Bloch - Ogus theory and the coniveau spectral sequence 

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## §1. A little K-theory

$X$ : smooth algebraic variety over a field $k$.

$M(X)$ : category of coherent $\mathcal{O}_X$-modules

$M(X) \xrightarrow{\text{Quillen's Q-construction}} QM(X) \xrightarrow{\text{classifying space}} BQM(X) \xrightarrow{\text{homotopy groups}} \pi_{*+1}(BQM(X)) =: K'_*(X)$

$P(X)$ : category of locally free coherent $\mathcal{O}_X$-modules.

the same approach gives $K_*(X)$

$Y \hookrightarrow X \hookleftarrow U : X \setminus Y$

$M_Y(X)$ : category of coherent $\mathcal{O}_X$-modules whose supports are contained in $Y$

$BQM_Y(X)$ gives $K'_{Y,*}(X)$ K-theory with support

properties :

(i). $BQM_Y(X) \longrightarrow BQM(X) \longrightarrow BQM(U)$ is a fibration sequence

so that one has long exact sequence

$$\cdots\cdots K'_{Y,m}(X) \longrightarrow K'_m(X) \longrightarrow K'_m(U) \longrightarrow K'_{Y,m-1}(X) \longrightarrow \cdots\cdots$$

$$\cdots \longrightarrow K'_{Y,0}(X) \to K'_0(X) \to K'_0(U) \to 0$$

(ii). $X$ smooth $\Rightarrow$ $BQP(X) \xrightarrow{\sim} BQM(X)$ homotopy equivalence

so $K_*(X) \xrightarrow{\sim} K'_*(X)$ isomorphism

(iii). $i: Y \hookrightarrow X$ closed immersion, then $i_* : BQM(Y) \xrightarrow{\sim} BQM_Y(X)$

homotopy equivalence (comes from Quillen's "dévissage" theorem)

so $K'_*(Y) \cong K'_{Y,m}(X)$.

$M_p(X)$ : category of coherent $\mathcal{O}_X$-modules whose supports have codim $\geq p$

we have a filtration : $M(X) = M_0(X) \supset M_1(X) \supset M_2(X) \supset \cdots$

Quillen's Localization theory $\Rightarrow$ Long exact sequence

$$\cdots \to K_i(M_{p+1}) \to K_i(M_p) \to \coprod_{x \in X^p} K_i(k(x)) \to K_{i-1}(M_{p+1}) \to \cdots$$

where $X^p$ is the set of points of $X$ of codim $p$

This Long exact sequence gives us an exact couple which induces a spectral sequence

$$E_1^{p,q} := \coprod_{x \in X^p} K_{-p-q}(k(x))$$

differential is given by $d_1 : \coprod_{x \in X^p} K_i(k(x)) \to K_{i-1}(M_{p+1}) \to \coprod_{x \in X^{p+1}} K_{i-1}(k(x))$

Question : $E_2$ ?

<u>Prop A</u> TFAE

(i). For every $p \geq 0$, the inclusion $M_{p+1}(X) \to M_p(X)$ induces zero maps on K-groups

(ii). For every $q$, the sequence

$$0 \to K_q'(X) \xrightarrow{e} \coprod_{x \in X^0} K_q(k(x)) \xrightarrow{d_1} \coprod_{x \in X^1} K_{q-1}(k(x)) \xrightarrow{d_1} \cdots \xrightarrow{d_1} \coprod_{x \in X^q} K_0(k(x)) \to 0$$

is exact. Here $e$ is the pull back map w.r.t $i_x : \operatorname{Spec} k(x) \to X$.

(iii) For all $q$, $E_2^{p,q}(X) = 0$ if $p \neq 0$, and the edge homomorphism $K_q'(X) \to E_2^{0,q}(X)$ is an isomorphism.

<u>proof</u>. (i) $\Rightarrow$ (ii) Since $M_{p+1}(X) \longrightarrow M_p(X)$ induces zero maps on K-groups, $K_{i-1}(M_{p+1}) \longrightarrow \coprod_{x\in X^{p+1}} K_{i-1}(k(x))$ is injective

So $\ker d_1 = \ker\left(\coprod_{x\in X^p} K_i(k(x)) \longrightarrow K_{i-1}(M_{p+1})\right)$

On ~~the~~ other hand, $\coprod_{x\in X^{p-1}} K_{i+1}(k(x)) \longrightarrow K_i(M_p)$ is surjective

So $\operatorname{Im} d_1 = \operatorname{Im}\left(K_i(M_p) \longrightarrow \coprod_{x\in X^p} K_i(k(x))\right)$

$= \ker\left(\coprod_{x\in X^p} K_i(k(x)) \longrightarrow K_{i-1}(M_{p+1})\right)$

Therefore $\ker d_1 = \operatorname{Im} d_1$, the sequence

$$0 \longrightarrow K_q'(X) \longrightarrow \coprod_{x\in X^0} K_q(k(x)) \xrightarrow{d_1} \coprod_{x\in X^1} K_{q-1}(k(x)) \xrightarrow{d_1} \cdots \longrightarrow \coprod_{x\in X^q} K_0(k(x))$$

is exact.

(ii) $\Rightarrow$ (iii) $$E_2^{p,q} = \frac{\ker\left(d_1 : \coprod_{x\in X^p} K_{-p-q}(k(x)) \longrightarrow \coprod_{x\in X^{p+1}} K_{-p-q-1}(k(x))\right)}{\operatorname{Im}\left(d_1 : \coprod_{x\in X^{p-1}} K_{-p-q+1}(k(x)) \longrightarrow \coprod_{x\in X^p} K_{-p-q}(k(x))\right)}$$

So it is clear ~~that~~ If $p>0$, $E_2^{p,q}(X) = 0$

If $p=0$, $E_2^{0,q}(X) \cong K_{-q}'(X)$

(iii) $\Rightarrow$ (i). Induction on $p$. Let $p$ be the ~~first~~ greatest one such that $M_p \to M_{p-1}$ doesn't induce zero maps on K-groups.

say $K_i(M_p) \longrightarrow K_i(M_{p-1})$ is not zero map.

Clearly $p \le d$ where $d = \dim(X)$.

$\operatorname{Im}\left(\coprod_{x\in X^{p-1}} K_{i+1}(k(x)) \longrightarrow K_i(M_p)\right) = \ker\left(K_i(M_p) \to K_i(M_{p-1})\right)$

$\subsetneq K_i(M_p)$

but $\ker\left(K_i(M_p) \longrightarrow \coprod_{x\in X^p} K_i(k(x))\right) = \operatorname{Im}\left(K_i(M_{p+1}) \to K_i(M_p)\right) = 0$ by Induction.

So. in this case.

$$\operatorname{Im} d_1 \subsetneqq \operatorname{Im}\Big( K_i(M_p) \longrightarrow \coprod_{x\in X^p} K_i(k(x)) \Big)$$
$$= \ker\Big( \coprod_{x\in X^p} K_i(k(x)) \longrightarrow K_{i-1}(M_{p+1}) \Big)$$
$$\subseteq \ker d_1$$

Contradiction. □.

<u>prop B</u>. Let $K_q'$ be the sheaf on $X$ associated to the presheaf $U \longmapsto K_q'(U)$. Assume that $\operatorname{Spec}(\mathcal{O}_{X,x})$ satisfies the equivalent conditions in <u>prop A</u>. for all $x\in X$. Then there is a canonical iso $\quad E_2^{p,q}(X) \cong H^p(X, \mathcal{K}_{-q}')$

<u>proof</u>: We have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{K}_{-q}' \longrightarrow \coprod_{x\in X^0} (i_x)_* K_{-q}(k(x)) \longrightarrow \coprod_{x\in X^1} (i_x)_* K_{-q-1}(k(x)) \longrightarrow \cdots\cdots$$

Since the sheaves $i_{x*}K_{-q}(k(x))$ are flasque this is a flasque resolution of $\mathcal{K}_{-q}'$

So $\quad E_2^{p,q}(X) \cong H^p(X, \mathcal{K}_q')$

<u>Cor</u>: $E_2^{p,-p}(X) \cong H^p(X, \mathcal{K}_p')$ but $E_2^{p,-p}$ is the cokernel of

$$\coprod_{x\in X^{p-1}} K_1(k(x)) \longrightarrow \coprod_{x\in X^p} K_0(k(x))$$

which is isomorphic to $CH^p(X)$. So $H^p(X, \mathcal{K}_p') \cong CH^p(X)$

<u>Conj</u> (Gersten) The equivelent conditions in prop A are satisfied for the spectrum of any regular local ring.

<u>Thm</u> (Quillen) Gersten's conj is true for $\operatorname{Spec}(\mathcal{O}_{X,x})$ where $X$ is a smooth varety $/ k$.

## §2 Bloch-Ogus theory:

$K'_*$ "like" homology.

$K_*$ "like" cohomology.

$X$ smooth, $K^{\oplus}_{Y,*}(X) \cong K'_*(Y)$ "like" Poincaré duality

Bloch-Ogus want to generalize §1 to general "cohomology-homology" theory.

(2.1) Definition. Let $\mathcal{V}$ be a category of algebraic scheme of finite type over $k$. Let $\mathcal{V}^*$ be the category whose objects are closed immersions $Y \hookrightarrow X$ and whose morphisms are Cartesian squares.

$$f_Y \hookrightarrow f_X : (Y \hookrightarrow X) \to (Y' \hookrightarrow X') : \quad \begin{tikzcd} Y \arrow[r, hook] \arrow[d, "f_Y"'] & X \arrow[d, "f_X"] \\ Y' \arrow[r, hook] & X' \end{tikzcd}$$

A twisted cohomology theory with supports is a sequence (indexed by $n \in \mathbb{Z}$) of contravariant functors $\mathcal{V}^* \to$ (graded abelian groups)

$$(Y \hookrightarrow X) \longmapsto \bigoplus_i H^i_Y(X, n).$$

Satisfying the flowing axioms:

(2.1.1). For $Z \subseteq Y \subseteq X$, there is a long exact sequence

$$\cdots \to H^i_Z(X,n) \to H^i_Y(X,n) \to H^i_{Y-Z}(X-Z,n) \to H^{i+1}_Z(X,n) \to \cdots$$

(2.1.2). If $f: (Y \hookrightarrow X) \to (Y' \hookrightarrow X')$ and $g: (Z \hookrightarrow Y) \to (Z' \hookrightarrow Y')$ are arrows in $\mathcal{V}^*$, and $h$, $k$ are the induced arrow

$$h: (Z \hookrightarrow X) \to (Z' \hookrightarrow X'), \quad k: (Y-Z \hookrightarrow X-Z) \to (Y'-Z' \hookrightarrow X'-Z')$$

then the following long exact sequences are commutative.

$$\begin{array}{ccccccc} \cdots \to & H^i_Z(X,n) & \to & H^i_Y(X,n) & \to & H^i_{Y-Z}(X-Z,n) & \to \cdots \\ & \uparrow H^*(h) & & \uparrow H^*(f) & & \uparrow H^*(k) & \\ \cdots \to & H^i_{Z'}(X',n) & \to & H^i_{Y'}(X',n) & \to & H^i_{Y'-Z'}(X'-Z',n) & \to \cdots \end{array}$$

(2.1.3). If $Z \hookrightarrow X \in ob\, \mathcal{V}^*$ and if $U \hookrightarrow X$ is open in $X$ which contains $Z$, the map $H^i_Z(X,n) \to H^i_Z(U,n)$ is an isomorphism.

(2.2) <u>Definition</u>. Let $\mathcal{V}_*$ be the category with $ob\,\mathcal{V}_* = ob\,\mathcal{V}$ but whose arrows consist only proper morphisms. A twisted homology theory is a sequence of covariant functors $\mathcal{V}_* \to$ (graded abelian groups), written $H_i(X,n)$ for $X \in \mathcal{V}_*$ satisfying the following axioms:

(2.2.1). $H_*$ is a presheaf in the étale topology

$\alpha: X' \to X$ étale, $\alpha^*: H_i(X,n) \to H_i(X',n)$

(2.2.2).
$$\begin{tikzcd} X' \arrow[r, "\beta"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\alpha"] & Y \end{tikzcd}$$
$\alpha, \beta$ étale, $f, g$ proper $\Longrightarrow$
$$\begin{tikzcd} H_i(X',n) \arrow[d, "H_i(g,n)"'] \arrow[dr, phantom, "\circlearrowright"] & H_i(X,n) \arrow[l, "\beta^*"'] \arrow[d, "H_i(f,n)"] \\ H_i(Y',n) & H_i(Y,n) \arrow[l, "\alpha^*"] \end{tikzcd}$$

(2.2.3). $i: Y \hookrightarrow X$ closed immersion $\alpha: (X-Y) \hookrightarrow X$ then

$$\cdots \to H_i(Y,n) \xrightarrow{i_*} H_i(X,n) \xrightarrow{\alpha^*} H_i(X-Y,n) \to H_{i-1}(Y,n) \to$$

is exact.

(2.2.4). $f: X' \to X$ proper, $Z = f(Z')$, $\alpha: X' - f^{-1}(Z) \hookrightarrow X' - Z'$. Then the diagram commutes:

$$\begin{array}{ccccccccc} \cdots \to & H_i(Z',n) & \to & H_i(X',n) & \to & H_i(X'-Z',n) & \to & H_{i-1}(Z',n) & \to \cdots \\ & \downarrow f_* & & \downarrow f_* & & \downarrow f_*\alpha^* & & \downarrow f_* & \\ \cdots \to & H_i(Z,n) & \to & H_i(X,n) & \to & H_i(X-Z,n) & \to & H_{i-1}(Z,n) & \to \cdots \end{array}$$

(2.3). <u>Definition</u>. A Poincaré duality theory with supports is a twisted cohomology theory $H^*$, together with the following structure:

(2.3.1). (Cap product with supports) For any $Y \hookrightarrow X$ in $\mathrm{Ob}\,\mathcal{V}^*$, a pairing : $H_i(X, m) \otimes H^j_Y(X, n) \longrightarrow H_{i-j}(Y, m-n)$

(2.3.2). (Compatibility of cap product with restriction)

If $Y \hookrightarrow X \in \mathrm{ob}\,\mathcal{V}^*$ and if $(\beta \hookrightarrow \alpha) : (Y' \hookrightarrow X') \to (Y \hookrightarrow X) \in \mathrm{Arr}\,\mathcal{V}^*$ and is étale, then for $a \in H^i_Y(X, n)$ and $z \in H_i(X, m)$

$$\alpha^*(a) \cap \alpha^*(z) = \beta^*(a \cap z)$$

(2.3.3). (projection formula) If $f$ is proper. $f : (Y_1 \hookrightarrow X_1) \to (Y_2 \hookrightarrow X_2)$ then for $a \in H^j_{Y_2}(X_2, n)$ and $z \in H_i(X_1, m)$,

$$H_i(f_X)(z) \cap a = H_i(f_Y)(z \cap H^j(f)(a))$$

(2.3.4). (Fundamental class) If $X \in \mathrm{ob}\,\mathcal{V}$ is irreducible and of dimension $d$, then $\exists$ a global section $\eta_X$ of $H_{2d}(X, d)$ if $\alpha : X' \to X$ étale, $\alpha^* \eta_X = \eta_{X'}$.

(2.3.5). (Poincaré duality) If $X \in \mathrm{Ob}\,\mathcal{V}$ is smooth of dim $d$. ~~then~~ and $Y \hookrightarrow X$ closed immersion, then cap-product induces an isomorphism : $\eta_X \cap : H^{2d-i}_Y(X, d-n) \to H_i(Y, n)$.

## §3 <u>Niveau and Coniveau spectral sequence</u>

$X \in \mathrm{ob}\,\mathcal{V}$. let $Z_d = Z_d(X)$ be the set of all closed subsets $Z \subset X$ of dim $\leq d$, ordered by inclusion. Let $Z_d/Z_{d-1}$ denote the ordered set of pairs $(Z, Z') \in Z_d \times Z_{d-1}$ s.t $Z' \subset Z$

$$(Z, Z') \leq (Z_1, Z_1') \text{ if } Z \subseteq Z_1 \text{ and } Z' \subseteq Z_1'$$

define $H_i(Z_d(X), n) = \varinjlim_{z \in Z_d} H_i(z, n)$

$$H_i(Z_d/Z_{d+1}, n) = \varinjlim_{(z,z') \in Z_d/Z_{d+1}} H_i(z - z', n)$$

when $(z, z') < (z_1, z_1')$, $z - z' \underset{\text{closed}}{\overset{u}{\hookrightarrow}} z_1 - z' \underset{\text{open}}{\overset{v}{\hookleftarrow}} z_1 - z_1'$

the transition map is given by $v^* u_*$

Def: The filtration by niveau is the ascending filtration $N_d H_i(X, n)$ on $H_i(X, n)$: $N_d H_i(X, n) = \operatorname{Im}(H_i(Z_d(X), n) \to H_i(X, n))$.

For $(z, z') \in Z_d/Z_{d+1}$, we have long exact sequence

$$\cdots \to H_m(z', n) \to H_m(z, n) \to H_m(z - z', n) \to H_{m-1}(z', n) \to \cdots$$

By taking limit

$$\cdots \to H_m(Z_{d+1}, n) \to H_m(Z_d, n) \to H_m(Z_d/Z_{d+1}, n) \to H_{m-1}(Z_{d+1}, n) \to \cdots$$

So we get an exact couple which induces a spectral sequence

$$E^1_{p,q} = H_{p+q}(Z_p/Z_{p-1}, n) \Longrightarrow N_\cdot H_{p+q}(X, n)$$

Remark. $H_m(Z_p/Z_{p-1}, n) \cong \bigoplus_{x \in X^p} H_m(x, n)$

where $H_m(x, n) \triangleq \varinjlim_{\substack{U \subseteq \overline{\{x\}} \\ U \text{ open}}} H_m(U, n)$

Prop. This spectral sequence is covariant w.r.t proper morphism and contravariant w.r.t. étale maps.

Let $Z^p = \{ Z \subset X \text{ closed}, \ \mathrm{codim}_X Z \geq p \}$.

<u>Def</u>. The filtration by coniveau is

$$N^p H^*(X,n) = \mathrm{Im}\Big( \varinjlim_{Z \in Z^p} H_Z^*(X,n) \to H^*(X,n) \Big)$$

<u>Prop</u>. Assume that $k$ is perfect, and that $X$ is smooth$/k$.

For $x \in Z^p$, define $H^*(x,n) = \varinjlim_{U \subseteq \overline{\{x\}}} H^*(U,n)$. Then there is a cohomological spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in Z^p/Z^{p+1}} H^{q-p}(x, n-p) \Rightarrow N^\cdot H^{p+q}(X,n).$$

<u>Proof</u>. For $x \in Z^p/Z^{p+1}$ $\exists$ small open $U \subset X$ s.t. $U \cap \overline{\{x\}}$ is smooth.

so poincaré duality gives $H_{p+q}(x,n) \xrightarrow{(\cap \eta_x)^{-1}} H^{pq}(x, p-n)$.

$\square$

Let $\mathcal{H}_*(n)$, $\mathcal{H}^*(n)$ be the zariski sheaves on $X$ associated to

$$U \mapsto H_*(U,n) \quad ; \quad U \mapsto H^*(U,n).$$

for $x \in X$, $i_x A$ denote the constant sheaf $A$ on $\overline{\{x\}}$ extended by zero to all of $X$.

<u>Main Theorem</u>: $k$ perfect field. $X \in \mathrm{Ob}\,\mathcal{V}$ smooth$/k$, $H^*, H_*$ a poincaré duality theory with supports. Then:

(i). The spectral sequence of sheaves $\mathcal{E}_1^{p,q} = \bigoplus_{x \in Z^p/Z^{p+1}} i_x H^{q-p}(x, n-p) \Rightarrow \mathcal{H}^{p+q}(n)$ is degenerate at $\mathcal{E}_2$, $\mathcal{E}_2^{p,q} = (0)$ for $p > 0$;

(ii) $0 \to \mathcal{H}^q(n) \to \bigoplus_{x \in Z^0/Z^1} i_x H^q(x,n) \to \bigoplus_{x \in Z^1/Z^2} i_x H^{q-1}(x, n-1) \to \cdots$

$\to \bigoplus_{x \in Z^q/Z^{q+1}} i_x H^0(x, n-q) \to 0$

is exact.

(iii) Let $\mathcal{H}^q_{Z^p}(n)$ be the sheaf associated to $U \mapsto \varinjlim_{Z \in Z^p} H^q_{Z \cap U}(U,n)$

The map $\mathcal{H}^q_{Z^{p+1}}(n) \to \mathcal{H}^q_{Z^p}(n)$ is zero for all $(p,q,n)$

Cor: (i). $H^p(X, \mathcal{H}^q(n)) = (0)$ for $p > q$

(ii) The $Z_2$ term of the spectral sequence

$$Z_1^{p,q} = \bigoplus_{x \in Z^p/Z^{p+1}} H^{q-p}(x, n-p) \Rightarrow N^{\cdot} H^{q+p}(X, n)$$

is given by $Z_2^{p,q} = H^p(X, \mathcal{H}^q(n))$.

Example: Let $\mathcal{H}^*_{ét}(n)$ be the Zariski sheaf on $X$ associated to $U \longmapsto H^*_{ét}(U, \mu_r^{\otimes n})$ where $\mu_r$ is the étale sheaf of $r$-th roots of unity. $r$ prime to char$(k)$.

Then $H^p(X, \mathcal{H}^p_{ét}(p)) \cong CH^p(X) \otimes \mathbb{Z}/r\mathbb{Z}$.

proof: We have

$$\bigoplus_{x \in Z^{p-1}/Z^p} H^1_{ét}(x, 1) \to \bigoplus_{x \in Z^p/Z^{p+1}} \mathbb{Z}/r\mathbb{Z} \to H^p(X, \mathcal{H}^p_{ét}(p)) \to 0$$

From Hilbert's Thm 90, we have

$$H^1_{ét}(x, 1) = H^1_{Galois}(k(x), \mu) \cong k(x)^*/k(x)^{*r}.$$

So

$$\begin{tikzcd}
\bigoplus_{x \in Z^{p-1}/Z^p} k(x)^* \arrow[r] \arrow[d] & \bigoplus_{x \in Z^p/Z^{p+1}} \mathbb{Z} \arrow[r] \arrow[d] & H^p(X, \mathcal{K}_p) \arrow[r] \arrow[d] & 0 \\
\bigoplus_{x \in Z^{p-1}/Z^p} k(x)^*/k(x)^{*r} \arrow[r] & \bigoplus_{x \in Z^p/Z^{p+1}} \mathbb{Z}/r\mathbb{Z} \arrow[r] & H^p(X, \mathcal{H}^p_{ét}(p)) \arrow[r] & 0
\end{tikzcd}$$

This implies that $H^p(X, \mathcal{H}^p_{ét}(p)) \cong CH^p(X) \otimes \mathbb{Z}/r\mathbb{Z}$