Albanese map for zero cycles

Xiaowen Hu

In the first lecture by Jilong Tong, Yong Hu asked a question on the existence of Albanese maps on zero cycles, especially for a closed point with an inseparable residue field. I write this note trying to answer this question. I have discussed this proof with Yong Hu in SUSTC during the first meeting of the workshop.

Welcome for any comments. The statement of lemma 0.2 is very simple, if someone knows any references about it, please tell me.

Theorem 0.1. *Let k be a field, and A an abelian variety over k. Then there is a natural homomorphism*

$$\mathrm{alb}_A : Z_0(A) \to A(k).$$

Moreover,

(i) for any finite extension of fields L/k, denoting $\pi : A_L \to A$ the projection from the base change $A_L = A \times_k L$ to A, the following diagram is commutative

$$\begin{tikzcd} Z_0(A) \arrow[r, "\mathrm{alb}_A"] \arrow[d, "\pi^*"] & A(k) \arrow[d, "\pi^{-1}"] \\ Z_0(A_L) \arrow[r, "\mathrm{alb}_{A_L}"] & A_L(L); \end{tikzcd} \tag{1}$$

(ii) alb_A *induces a homomorphism*

$$\mathrm{CH}_0(A) \to A(k).$$

Proof : Let $x : \mathrm{Spec}(L) \to A$ be a closed point of A. Thus L is a finite extension of k. We first treat the two cases L/k is separable or pure inseparable.

(i) L/k is separable. Let F be a finite Galois extension containing L. Let $A_F = A \times_k F$, and denote the projection $A_F \to A$ by π. The inverse image $\pi^{-1}(x)$ in A_F is a finite set of F-points, upon which there is a transitive action by $\mathrm{Gal}(F/k)$. The image of the zero cycle $\sum_{z \in \pi^{-1}(x)}[z]$ in $A_F(F) = A(F)$, is invariant under $\mathrm{Gal}(F/k)$, thus descends to a k-point of A. It is not hard to see that this map is independent of the choice of the finite Galois extension F. (A related question is whether the formation of $\mathrm{Alb}_{X/k}$ commutes with Galois base changes. Jilong Tong suggested that it might be shown also by the Galois descent. I have not checked that.)

(ii) L/k is purely inseparable. For brevity we suppose $[L : k] = p$; the generalization to the general case will be obvious. First we need to understand what the natural map will be in this case. In the same spirit as the separable case, we consider the base change $\pi : A_L \to A$. Then the inverse image of $[x]$, as a zero cycle, is an L-point of A_L with multiplicity p. So the natural definition of $\mathrm{alb}([x])$ is the composition $\mathrm{Spec}(L) \xrightarrow{x} A \xrightarrow{\times p} A$, which we expect to factor through a unique k-point of the right-handside A. This is true by lemma 0.2.

Now the general case is clear. Let L_i be the inseparable closure of k in L, and denote $p^i = [L_i : k]$. Let F be a Galois of L_i containing L. Let $\pi : A_F \to A$. Then $\mathrm{alb}(\pi^{-1}[x])$ is a L_i-point of A, and denote it by y. The composition

$$\mathrm{Spec}(L_i) \xrightarrow{y} A \xrightarrow{\times p^i} A$$

factors through a unique k-point z of the right-handside A. Then define

$$\mathrm{alb}([x]) = z.$$

The commutativity of the diagram (1) follows from the construction of alb_A.

For (ii), we need to show that

$$\mathrm{alb}_A(\alpha) = 0. \tag{2}$$

for any zero cycle α of A that is rationally equivalent to zero. But a k-point y of A is equal to zero if and only if its inverse image $\pi^{-1}(y)$, as an L-point, of A_L is zero. So by (i) and a standard limiting process we can pass to the algebraic closure $\bar{k}$, for which (2) is well-known.

Lemma 0.2. *Let k be a field of characteristic p, and L a purely inseparable finite extension of k, and suppose $[L : k] = p^r$. Let A be an abelian variety over k, x an L-point of A. Then the composition*

$$f : \mathrm{Spec}(L) \xrightarrow{x} A \xrightarrow{\times p^r} A$$

factors through a unique k-point of A.

Proof : For brevity we assume $r = 1$. The general case will be clear. Since $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is a fppf covering, by descent theory, the following is an exact sequence of sets

$$A(k) \to A(L) \rightrightarrows A(L \otimes_k L).$$

Thus to show that f factors through a unique k-point, it suffices to show that the images of f by the two maps $A(L) \rightrightarrows A(L \otimes_k L)$ coincide. By assumption and field theory, there exists $\alpha \in k - k^p$ such that $L = k[t]/(t^p - \alpha)$. Then

$$L \otimes_k L \cong k[t,s]/(t^p - \alpha, s^p - \alpha) \cong k[t,s]/(t^p - \alpha, (t-s)^p). \tag{3}$$

Recall that ([SGA3, Exp VII, 4.3]) for any flat commmutative group scheme G/S, one has

$$\mathrm{Ver}_{G/S} \circ \mathrm{Fr}_{G/S} = p \cdot \mathrm{id}_G,$$

where $\mathrm{Ver}_{G/S} : G^{(p/S)} \to G$ is the Verschiebung, $\mathrm{Fr}_{G/S} : G \to G^{(p/S)}$ is the relative Frobenius, and $G^{(p/S)} = G \times_{S,\mathrm{Fr}} S$. Notice that $\mathrm{Ver}_{G/S}$ and $\mathrm{Fr}_{G/S}$ are both finite morphisms.

Let $\mathrm{Spec}(R)$ be an affine open subscheme of A which contains the L-point x. Denote by v and $\mathrm{fr}_{R/k}$ the restriction of $\mathrm{Ver}_{A/k}$ and $\mathrm{Fr}_{A/k}$ to $\mathrm{Spec}(R)$, i.e.,

$$R \xrightarrow{v} R^{(p/k)} = k \otimes_{\mathrm{Fr},k} R \xrightarrow{\mathrm{fr}_{R/k}} R.$$

The L-point x corresponds to a k-homomorphism $g : R \to L$. Denote the two imbeddings of L into $L \otimes_k L$ by σ and τ, which are both k-homomorphisms. Let $r \in R$, and suppose

$$v(r) = \sum_i \lambda_i \otimes r_i,$$

where $\lambda_i \in k$, $r_i \in R$. Then by the definition of the relative Frobenius,

$$\mathrm{fr}_{R/k} \circ v(r) = \sum_i \lambda_i r_i^p.$$

Since g is a k-homomorphism,

$$g \circ \mathrm{fr}_{R/k} \circ v(r) = \sum_i \lambda_i g(r_i)^p.$$

Since σ and τ are k-homomorphisms

$$\sigma \circ g \circ \mathrm{fr}_{R/k} \circ v(r) - \tau \circ g \circ \mathrm{fr}_{R/k} \circ v(r_i) = \sum_i \lambda_i (\sigma \circ g(r_i) - \tau \circ g(r_i))^p.$$

By (3), it is easily seen that $(\sigma \circ g(r_i) - \tau \circ g(r_i))^p = 0$. Therefore the two compositions

$$R \xrightarrow{v} R^{(p/k)} \xrightarrow{\mathrm{fr}_{R/k}} R \xrightarrow{g} L \rightrightarrows L \otimes_k L$$

coincide. By fppf descent, as we have said at the beginning, we are done. □

Corollary 0.3. *Let k be a field, X a smooth, proper and geometrically connected scheme over k, with a k-point x. Then there is a natural homomorphism*

$$\mathrm{CH}_0(X) \to \mathrm{Alb}(X) := \mathrm{Alb}_{X/k}(k).$$

Proof : By [Wittenberg08, Appendix], or the lecture notes of Jilong, such X has an Albanese variety $\mathrm{Alb}_{X/k}$. The conclusion follows from theorem 0.1. □

References

[SGA3] Séminaire de Géométrie Algébrique 3 : Schémas en groupes, par M. Demazure et A. Grothendieck, Lecture Notes in Math. 151, Springer-Verlag, 1970.

[Wittenberg08] O. Wittenberg, On Albanese torsors and the elementary obstruction. Mathematische Annalen 340 (2008), no. 4, 805-838.