Tentative plan for the joint seminar on

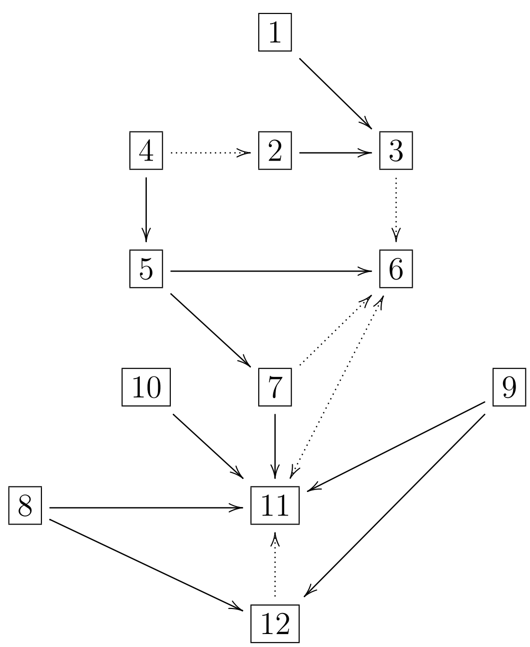
"Chow group of zero-cycles on varieties over local fields"

This joint seminar aims to understand the paper [SS10] by Shuji Saito and Kanetomo Sato, in which they proved a conjecture of Colliot-Thélène on the structure of the Chow group of zero-cycles of p-adic varieties. An excellent exposition of this seminal work is the Bourbaki seminar talk [CT09]. At the number theory seminar of CNU (首师大数论讨论班), Yong Hu (胡勇) gave an overview on the subject. His note is available upon request by email to huy@sustc.edu.cn

Ideally, each talk should last **90-120 minutes** with a 10-minute break if needed.

Leitfaden

A dashed arrow indicates a weak dependence.



Part I: Warm up — Albanese map and Bloch–Ogus theory

Talk 1: *The Albanese map on zero-cycles*

- Goal : recall basics about the Albanese variety Alb_X

  (definition of CH_0)

  define (at least when the ground field is perfect) the Albanese map alb_X from $A_0 = \mathrm{Ker}(\deg : CH_0 \to \mathbb{Z})$ to rational points of Alb_X

  Statement of some classical results about the map alb_X (over $\mathbb{C}$, $\overline{\mathbb{F}}_p$): Rojtman's theorem [Rojt80], [Milne82]

  Statement of Mumford's theorem [Mum68]: nonzero $P_g := h^0(\Omega^2)$ implies A_0 "very large"

  The proof of Mumford's theorem is not required, but it will be great if some of the key ideas can be sketched.

- Prerequisites : general knowledge of algebraic geometry; generalities on abelican varieties

- References :

  Abelian varieties: standard references: books of Mumford, Lang, van der Geer-Moonen; Milne's online notes, etc.

  Chow groups: [Fulton98]

  Albanese variety and the Albanese map: [Serre60], [Blo80,§1] ([Wittenberg08, Appendix] is useful in the case of imperfect base fields)

  Mumford's theorem: [Blo80, §1], [Voi02]

Talk 2: *Bloch-Ogus theory (over a field) and the coniveau spectral sequence*

- Goal : For smooth varieties X/K, explain the construction of the coniveau spectral sequence

  $$E_1^{pq} = \bigoplus_{x \in X^p} H^{q-p}(x, \mathbb{Z}/n) \Longrightarrow H^{p+q}(X, \mathbb{Z}/n)$$

  and the Bloch-Ogus spectral sequence

  $$E_2^{pq} = H^p_{Zar}(X, \mathscr{H}^q(\mathbb{Z}/n(j))) \Longrightarrow H^{p+q}(X, \mathbb{Z}/n(j)) .$$

  Remark: Here the functor H^* with no subscript can be a general functor satisfying a set of axioms described in [BO74, §1]. To be independent of Talk 4, for this talk we shall admit that this H can be the usual étale cohomology theory.

- Prerequisites : basic knowledge of étale cohomology; some familiarity with spectral sequences
- References : [BO74], [Kato86], [CTHK97], ([Jan00]? [And04]?)

Talk 3: *Applications of Bloch-Ogus theory: Proof of Rojtman's theorem; K-theoretic methods for CH^2*

- Goal : Proof of Rojtman's theorem

  Give some applications of the K-theoretic method to CH^2
- Prerequisites : basic étale cohomology theory; a little K-theory
- References :

  Rojtman's theorem: [Blo79], [CT93]

  Applications of K-theory to CH^2: [CT04](a recent survey); [CT83], [CTSS83], [CTR85], [CTR91], [Mur94],

Part II: Statement of main result – Homology theory and cyclc class map

Talk 4: *Some preliminaries on étale cohomology*

- Goal : Review sufficiently much material (derived functors, trace maps, purity, etc.) from étale cohomology to define the cohomology class of algebraic cycles.
- Prerequisites : a second course on étale cohomology
- References : Standard references on étale cohomology: [SGA4], [SGA4 1/2, Chapter Cycle], [FuLei11, Chap. 8],

  Gabber's proof of absolute purity: [Fuj02]

Talk 5: *Homological cycle class map and Kato homology*

- Goal : Explain the notation of "homology theory" in the sense of [JS03, §2]

  Introduce the homology theory we need and prove the niveau spectral sequence (=Cor.1.10 of [SS10])

  Prove Prop. 2.14 of [SS10], which computes low degree terms of the Kato homology groups.

- Prerequisites : a second course on étale cohomology
- References : [JS03], [SS10, §§1-2]

Talk 6: *The main theorem: statement and applications*

- Goal : Statement of [SS10, Thm.0.6]
  Prove the surjectivity part ([SS10, §5]).
  Prove the corollaries ([SS10, §9])
- Prerequisites : general knowledge of homological algebra and étale cohomology
- References : [SS10, §§0, 5, 9]; [CT09, §3]

Part III: Technical details of the proof

Talk 7: *The vanishing theorem*

- Goal : Prove Theorem 3.2 of [SS10].
- Prerequisites : a second course on étale cohomology leading to a good grasp of notions like Gysin map, ℓ-adic cohomology, weights, etc;
- References : [SS10, §3] (and perhaps also [Fuj02] for absolute purity)

Talk 8: *Bertini theorem over discrete valuation rings*

- Goal : recall classical Bertini type theorems over infinite fields,
  state Bertini theorems over finite fields
  Prove Thm. 4.2 of [SS10]
- Prerequisites : general knowledge of algebraic geometry;
- References : Bertini over infinite fields: [Jou83], [KA79]
  over finite fields: [Poo04], [Poo08], [ChPo16]
  over DVR: [JS12], [SS10, §3]

Talk 9: *Moving lemma for* 1*-cycles*

- Goal : Prove Prop.7.1 of [SS10].

  Find and explain some interesting generalizations in [GLL13] and [GLL15] (e.g. Thm. 2.3 of [GLL13]).

- Prerequisites : general knowledge of algebraic geometry; some hard commutative algebra

- References : [SS10, §7], [GLL13, §2], [GLL15]

Talk 10: *Resolution of embedded singularities using blowups*

- Goal : Prove Thm.A.1 of [SS10]

- Prerequisites : general knowledge of algebraic geometry; some familiarity with commutative algebra facts related to the blowup construction

- References : [SS10, Appendex A]

Part IV: End of proof and a subsequent work

Talk 11: *The blowup formula and proof of the injectivity*

- Goal : Prove the blowup formula ([SS10, Lemma 6.4]) and finish the proof of the injectivity part of the main theorem ([SS10, §8]).

- Prerequisites : general knowledge of algebraic geometry; understanding of main results in previous talks

- References : [SS10, §§6, 8] (if you want, compare also Bloch's appendix to [EW16]).

Talk 12: *A restriction isomorphism for cycles of relative dimension* 0

- Goal : Construction of the cycle group $\mathcal{C}(Y)$ of [KEW, §2]

  Statement of [KEW, Thm. 3.1]

  Proofs of [KEW, Thm. 5.1 and Cor. 5.2]

- Prerequisites : general knowledge of algebraic geometry; understanding of main results of Talks 8 and 9;

- References : [KEW16, §§1-5], [GLL13, §2]

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