SOME EXAMPLES OF DR-INDECOMPOSABLE SPECIAL FIBERS OF SEMI-STABLE REDUCTIONS OVER WITT RINGS

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Abstract. We answer negatively an open problem of Illusie on the DR-decomposability of the log de Rham complex of the special fiber of a semi-stable reduction over the Witt ring. We also show that E_1 degeneration of the Hodge to log de Rham spectral sequence does not imply DR-decomposability of semi-stable varieties.

1. Introduction

The work of Deligne-Illusie [5] is fundamental in Hodge theory since it gives a new method to establish the E_1-degeneration property of the Hodge to de Rham spectral sequence. Let k be a perfect field of positive characteristic and X_0 an algebraic variety over k. We have the following commutative diagram of Frobenius

$$\begin{array}{ccccc} X_0 & \xrightarrow{F=F_{X_0/k}} & X_0' & \xrightarrow{\pi} & X_0 \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}\ k & \xrightarrow{\sigma} & \mathrm{Spec}\ k, \end{array}$$

The variety X_0 is said to be *DR-decomposable* if the complex $\tau_{<p}F_*\Omega^\bullet_{X_0}$ is quasi-isomorphic to $\bigoplus_{i=0}^{\dim X}\Omega^i_{X_0'/k}[-i]$, where $\Omega^\bullet_{X_0}$ is the de Rham complex of X_0/k. The main result of Deligne-Illusie asserts that for *smooth* varieties, X_0 is $W_2 = W_2(k)$-liftable if and only if it is DR-decomposable. On the other hand, if X_0 is proper over k and $\dim X_0 < p$, the DR-decomposability of X_0 implies the E_1-degeneration of the Hodge to de Rham spectral sequence (for $\dim X_0 = p$, the E_1-degeneration also holds by the Grothendieck duality). Properness on X_0 is required because of finite dimensionality of Hodge cohomologies. However, it is not clear whether one can remove the assumption on the dimension of X_0: this is exactly one of the two open problems posed by Illusie [6]. It is neither clear whether E_1-degeneration would imply the DR-decomposability.

This note grew out from our study on the other open problem posed by Illusie in loc. cit, that is about the generalization of Deligne-Illusie's main result to semi-stable varieties over k. Note that semi-stable varieties appear naturally in algebraic geometry as very typical singular varieties. The problem is stated as follows: let k be as above and $W = W(k)$ the ring of Witt vectors. For a semi-stable reduction X over W, we set $X_0 = X \times_W k$, the special fiber of X, and $F : X_0 \to X_0'$ the relative Frobenius. Consider the complex of $\mathscr{O}_{X_0}$-modules:

$$\Omega^{\log\bullet}_{X_0} = \Omega^\bullet_X(\log X_0)|_{X_0}.$$

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Problem 1.1 (Illusie, Problem 7.14 [6]). *Is the complex $\tau_{<p}F_*\Omega^{\log\bullet}_{X_0}$ decomposable in $D(X_0')$?*

Our answer to this problem is NO. Indeed, we constructed explicit examples of semi-stable reductions over W negating the problem, whose dimension can be arbitrary large (in the curve case the answer is affirmative for cohomological reason) and the characteristic of k can be arbitrary. See §3 for the construction. We also examined the E_1-degeneration property of these examples. It turns out that all examples we constructed whose dimensions are less than or equal to the characteristic of the residue field have the E_1-degeneration property. This is a direct consequence of Theorem 4.1 and Deligne-Illusie's decomposation theorem. Therefore, the E_1-degeneration property is NOT equivalent to the DR-decomposability in the semi-stable (non-smooth) case. We are not aware of similar results in the smooth case.

2. DR-decomposability and log deformation

We use the log geometry as developed in the work [3] to study Problem 1.1, and the construction of our examples is mainly based on a simple criterion of the DR-decomposability in terms of the existence of a log smooth deformation over the log scheme $(W_2(k), 1 \mapsto 0)$ (Theorem 2.3).

Let X be a semi-stable reduction over W. Let M_{X_0} (resp. $M_{\mathrm{Spec}(k)}$) be the log structure on X (resp. $\mathrm{Spec}(W)$) attached to the reduced normal crossing divisor X_0 (resp. $\mathrm{Spec}(k)$) (Example (1.5) [3]). Then the extended morphism of log schemes $f : (X, M_{X_0}) \to (\mathrm{Spec}(W), M_{\mathrm{Spec}(k)})$ is smooth. Let $(X_0, M_0) \to \mathbf{k} := (k, 1 \mapsto 0)$ be the base change of f via the inclusion $\mathrm{Spec}(k) \to \mathrm{Spec}(W)$. When the context is clear, we denote the log scheme (X_0, M_0) simply by X_0 (in some other occasion, we use $\underline{X}$ to denote the underlying scheme of a log scheme X). It is known that the morphism $X_0 \to \mathbf{k}$ is smooth, and the de Rham complex $\Omega^\bullet_{X_0/\mathbf{k}}$ of the log variety $X_0/\mathbf{k}$ is naturally isomorphic to the complex $\Omega^{\log\bullet}_{X_0}$ considered in §1 (1.7 [3]). Moreover, it is known that the log structure M_0 of X_0 is of semi-stable type:

Definition 2.1. ([15]) A log variety X over $\mathbf{k}$ is called semi-stable type if étale locally over each closed point $x \in \underline{X}$ it is strict smooth over

$$(\mathrm{Spec}(k(x)[x_1, \cdots, x_r]/(x_1 \cdots x_r)), \bigoplus_{i=1}^{r} \mathbb{N}e_i, e_i \mapsto x_i),$$

where the log structure is induced by the homomorphism of monoids $\bigoplus_{i=1}^{r} \mathbb{N}e_i \to \mathscr{O}_{\underline{X}}$ defined by $e_i \mapsto x_i$.

Let F be the absolute Frobenius of the log scheme $\mathbf{k}$ which is given by the commutative diagram

$$\begin{array}{ccc} k & \xrightarrow{F_k} & k \\ {\scriptstyle 0}\uparrow & & {\scriptstyle 0}\uparrow \\ \mathbb{N} & \xrightarrow{\times p} & \mathbb{N}. \end{array}$$

It is easy to verify that F is liftable to the log scheme $\mathbf{W}_2 := (W_2, 1 \mapsto 0)$ (but not to the log scheme $(W_2, 1 \mapsto p)$!), and an obvious lifting G over $\mathbf{W}_2$ is given by the following commutative diagram:

$$\begin{array}{ccc} W_2 & \xrightarrow{F_{W_2}} & W_2 \\ {\scriptstyle 0}\uparrow & & \uparrow{\scriptstyle 0} \\ \mathbb{N} & \xrightarrow{\times p} & \mathbb{N}, \end{array}$$

where F_{W_2} is the Frobenius automorphism of W_2. A special case of the Kato's decomposition theorem is the following

Theorem 2.2 (Theorem 4.12 [3])**.** *Let $X/\mathbf{k}$ be a log variety of semi-stable type and X' the base change of X via the absolute Frobenius of $\mathbf{k}$. Let $F_{X/\mathbf{k}} : X \to X'$ be the relative Frobenius. Then the complex $\tau_{<p}F_{X/\mathbf{k}*}\Omega^{\bullet}_{X/\mathbf{k}}$ is decomposable if and only if X' is liftable to $\mathbf{W}_2$.*

Remark that Kato's decomposition theorem works for a log variety of Cartier type which is more general than semi-stable type (Definition 4.8 [3]). In the following, we show further that X' is liftable to $\mathbf{W}_2$ if and only if X itself is liftable to $\mathbf{W}_2$, and hence we obtain the following criterion for DR-decomposability:

Theorem 2.3. *Notation and assumption as Theorem 2.2. Then the complex $\tau_{<p}F_{X/\mathbf{k}*}\Omega^{\bullet}_{X/\mathbf{k}}$ is decomposable if and only if X is liftable to $\mathbf{W}_2$.*

Proof. Via the base change by G, one obtains a $\mathbf{W}_2$-lifting of X' from that of X. Since G is not an isomorphism of log schemes, our argument is to show the converse nevertheless is still true. Let $\omega_X \in H^2(X, T_{X/\mathbf{k}})$ (resp. $\omega_{X'} \in H^2(X', T_{X'/\mathbf{k}})$) be the obstruction class of the lifting of X (resp. X') to $\mathbf{W}_2$. Recall that ω_X is constructed as follows: Let $\{U_i\}$ be an affine cover of X. Choosing for each U_i a log smooth lifting $\mathscr{U}_i$ on $\mathbf{W}_2$, we then have that on each overlap $U_{ij} = U_i \cap U_j$ there exists an isomorphism $\alpha_{ij} : \mathscr{U}_j|U_{ij} \to \mathscr{U}_i|U_{ij}$. Then ω_X is represented by $\{(U_i \cap U_j \cap U_k, \alpha_{ij}\alpha_{jk}\alpha_{ki})\}$. Because of the existence of G, $\{(G^{-1}(U_i \cap U_j \cap U_k), G^*\alpha_{ij}\alpha_{jk}\alpha_{ki})\}$ represents $\omega_{X'}$. Thus we have that $\sigma^*(\omega_X) = \omega_{X'}$ through the canonical map

$$\begin{array}{ccc} H^2(X', T_{X'/\mathbf{k}}) & = & H^2(X', \sigma^* T_{X/\mathbf{k}}) \, . \\ & & {\scriptstyle \sigma^*}\uparrow \\ & & H^2(X, T_{X/\mathbf{k}}) \end{array}$$

The above equality uses the fact that $\sigma^*\Omega_{X/\mathbf{k}} = \Omega_{X'/\mathbf{k}}$ and that both sheaves are locally free (see 1.7 and Proposition 3.10 [3]). However, since σ is an isomorphism of schemes, the map σ^* in the vertical line is bijective. It follows immediately that $\omega_X = 0$ under the assumption that X' is $\mathbf{W}_2$-liftable and hence X itself is $\mathbf{W}_2$-liftable. □

The following corollary ensures it is valid to assume k is algebraically closed in the study of Problem 1.1.

Corollary 2.4. *Let $f : X \to \mathbf{k}$ be a smooth morphism of semistable type and k' be a perfect field containing k. Denote by $\mathbf{k}'$ the field k' with the induced log structure from $\mathbf{k}$ and by $X_{\mathbf{k}'}$ the log base change. Then $\tau_{<p}F_{X/\mathbf{k}*}\Omega^{\bullet}_{X/\mathbf{k}}$ is decomposable if and only if $\tau_{<p}F_{X_{\mathbf{k}'}/\mathbf{k}'*}\Omega^{\bullet}_{X_{\mathbf{k}'}/\mathbf{k}'}$ is decomposable.*

Proof. By Theorem 2.3, it is enough to show that a $(W_2(k'), \mathbb{N} \mapsto 0)$-lifting of $X_{\mathbf{k}'}$ induces a $(W_2(k), \mathbb{N} \mapsto 0)$-lifting of X. By the flat base change, one has the isomorphism $H^2(X, T_{X/\mathbf{k}}) \otimes_k k' = H^2(X_{\mathbf{k}'}, T_{X/\mathbf{k}'})$ and hence the injection $\alpha : H^2(X, T_{X/\mathbf{k}}) \to H^2(X_{\mathbf{k}'}, T_{X/\mathbf{k}'})$. Then, by the same arguments in Theorem 2.3, the obstruction class ob_k to lifting X to $\mathbf{W_2(k)}$ is mapped to to the obstruction class $ob_{k'}$ of lifting $X_{\mathbf{k}'}$ to $(W_2(k'), \mathbb{N} \mapsto 0)$ via the map α. By the condition that $\alpha(ob_k) = ob_{k'} = 0$, it follows that $ob_k = 0$. □

Remark 2.5. After presenting our results, Weizhe Zheng provided us a more conceptual proof of Theorem 2.3: Denote by $\mathrm{Lift}(X)$ (resp.$\mathrm{Lift}(X')$) the groupoid of liftings of X (rsep. X') over $\mathbf{W}_2$. Let $G : \mathbf{W}_2 \to \mathbf{W}_2$ be a lifting of the log Frobenius morphism $F : \mathbf{k} \to \mathbf{k}$. Given a lifting $X^{(1)} \in \mathrm{Lift}(X)$, the pullback of $X^{(1)}$ along G gives an object in $\mathrm{Lift}(X')$. With the obvious assignments on morphisms, one can get a functor

$$A : \mathrm{Lift}(X) \to \mathrm{Lift}(X').$$

Conversely, let $X'^{(1)} \in \mathrm{Lift}(X')$ be a lifting of X'. Denote by $i : X' \hookrightarrow X'^{(1)}$ the canonical strict closed immersion and by $\sigma : X' \to X$ the base change of $F : \mathbf{k} \to \mathbf{k}$. Recall that $\underline{\sigma} : \underline{X'} \to \underline{X}$ is an isomorphism and $\mathscr{M}_{X'} \simeq \mathscr{M}_X \oplus_{\mathscr{K}_k} \mathscr{M}_k$. One can construct the pushout $X'^{(1)} \amalg_{X'} X$ of the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle i} & & \\ X'^{(1)} & & \end{array}$$

as follows:

- The underlying scheme $\underline{X'^{(1)} \amalg_{X'} X}$ is defined to be $\underline{X'^{(1)}}$,
- the log structure of $X'^{(1)} \amalg'_X X$ is defined to be $\mathscr{M}_{X'^{(1)}} \times_{\mathscr{M}_{X'}} \mathscr{M}_X$.

With the obvious assignments on morphisms, the pushout process along $\sigma : X' \to X$ gives a functor

$$B : \mathrm{Lift}(X') \to \mathrm{Lift}(X).$$

It is straightforward to check the following proposition.

Proposition 2.6. *The functor A gives an equivalence of groupoids, and the functor B is its quasi-inverse.*

3. Examples

In this section, k is an algebraically closed field of characteristic $p > 0$. We proceed to construct examples of semi-stable reductions over W whose special fibers do not admit log deformation to $\mathbf{W}_2$, which negate Problem 1.1 because of Theorem 2.3.

3.1. **More preparations.** The first lemma is another characterization of semi-stable reductions over $W = W(k)$.

Lemma 3.1. *Let K_0 be the fractional field of W. Then an W-scheme X is a semi-stable reduction over W if and only if the following two properties hold:*

1. *the generic fiber $X_{K_0} = X \times_W K_0$ is smooth over K_0,*
2. *the special fiber $X_k = X \times_W k$ is a normal crossing variety over k.*

Proof. See [4], 2.16. □

The second lemma is rather standard.

Lemma 3.2. *Let $X/\mathbf{k}$ be a log variety of semi-stable type. Assume the irreducible components $\{X_i, i \in I\}$ of the underlining variety $\underline{X}$ to be smooth. Let $\mathscr{X}$ be a smooth deformation X over $\mathbf{W}_2$. Then the underlying scheme of $\mathscr{X}$ is written into the schematic union of closed subschemes $\underline{\mathscr{X}} = \bigcup_{i\in I} \mathscr{X}_i$s with the property that, for each nonempty $J \subseteq I$, the schematic intersection $\bigcap_{j\in J} \mathscr{X}_j$ is a W_2-lifting of $\bigcap_{j\in J} X_j$.*

Proof. Set

$$\mathscr{I}_i = I_i + pI_i,$$

where I_i is the ideal sheaf of X_i in X. Then, $\mathscr{I}_i$ is an ideal sheaf of $\mathscr{O}_{\underline{\mathscr{X}}}$. We claim that the closed subschemes $\mathscr{X}_i$s defined by $\mathscr{I}_i$s have the property in the lemma. To show this it suffices to prove the following properties:

(1) $\mathscr{O}_{\underline{\mathscr{X}}}/\mathscr{I}_i$ is flat over W_2,
(2) $\bigcap \mathscr{I}_i = 0$, and
(3) for each nonempty $J \subseteq I$, $\mathscr{O}_{\underline{\mathscr{X}}}/\cup_{j\in J}\mathscr{I}_j$ is flat over W_2.

Since $\widehat{\mathscr{O}_{\underline{\mathscr{X}},x}}$ is faithfully flat over $\mathscr{O}_{\underline{\mathscr{X}},x}$ for each point $x \in \underline{\mathscr{X}}$, it suffices to verify the above claim after tensoring with $\widehat{\mathscr{O}_{\underline{\mathscr{X}},x}}$ for every $x \in \underline{\mathscr{X}}$. By ([3] Theorem 3.5, Proposition 3.14), there is an étale morphism $U \to \underline{\mathscr{X}}$ such that we have

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathrm{Spec}(W_2[x_1,\cdots,x_n]/(x_1\cdots x_r)) \\ & \searrow^{\pi'|_U} & \downarrow \\ & & \mathrm{Spec}(W_2) \end{array},$$

where f is an étale morphism. As a consequence, there is an isomorphism

$$\alpha : \widehat{\mathscr{O}_{\underline{\mathscr{X}},x}} \cong W_2[[x_1,\cdots,x_n]]/(x_1\cdots x_r)$$

such that each $\mathscr{I}_i\widehat{\mathscr{O}_{\underline{\mathscr{X}},x}}$ (whenever it is nonempty) is generated by $\alpha^{-1}(\Pi_{j\in J_i}x_j)$ for some nonempty set $J_i \subseteq \{1,\cdots,r\}$. Moreover, $\{1,\cdots,r\}$ is the disjoint union of J_is. Then the claim follows from direct calculations. □

By the above two lemmas, we can conclude the following

Proposition 3.3. *Let Z be a smooth scheme over W. Let Y_0 be a smooth closed subvariety of $Z_0 = Z \times_W k$. Set $X = Bl_{Y_0}Z$, the blowup of Z along the closed subscheme Y_0. Then X is a semi-stable reduction over W, whose special fiber X_0 is a simple normal crossing divisor consisting of two smooth components $Bl_{Y_0}Z_0$ and $\mathbb{P}(N_{Y_0/Z})$ (the projective normal bundle of Y_0 in Z) which intersect transversally along $\mathbb{P}(N_{Y_0/Z_0})$ (the projective normal bundle of Y_0 in Z_0). Furthermore, if the normal crossing variety X_0 over $\mathbf{k}$ admits a smooth deformation over $\mathbf{W}_2$, then both pairs $(Bl_{Y_0}Z_0, \mathbb{P}(N_{Y_0/Z_0}))$ and $(\mathbb{P}(N_{Y_0/Z}), \mathbb{P}(N_{Y_0/Z_0}))$ are $W_2(k)$-liftable.*

Proof. The first statement follows from Lemma 3.1 (the remaining fact is fairly standard and therefore omitted, see [8]). The second statement follows from Lemma 3.2. □

Proposition 3.4 (Cynk-van Straten, [1] Theorem 3.1)**.** *Let $\pi : Y \to X$ be a morphism of schemes over k and let $S = Spec(A)$, where A is artinian with residue field k. Assume that $\mathscr{O}_X = \pi_*\mathscr{O}_Y$ and $R^1\pi_*(\mathscr{O}_Y) = 0$. Then for every lifting $\mathscr{Y} \to S$ of Y there exists a preferred* *lifting $\mathscr{X} \to S$ making a commutative diagram*

$$\begin{array}{ccc} Y & \hookrightarrow & \mathscr{Y} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathscr{X} \end{array}$$

Corollary 3.5. *Notation as in Proposition 3.3. If Y_0 is not $W_2(k)$-liftable, then the special fiber X_0 of X (regarded as a log variety over $\mathbf{k}$) does not admit any smooth deformation over $\mathbf{W}_2$.*

Proof. Use Propositions 3.3 and 3.4 which assert that the W_2-liftability of $\mathbb{P}(N_{Y_0/Z})$ implies that of Y_0. □

3.2. **Example 1.** Corollary 3.5 provides direct examples: take a smooth projective variety Y_0 over k which is non W_2-liftable, and take a closed embedding $Y_0 \hookrightarrow Z_0$ over k into a smooth projective variety such that the codimension $\mathrm{Cod}_{Z_0} Y_0 \geq 2$ and Z_0 admits a smooth lifting Z over W (for example take Z_0 to be a projective space of high dimension). Set $X = \mathrm{Bl}_{Y_0} Z$, the blowup of Z along the closed subscheme Y_0. Then X is a semi-stable reduction over W whose special fiber $X_0/\mathbf{k}$ does not admit $\mathbf{W}_2$-deformation.

3.3. **Example 2.** Notice that Mukai [12] has obtained a nice generalization to higher dimension of Raynaud's classical example [16] of non W_2-liftable smooth projective surface over k. His construction, together with an idea of Liedtke-Satriano (Theorem 1.1 (a) [10]), allows us to make concrete examples of all relative dimensions ≥ 2.

Let us recall first the following

Definition 3.6 ([12])**.** A smooth curve C over k of genus ≥ 2 is called a Tango-Raynaud curve if there exists a rational function f on C such that $df \neq 0$ and that $(df) = pD$ for some ample divisor D.

A typical example of Tango-Raynaud curve is the plane curve defined by the affine polynomial

$$G(x^p) - x = y^{pe-1},$$

where G is a polynomial of degree $e \geq 1$ in the variable x. The following lemma is well known.

Lemma 3.7 ([12])**.** *Let C be a Tango-Raynaud curve, then there exists a rank two vector bundle E on C together with a smooth curve D in the projectification $\mathbb{P}_C(E)$ of E/C, such that the composite $D \to \mathbb{P}_C(E) \to C$ is the relative Frobenius $F_0 : D \to D^{(p)} = C$.*

Proposition 3.8. *Notation as in Lemma 3.7. Let $\mathscr{C}$ be a W-lifting of C and $\mathscr{E}$ a lifting of E over $\mathscr{C}$. For $d \geq 2$, set $Z_d = \mathbb{P}_{\mathscr{C}}(\mathscr{E} \oplus \mathscr{O}_{\mathscr{C}}^{d-2})$ and $X_d = Bl_D Z_d$. Then X_d is a semi-stable reduction over W of relative dimension d, whose special fiber, regarded as a log variety over $\mathbf{k}$, is non $\mathbf{W}_2$-liftable and therefore DR-indecomposable.*

Proof. We prove the statement for $d = 2$ only (the proof for $d \geq 3$ is the same). Denote

$$C_0 = C, \quad Y_0 = D, \quad Z_0 = \mathbb{P}_C(E), \quad Z = Z_2.$$

Assume the contrary that the special fiber X_0 of $Bl_{Y_0} Z$, regarded as a log variety over $\mathbf{k}$, admit a smooth deformation over $\mathbf{W}_2$. It follows from Proposition 3.2 that the pair

(Z_0, Y_0) consisting of the component $Z_0 = Bl_{Y_0}Z_0$ of X_0 together with the divisor $Y_0 = \mathbb{P}(N_{Y_0/Z}) \cap Z_0 \subset X_0$ lift to a pair (Z_1, Y_1) over W_2 (The scheme Z_1 is not necessarily the mod p^2-reduction of Z). On the other hand, Proposition 3.4 implies that the projection $Z_0 \to C_0$ is the reduction of a certain W_2-morphism $Z_1 \to C_1$. Therefore, the composite $F_0 : Y_0 \hookrightarrow Z_0 \to C_0$ lifts to the composite $F_1 : Y_1 \hookrightarrow Z_1 \to C_1$ over W_2. But this leads to a contradiction: the nonzero morphism $dF_1 : F_1^*\Omega_{C_1} \to \Omega_{Y_1}$ is divisible by p and it induces a nonzero morphism over k

$$\frac{dF_1}{p} : F_0^*\Omega_{C_0} \to \Omega_{Y_0},$$

which is impossible because of the degree. Therefore, $X_0/\mathbf{k}$ is indeed non $\mathbf{W}_2$-liftable as claimed. □

4. An E_1-degeneration result

This section is devoted to prove the following

Theorem 4.1. *Let k be an algebraically closed field and R a DVR with the residue field k. Let Z/R be a smooth proper R-scheme and X/R be a blow-up of X along a closed regular center Y_0 supported in $Z_0 = Z \times_R k$. If the Hodge to de Rham spectral sequence*

$$E_1^{pq} = H^q(Z_0, \Omega^p_{Z_0}) \Rightarrow H^{p+q}(\Omega^\bullet_{Z_0})$$

degenerates at E_1 (e.g. when $char(k) = 0$ or $\dim Z_0 \leq char(k)$ and R is of mixed characteristic), then the Hodge to log de Rham spectral sequence

$$E_1^{pq} = H^q(X_0, \wedge^p\Omega^{\log}_{X_0}) \Rightarrow H^{p+q}(\Omega^{\log\bullet}_{X_0})$$

degenerates at E_1.

Recall from Proposition 3.3 that X_0 is a simple normal crossing divisor consisting of two smooth components $X_1 = Bl_{Y_0}Z_0$ and $X_2 = \mathbb{P}(N_{Y_0/Z})$ which intersect transversally along $D = \mathbb{P}(N_{Y_0/Z_0})$. The blowdown morphism of the log pairs $(Z, Z_0) \to (X, X_0)$ restricts on the special fiber to a log morphism $\pi : X_0 \to (Z_0, 1 \mapsto 0)$ between log varieties over $(\mathrm{Spec}(k), 1 \mapsto 0)$. This induces a canonical morphism

$$\pi^{*i} : \Omega^i_{Z_0} \to R\pi_* \bigwedge^i \Omega^{\log}_{X_0}.$$

Our main technical step in proving Theorem 4.1 is the following

Proposition 4.2. *Let Z/R be a smooth proper R-scheme and X/R be a blow-up of X along a closed regular center Y_0 supported in Z_0. Denote by $\pi : X_0 \to Z_0$ the restriction morphism. Then for each i, the canonical morphism*

$$\Omega^i_{Z_0} \to R\pi_* \wedge^i \Omega^{log}_{X_0}$$

is an isomorphism in $D^b(Z_0)$.

From Proposition 4.2, we may derive the main result of the section.

Proof of Theorem 4.1. We actually prove that the two spectral sequences

$$E_1^{pq} = H^q(Z_0, \Omega^p_{Z_0}) \Rightarrow H^{p+q}(\Omega^\bullet_{Z_0}) \tag{1}$$

and

$$E_1^{pq} = H^q(X_0, \wedge^p\Omega^{\log}_{X_0}) \Rightarrow H^{p+q}(\Omega^{\log\bullet}_{X_0}) \tag{2}$$

are isomorphic. First recall that (1) is induced by the hypercohomology of the complex $\Omega^\bullet_{Z_0}$ with respect to the truncated filtration

$$F^i=\tau^{\rm st}_{\geq i}\Omega^\bullet_{Z_0},$$

where $\tau^{\rm st}$ is the stupid truncation. (2) is induced by the hypercohomology of the complex $\Omega^{\log\bullet}_{X_0}$ with respect to the truncated filtration

$$F^i=\tau^{\rm st}_{\geq i}\Omega^{\log\bullet}_{X_0}.$$

By Proposition 4.2, there are natural quasi-isomorphisms

$$R\pi_*\Omega^{\log\bullet}_{X_0}\simeq\pi_*\Omega^{\log\bullet}_{X_0}\simeq\Omega^\bullet_{Z_0},$$

and the isomorphisms respect the filtration

$$F^i=R\pi_*\tau^{\rm st}_{\geq i}\Omega^{\log\bullet}_{X_0}\simeq\pi_*\tau^{\rm st}_{\geq i}\Omega^{\log\bullet}_{X_0}$$

in the left, middle and

$$F^i=\tau^{\rm st}_{\geq i}\Omega^\bullet_{Z_0}$$

in the right. As a consequence, the two spectral sequences (1) and (2) are naturally isomorphic. □

To prove Proposition 4.2, we make some preparations. Let $X_0=X_1\cup_D X_2$ be a variety consisting of two smooth projective components X_1 and X_2 such that they intersect transversely along a smooth divisor D. Assume that X_0 has a log structure of semi-stable type (Definition 2.1). Then the normalization $X_1\cup X_2\to\underline{X_0}$ and the diagonal immersion $D\to X_1\cup X_2$ lift to log morphisms

$$(X_1\cup X_2,D_1\cup D_2\oplus(1\mapsto 0))\to X_0$$

and

$$(D,(1\mapsto 0)^{\oplus 2})\to(X_1\cup X_2,D_1\cup D_2)$$

over the base $(\mathrm{Spec}(k),1\mapsto 0)$. These log morphisms induce morphisms of sheaves on X_0

$$\Omega^i_{X_0^{\log}}\to\Omega^i_{X_1}(\log D)\oplus\Omega^i_{X_2}(\log D)\tag{3}$$

and

$$\Omega^i_{X_1}(\log D)\oplus\Omega^k_{X_2}(\log D)\to\Omega^k_{(D,(1\mapsto 0)^{\oplus 2})/(\mathrm{Spec}(k),1\mapsto 0)}.\tag{4}$$

for each i. By the definition of log cotangent sheaf,

$$\Omega_{(D,(1\mapsto 0)^{\oplus 2})/(\mathrm{Spec}(k),1\mapsto 0)}\simeq\Omega_D\oplus(\mathbb{Z}^{\oplus 2}/\mathbb{Z}\otimes_{\mathbb{Z}}\mathscr{O}_D)/\alpha(m)\otimes m-d\alpha(m)\otimes 1.$$

Thanks to the log structure of $(D,(1\mapsto 0)^{\oplus 2})$, $\alpha(m)\otimes m-d\alpha(m)\otimes 1$ are null relations. Therefore

$$\Omega_{(D,(1\mapsto 0)^{\oplus 2})/(\mathrm{Spec}(k),1\mapsto 0)}\simeq\Omega_D\oplus\mathscr{O}_D.$$

This isomorphism induces the forgetful morphism

$$\Omega_{(D,(1\mapsto 0)^{\oplus 2})/(\mathrm{Spec}(k),1\mapsto 0)}\to\Omega_D$$

and the log residue morphism

$$\Omega_{(D,(1\mapsto 0)^{\oplus 2})/(\mathrm{Spec}(k),1\mapsto 0)}\to\mathscr{O}_D.$$

Therefore

$$\Omega^k_{(D,(1\mapsto 0)^{\oplus 2})/(\mathrm{Spec}(k),1\mapsto 0)}\cong\bigwedge^k(\Omega_D\oplus\mathscr{O}_D)\simeq\Omega^k_D\oplus\Omega^{k-1}_D$$

and by local calculation the restriction morphism

$$\Omega^k_{X_1}(\log D) \to \Omega^k_{(D,(1\mapsto 0)^{\oplus 2})/(\mathrm{Spec}(k),1\mapsto 0)}$$

is equivalent to

$$(\iota, \mathrm{res}_D) : \Omega^k_{X_1}(\log D) \to \Omega^k_D \oplus \Omega^{k-1}_D$$
$$\beta + \gamma \frac{dz}{z} \mapsto (\beta, \gamma).$$

Here we use a local chart of X_1 where $D = \{z = 0\}$ and β, γ does not contain dz. This phenomenon is interesting in itself. The residue map res_D is a part of the restriction map of log cotangent sheaves. It makes log geometry a convenient and natural language in such a situation.

Assume locally X_0 is embedded into the affine space with a system of local coordinates $(z_1, z_2, \cdots, z_n)$ such that $X_1 = \{z_1 = 0\}$, $X_2 = \{z_2 = 0\}$. Since

$$\frac{dz_1}{z_1} + \frac{dz_2}{z_2} = 0$$

on X_0, a log form on X_0 is of the form

$$\beta + \gamma_1 \frac{dz_1}{z_1} = \beta - \gamma_1 \frac{dz_2}{z_2}.$$

Therefore two k-forms $\beta_1 + \gamma_1 \frac{dz_1}{z_1}$ on X_2 and $\beta_2 + \gamma_2 \frac{dz_2}{z_2}$ on X_2 glue to a log k-form on X_0 if and only if

$$\beta_1|_D = \beta_2|_D$$

and

$$\gamma_1|_D + \gamma_2|_D = 0.$$

This proves

Lemma 4.3. *For each $k \geq 0$ there is a short exact sequence of sheaves*

$$0 \to \Omega^k_{X^{log}} \to \Omega^k_{X_1}(logD) \oplus \Omega^k_{X_2}(logD) \xrightarrow{\varphi} \Omega^i_D \oplus \Omega^{i-1}_D \to 0$$

where φ is defined by

$$\begin{pmatrix} \iota & res_D \\ -\iota & res_D \end{pmatrix}.$$

Here Ω^{-1}_D is defined to be 0.

The following well-known lemma will be used several times in the sequel.

Lemma 4.4. *Let $\mathbb{P}^n$ be the projective space over k. The following vanishing results hold:*

(1)

$$H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}) = 0, p \neq q$$

(2) *If $i \neq 0$, then*

$$H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(i)) = 0,$$

for $q = 0$, $i \leq p$ or $q = n$, $i \geq p - n$ or $q \neq 0, n$.

(3) *Let H be a hyperplane in $\mathbb{P}^n$, then*

$$H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(logH)) = \begin{cases} k, & p=q=0 \\ 0, & otherwise. \end{cases}$$

Lemma 4.5. *Let Z be a smooth variety and $\pi : P \to Z$ be a projective bundle of relative dimension r. Let $D \subset P$ be a relative hyperplane. Then for each $i \geq 0$ there is a canonical isomorphism*

$$\Omega^i_Z \simeq R\pi_*\Omega^i_P(logD)$$

in $D(Z)$.

Proof. The exact sequence

$$0 \to \pi^*\Omega_Z \to \Omega_P(\log D) \to \Omega_{P/Z}(\log D) \to 0$$

induces a decreasing filtration

$$F^p = \pi^*\Omega^p_Z \wedge \Omega^{i-p}_P(\log D) \subset \Omega^i_P(\log D)$$

such that

$$F^p/F^{p+1} \simeq \pi^*\Omega^p_Z \otimes \Omega^{i-p}_{P/Z}(\log D).$$

Therefore we have a spectral sequence

$$E_1^{pq} = R^q\pi_*(\pi^*\Omega^p_Z \otimes \Omega^{i-p}_{P/Z}(\log D)) \Rightarrow R^{p+q}\pi_*(\Omega^i_P(\log D)).$$

By Lemma 4.4, we see that

$$E_1^{pq} \simeq \Omega^p_Z \otimes R^q\pi_*(\Omega^{i-p}_{P/Z}(\log D)) = \begin{cases} \Omega^i_Z, & p=i, q=0 \\ 0, & \text{otherwise}. \end{cases}$$

This proves the lemma. □

Lemma 4.6. *Let Z_0 be a smooth projective variety and Y_0 be a smooth closed subvariety of Z_0. Denote $\pi : X_1 \to Z_0$ be the blowup along Y_0 with exceptional divisor D. Then for each $k \geq 0$, there is a distinguished triangle in $D^b(Z_0)$ induced by natural morphisms:*

$$\Omega^k_{Z_0} \xrightarrow{u} R\pi_*\Omega^k_{X_1} \oplus \Omega^k_{Y_0} \xrightarrow{v} R\pi_*\Omega^k_D \to \Omega^k_{Z_0}[1].$$

In other words, we have the short exact sequence

$$0 \to \Omega^k_{Z_0} \to \pi_*\Omega^k_{X_1} \oplus \Omega^k_{Y_0} \to \pi_*\Omega^k_D \to 0 \tag{5}$$

and the isomorphism

$$R^i\pi_*\Omega^k_{X_1} \to R^i\pi_*\Omega^k_D \tag{6}$$

for each $i > 0$.

Proof. Denote the following automorphism of $\pi_*\Omega^k_{X_1} \oplus \Omega^k_{Y_0}$ by ϕ:

$$(a, b) \mapsto (a, a-b),$$

By composing with ϕ, the exactness of the sequence (5) is reduced to the following isomorphisms

$$\Omega^k_{Z_0} \cong \pi_*\Omega^k_{X_1}; \quad \Omega^k_{Y_0} \cong \pi_*\Omega^k_D.$$

For $k = 0$, these are obvious. For $k \geq 1$, their truth can be easily seen by considering the local model of a blow-up along a smooth center: we assume that X_1 is the blow up of

$Z_0 = \mathbb{A}^n$ along $Y_0 = \mathbb{A}^r$ defined by the intersection of some coordinate hyperplanes. Then the map

$$\pi : D \to Y_0$$

is the projection

$$\mathbb{A}^r \times \mathbb{P}^s \to \mathbb{A}^r.$$

Thus, it is trivial to get $\pi_*\Omega^k_D = \Omega^k_{Y_0}, k \geq 0$ by this description. For the first isomorphism, we use the following estimation:

$$\pi^*\Omega^k_{Z_0} \subset \Omega^k_{X_1} \subset \pi^*\Omega^k_{Z_0}(kD).$$

From this, it follows that

$$\Omega^k_{Z_0} \subset \pi_*\Omega^k_{X_1} \subset \Omega^k_{Z_0} \otimes \pi_*\mathscr{O}_{X_1}(kD) = \Omega^k_{Z_0},$$

and hence $\pi_*\Omega^k_{X_1} = \Omega^k_{Z_0}$.

The proof of (6) is divided into two parts. First we show that the natural map

$$R^i\pi_*\Omega^k_{X_1}|_D \to R^i\pi_*\Omega^k_D$$

is an isomorphism for each $i > 0$. Considering the long exact sequence associated to

$$0 \to \mathscr{O}_D(1) \otimes \Omega^{k-1}_D \to \Omega^k_{X_1}|_D \to \Omega^k_D \to 0$$

where $\mathscr{O}_D(1)$ is the tautological bundle of the projective bundle $D \to Y_0$, we see that it sufficient to prove that

$$R^i\pi_*(\mathscr{O}_D(1) \otimes \Omega^k_D) = 0, \quad i > 0. \tag{7}$$

Notice that the short exact sequence

$$0 \to \pi^*\Omega_{Y_0} \to \Omega_D \to \Omega_{D/Y_0} \to 0$$

induces a decreasing filtration

$$F^p = \pi^*\Omega^p_{Y_0} \wedge \Omega^{k-p}_D \subset \Omega^k_D$$

such that

$$F^p/F^{p+1} \simeq \pi^*\Omega^p_{Y_0} \otimes \Omega^{k-p}_{D/Y_0}.$$

Therefore we have a spectral sequence

$$E_1^{pq} = R^q\pi_*(\pi^*\Omega^p_{Y_0} \otimes \Omega^{k-p}_{D/Y_0} \otimes \mathscr{O}_D(1)) \Rightarrow R^{p+q}\pi_*(\mathscr{O}_D(1) \otimes \Omega^k_D).$$

Since $D \to Y_0$ is a projective bundle, we obtain that

$$E_1^{pq} = \Omega^p_{Y_0} \otimes R^q\pi_*(\Omega^{k-p}_{D/Y_0} \otimes \mathscr{O}_D(1))$$

for $p + q \geq 1$ and $p, q \geq 0$, thanks to the Lemma 4.4. This proves (7) and thus

$$R^i\pi_*\Omega^k_{X_1}|_D \to R^i\pi_*\Omega^k_D$$

is an isomorphism for each $i > 0$.

Next we show that the canonical morphism

$$R^i\pi_*\Omega^k_{X_1} \to R^i\pi_*(\Omega^k_{X_1}|_D)$$

is an isomorphism for each $i > 0$. By the long exact sequence associated to

$$0 \to \Omega^k_{X_1} \otimes \mathscr{O}_{X_1}(-D) \to \Omega^k_{X_1} \to \Omega^k_{X_1}|_D \to 0,$$

we see that it sufficient to show the vanishing

$$R^i\pi_*(\Omega^k_{X_1}\otimes \mathscr{O}_{X_1}(-D))=0 \tag{8}$$

for each $i>0$.

Notice that the short exact sequence

$$0\to \pi^*\Omega_{Z_0}\to \Omega_{X_1}\to \Omega_{X_1/Z_0}\simeq \Omega_{D/Y_0}\to 0$$

induces a decreasing filtration

$$F^p=\pi^*\Omega^p_{Z_0}\wedge \Omega^{k-p}_{X_1}\subset \Omega^k_{X_1}$$

such that

$$F^p/F^{p+1}\simeq \pi^*\Omega^p_{Z_0}\otimes \Omega^{k-p}_{D/Y_0}.$$

Therefore we have a spectral sequence

$$E_1^{pq}=R^q\pi_*(\pi^*\Omega^p_{Z_0}\otimes\Omega^{k-p}_{D/Y_0}\otimes \mathscr{O}_{X_1}(-D))\Rightarrow R^{p+q}\pi_*(\Omega^k_{X_1}\otimes\mathscr{O}_{X_1}(-D)).$$

Since $D\to Y_0$ is a projective bundle, we obtain that

$$E_1^{pq}=\Omega^p_{Z_0}\otimes R^q\pi_*(\Omega^{k-p}_{D/Y_0}\otimes \mathscr{O}_D(1))$$

for $p+q\geq 1$ and $p,q\geq 0$, thanks again to the Lemma 4.4. This proves (8) and thus

$$R^i\pi_*\Omega^k_{X_1}\to R^i\pi_*(\Omega^k_{X_1}|_D)$$

is an isomorphism for each $i>0$. So we finish the proof of (6). □

Now we are ready to prove Proposition 4.2.

Proof. By Lemma 4.3 and 4.5, we have a distinguished triangle

$$R\pi_*\bigwedge^i\Omega^{\log}_X\to R\pi_*\Omega^i_{X_1}(\log D)\oplus\Omega^i_{Y_0}\xrightarrow{R\pi_*\varphi} R\pi_*\Omega^i_D\oplus R\pi_*\Omega^{i-1}_D\to R\pi_*\Omega^i_{X^{\log}}[1]$$

in $D^b(Z_0)$. This triangle fills in the following diagram in $D^b(Z_0)$

(9)

$$\begin{array}{ccccccc}
R\pi_*\Omega^i_D & \longrightarrow & 0 & \longrightarrow & R\pi_*\Omega^i_D[1] & \xrightarrow{\mathrm{Id}} & R\pi_*\Omega^i_D[1]\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
R\pi_*\Omega^i_{X_1}\oplus\Omega^i_{Y_0} & \longrightarrow & R\pi_*\Omega^i_{X_1}(\log D)\oplus\Omega^i_{Y_0} & \longrightarrow & R\pi_*\Omega^{i-1}_D & \longrightarrow & R\pi_*\Omega^i_{X_1}[1]\oplus\Omega^i_{Y_0}[1]\\
\uparrow{\scriptstyle p} & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle \mathrm{pr}} & & \uparrow\\
R\pi_*\wedge^i\Omega^{\log}_{X_0} & \longrightarrow & R\pi_*\Omega^i_{X_1}(\log D)\oplus\Omega^i_{Y_0} & \xrightarrow{R\pi_*\varphi} & R\pi_*\Omega^i_D\oplus R\pi_*\Omega^{i-1}_D & \longrightarrow & R\pi_*\wedge^i\Omega^{\log}_{X_0}[1]\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
R\pi_*\Omega^i_D[-1] & \longrightarrow & 0 & \longrightarrow & R\pi_*\Omega^i_D & \xrightarrow{\mathrm{Id}} & R\pi_*\Omega^i_D,
\end{array}$$

which is generated from the centered commutative square

$$\begin{array}{ccc} R\pi_*\Omega^i_{X_1}(\log D)\oplus\Omega^i_{Y_0} & \longrightarrow & R\pi_*\Omega^{i-1}_D \\ \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow \\ R\pi_*\Omega^i_{X_1}(\log D)\oplus\Omega^i_{Y_0} & \xrightarrow{R\pi_*\varphi} & R\pi_*\Omega^i_D\oplus R\pi_*\Omega^{i-1}_D. \end{array}$$

In the diagram (9), p is induced (non-canonically) by the above commutative square. The second horizontal line is the direct sum of the distinguished triangles

$$R\pi_*\Omega^i_{X_1}\to R\pi_*\Omega^i_{X_1}(\log D)\to R\pi_*\Omega^{i-1}_D\to R\pi_*\Omega^i_{X_1}[1]$$

and

$$\Omega^i_{Y_0}\xrightarrow{\mathrm{Id}}\Omega^i_{Y_0}\to 0\to\Omega^i_{Y_0}[1].$$

The horizontal lines of (9) are distinguished triangles. The second and third vertical lines are also distinguished. By the 3×3 lemma of triangulated categories, the first vertical line induces a distinguished triangle

$$R\pi_*\wedge^i\Omega^{\log}_{X_0}\to R\pi_*\Omega^k_{X_1}\oplus\Omega^i_{Y_0}\to R\pi_*\Omega^k_D\to R\pi_*\wedge^i\Omega^{\log}_{X_0}[1].$$

Comparing with Lemma 4.6, we see that there is a quasi-isomorphim

$$R\pi_*\wedge^i\Omega^{\log}_{X_0}\simeq\Omega^i_{Z_0}.$$

Note that this isomorphism may not be the natural one induced by the morphism π. However, we obtain as a consequence of the abstract quasi-isomorhism that

$$R^k\pi_*\wedge^i\Omega^{\log}_{X_0}\simeq 0,\quad k>0.$$

It remains to show that the natural morphism of sheaves

$$\Omega^i_{Z_0}\to\pi_*\wedge^i\Omega^{\log}_{X_0} \tag{10}$$

is an isomorphism.

Let us consider the cohomologies at place 0 of the diagram (9),

$$\begin{array}{ccccccc} \pi_*\Omega^i_{X_1}\oplus\Omega^i_{Y_0} & \longrightarrow & \pi_*\Omega^i_{X_1}(\log D)\oplus\Omega^i_{Y_0} & \longrightarrow & \pi_*\Omega^{i-1}_D & & \\ \uparrow{\scriptstyle p^0} & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow{\scriptstyle \mathrm{pr}} & & \\ \pi_*\wedge^i\Omega^{\log}_{X_0} & \longrightarrow & \pi_*\Omega^i_{X_1}(\log D)\oplus\Omega^i_{Y_0} & \xrightarrow{\pi_*\varphi} & \pi_*\Omega^i_D\oplus R\pi_*\Omega^{i-1}_D & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \pi_*\Omega^i_D & \xrightarrow{\mathrm{Id}} & \pi_*\Omega^i_D \end{array} \tag{11}$$

The two vertical sequences in the middle are short exact sequences. Therefore, by the snake lemma, there is an exact sequence

$$0\to\pi_*\wedge^i\Omega^{\log}_{X_0}\xrightarrow{p^0}\pi_*\Omega^i_{X_1}\oplus\Omega^i_{Y_0}\xrightarrow{\delta}\pi_*\Omega^i_D$$

where δ is the boundary map which is identical to the one in (5). Hence by (5) we see that the natural map (10) is an isomorphism. □

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