. By induction, we have

$$a_1^{(n-1)} = f_{21}' f_{32}' \cdots f_{n,n-1}' a_n \tag{50}$$

$$+ ((n-2) f_{21}'' f_{32}' \cdots f_{n-1,n-2}' + (n-3) f_{21}' f_{32}'' \cdots f_{n-1,n-2}' + \cdots + f_{21}' f_{32}' \cdots f_{n-1,n-2}'') a_{n-1} \tag{51}$$

$$+ O \tag{52}$$

$$a_1^{(n)} = ((n-1) f_{21}'' f_{32}' \cdots f_{n,n-1}' + (n-2) f_{21}' f_{32}'' \cdots f_{n,n-1}' + \cdots + f_{21}' f_{32}' \cdots f_{n,n-1}'') a_{n-1} + O \tag{53}$$

Here O means terms involving lower index a_i. So the coefficient of a_n in the expansion of $(n+1)C'a_1^{(n-1)} + Ca_1^{(n)}$ is zero.

□

The same as in the previous cases, we can recover the generalized projective structure by solving higher order differential equations

$$u^{(n)} - S_{n-1}u^{(n-2)} - \cdots - S_1 = 0, \tag{54}$$

which gives the map to the $\mathbb{P}^{n-1}$. Since the complete flag is determined by the oscillating filtration. This determines the map to G/B. More explicitly, we take the linear independent solutions $f_1, \cdots, f_n$ such that the Wronskian determinant is equal to 1. Then we have a map to G/B by the Wronskian matrix W. Denote the i-th column vector by $\tilde{e}_i = (f_1^{(i)}, \cdots, f_n^{(i)})^T$. Then $f_*(\frac{\partial}{\partial z}) = \tilde{e}_1^* \otimes \tilde{e}_2 + \tilde{e}_2^* \otimes \tilde{e}_3 + \cdots + \tilde{e}_{n-1}^* \otimes \tilde{e}_n$. So the lifting $\varphi = \tilde{e}_1$. This can be seen use the same trick as before by applying $g \in SL(n,\mathbb{C})$, which is the inverse of W at z_0. So we have a bijective correspondence between generalized projective structure on U and $n-1$-tuple of holomorphic functions $(S_1, \cdots, S_{n-1})$.

The discussion also gives another criterion for the canonical lifting. The map $f \to G/B$ is determined by the oscillating filtration from the map to $\mathbb{P}^{n-1}$, written as $[f_1, \cdots, f_n]$. Then $\varphi = (f_1, \cdots, f_n)^T$ is the canonical lifting if and only if the $\det(\varphi, \frac{\partial}{\partial z}\varphi, \cdots, \frac{\partial^{n-1}}{\partial z^{n-1}}\varphi)$is equal 1. This also gives another proof of Proposition 7.2, since the connection matrix is in $\mathfrak{sl}(n,\mathbb{C})$.

Remark 7.4. The transformation law for S_i under coordinate change can be deduced from the change of lifting φ. If $w = w(z)$ is another coordinate, then $s_w = \frac{\partial}{\partial w} = w'^{-1}s_z$. So $s_{i,w} = w'^{-1}s_{i,z}$. Hence $\varphi_w = w^{\frac{\prime n(n-1)}{2}}\varphi_z$. The transformation law for S_i is the same as the gauge transformation under the base change for φ and its higher derivatives.

Remark 7.5. In [2], Indranil Biswas studied a class of equivariant immersions of universal cover of a compact Riemann surface into a projective space. Notice that the map f to the complete flag is determined by the oscillating filtration of the projection to the first subspace F^{n-1}, i.e. the map to $\mathbb{P}^{n-1}$. Indranil Biswas' condition for the immersion exactly picks out the maps to $\mathbb{P}^{n-1}$ that can recover a horizontal and strictly transverse map to G/B by taking oscillating filtration.

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