§1. Homology theory

Defin. $\mathcal{C}$: a category of noetherian schemes such that for any $X \in ob\,\mathcal{C}$, every immersion $Y \longrightarrow X$ is a morphism in $\mathcal{C}$.

1) $\mathcal{C}_*$: ~~the~~ subcategory of $\mathcal{C}$ with same objects, but where morphisms are just proper morphisms.

A homology theory on $\mathcal{C}$ is a sequence of covariant functors

$$H_a : \mathcal{C}_* \longrightarrow Ab \qquad (a \in \mathbb{Z})$$

satisfying the following condition:

(i) for each open immersion $j: U \hookrightarrow X$ in $\mathcal{C}$, there is a map $j^*: H_a(X) \longrightarrow H_a(U)$ in a functorial way

(ii) If $i: Y \to X$ is a closed immersion in $\mathcal{C}$ with complement $j: U \to X$, there is a long exact sequence

$$\cdots \xrightarrow{\delta} H_a(Y) \xrightarrow{i_*} H_a(X) \xrightarrow{j^*} H_a(U) \xrightarrow{\delta} H_{a-1}(Y) \longrightarrow \cdots$$

This sequence is functorial w.r.t. proper or open morphisms.

2) A morphism between homology theory H and H' is a morphism $\phi: H \to H'$ of functors on $\mathcal{C}_*$ which is compatible with the long exact sequence from (ii).

Basic example

~~1) Let S be a noeth~~

1) S: noetherian scheme
$\mathcal{C} = Sch_{sft}/S$: the category of schemes separated and of finite type over S
$\Lambda = \Lambda_S \in D^b(S_{et})$: a bounded complex of etale sheaves on S. Define

$$H_a^\Lambda(X) = H^{-a}(X_{et}, Rf^!\Lambda) \quad \text{for any } f: X \to S \in Sch_{sft}/S.$$

2) X: a noetherian scheme
$\mathcal{C} = Sub(X)$
$\Lambda = \Lambda_X \in D^+(X_{et})$. ~~Define~~ For any $Z \in Sub(X)$

$$H_a^\Lambda(Z) = H_Z^{-a}(U_{et}, \Lambda|U)$$

as the etale cohomology with support in Z, where U is an open subscheme of X containing Z as a closed subschemes

Let H be a homology theory on $\mathcal{C}$.

1) For any $N \in \mathbb{Z}$, the shifted homology theory $H[N]$ is defined by setting $H[N]_a(X) = H_{a+N}(X)$ and multiplying the connection maps δ by $(-1)^N$.

2) For any $X \in ob\,\mathcal{C}$, the restrict. $H/_X$ is again a homology theory on $\mathcal{C}/_X$

3) Let H be a homology theory on $\mathcal{C}/X$ and $Z \hookrightarrow X$ be an immersion.
For any $T/X \in \mathcal{C}/X$ define $H_a^{(Z)}(T) = H_a(T \times_X Z)$. Then $H_a^{(Z)}$ is a homology theory on $\mathcal{C}/X$.
For any open immersion $j: U \hookrightarrow X$ (resp. closed immersion $i: Y \to X$), we get a morph of homology theories $j^*: H \to H^{(U)}$ (resp. $i_* H^{(Z)} \to H$).
If j is the complement of i, the connect map $\delta: H_a(T\times_X U) \to H_{a-1}(T\times_X Y)$ for any $T \in \mathcal{C}/X$ and $a \in \mathbb{Z}$ defines a morphism of homology theories $\delta: H^{(U)}[1] \to H^{(Y)}$.

Lemma: Let H be a homology theory on $\mathcal{C}$ and let $X \in \mathrm{ob}\,\mathcal{C}$. Let Y and Z be two closed subschemes of X with complement U and V respectively. The diagrams

$$\begin{array}{ccc} H_a(U\cap V) & \xrightarrow{-\delta} & H_{a-1}(U\cap Z) \\ \downarrow \delta & & \downarrow \delta \\ H_{a-1}(Y\cap V) & \xrightarrow{\delta} & H_{a-2}(Y\cap Z) \end{array}$$

commutes.

Spectral sequences.

Theorem. If H is a homology theory on $\mathcal{C}$, then for any $X \in \mathrm{ob}(\mathcal{C})$, there is a spectral sequence of homological type

$$E^1_{p,q}(X) = \bigoplus_{x \in X_p} H_{p+q}(x) \Longrightarrow H_{p+q}(X).$$

$X_p = \{x \in X \mid \dim \overline{\{x\}} = p\}$

$H_a(x) = \varinjlim_U H_a(U)$ where U runs over all nonempty open subschemes of $\overline{\{x\}}$.
This spectral sequence is covariant w.r.t. proper morphism in $\mathcal{C}$ and contravariant w.r.t. open immersion.

Proof: Let $\mathcal{Z}_p(X)$ be the set of closed subsets of X of dimension p.
For $r<p$, let $\mathcal{Z}_p/\mathcal{Z}_r(X)$ be the set of pairs $(Z, Z') \in \mathcal{Z}_p(X) \times \mathcal{Z}_r(X)$ such that $Z \supset Z'$.
For $(Z, Z') \in \mathcal{Z}_p/\mathcal{Z}_r(X)$, we have a long exact sequence

$$\cdots \to H_a(Z') \to H_a(Z) \to H_a(Z\setminus Z') \xrightarrow{\delta} H_{a-1}(Z') \to \cdots$$

By taking limit over $\mathcal{Z}_p/\mathcal{Z}_r(X)$, we get a long exact sequence

$$\cdots \to H_a(\mathcal{Z}_r(X)) \to H_a(\mathcal{Z}_p(X)) \to H_a(\mathcal{Z}_p/\mathcal{Z}_r(X)) \to H_{a-1}(\mathcal{Z}_r(X)) \to \cdots$$

Fact: $H_a(\mathcal{Z}_p(X)) = 0$ for $p<0$ and $H_a(\mathcal{Z}_p(X)) = H_a(X)$ for $p \geq \dim X$.
For any $p, q \in \mathbb{Z}$, $r \in \mathbb{Z}_+^*$, define

$$E^r_{p,q}(X) = \mathrm{Im}\left(H_{p+q}(\mathcal{Z}_p/\mathcal{Z}_{p-r}(X)) \to H_{p+q}(\mathcal{Z}_{p+r-1}/\mathcal{Z}_{p-1}(X))\right)$$

define $d_{p,q}^r: E_{p,q}^r \longrightarrow E_{p-r,q+r-1}^r$ by the commutative diagrame

$$\begin{array}{ccc} H_{p+q}(\mathcal{Z}_p/\mathcal{Z}_{p+r}) & \longrightarrow & H_{p+q}(\mathcal{Z}_{p+r-1}/\mathcal{Z}_{p-1}) \\ \downarrow \delta & & \downarrow \delta \\ H_{p+q-1}(\mathcal{Z}_{p-r}/\mathcal{Z}_{p-2r}) & \longrightarrow & H_{p+q-1}(\mathcal{Z}_{p-1}/\mathcal{Z}_{p-r-1}) \end{array}$$

In particular, $E_{p,q}^1 = H_{p+q}(\mathcal{Z}_p/\mathcal{Z}_{p-1})$ and $E_{p,q}^\infty = \mathrm{Im}\left(H_{p+q}(\mathcal{Z}_p) \longrightarrow H_{p+q}(\mathcal{Z}_\infty/\mathcal{Z}_{p-1})\right)$

For any $Z \in \mathcal{Z}_p(X)$, there is $Z' \in \mathcal{Z}_{p-1}(X)$ such that $Z' \subset Z$ and $Z \setminus Z'$ is a disjoint union of irreducible subschemes of Z of dimension r. Then we have

$$E_{p,q}^1(X) = \bigoplus_{x \in X_p} H_{p+q}^\theta(x);$$

The map $d_{p,q}^1: \bigoplus_{x \in X_p} H_{p+q}(x) \longrightarrow \bigoplus_{y \in X_{p-1}} H_{p+q-1}(y)$ $\quad (d_{p,q}^1: E_{p,q}^1 \longrightarrow E_{p,q}^1)$

is induced by $\quad \delta_{xy}: H_{p+q}(x) \longrightarrow H_{p+q-1}(y)$ which is defined by the connecting map

$$H_{p+q}(U) \xrightarrow{\delta} H_{p+q-1}(\overline{\{x\}} \setminus U) \longrightarrow H_{p+q-1}(y).$$

If $y \notin \overline{\{x\}}$, then $\delta_{xy} = 0$.

Since $E_{p,q}^1 = 0$ for any $p < 0$, then the edge morphism is

$$E_{0,q}^2(X) = \mathrm{Im}\left(H_q(\mathcal{Z}_0) \longrightarrow H_q(\mathcal{Z}_1)\right) \longrightarrow H_q(X).$$

The case $S = \mathrm{spec}\, F$ for a field F.

Proposition: Let X be separated of finite type over F, and let H be a homology theory on $\mathcal{C}$. If $i: Y \to X$ is a closed immersion with the complement $j: U \to X$, then

(a) for all p, q, we have an exact sequence

$$0 \to E_{p,q}^1(Y) \xrightarrow{i_*} E_{p,q}^1(X) \xrightarrow{j^*} E_{p,q}^1(U) \to 0$$

(b): The connecting map $\delta: H_a(Z \cap U) \longrightarrow H_{a-1}(Z \cap Y)$ for any $Z \in \mathrm{sub}(X)$ induces a morphism of spectral sequences

$$\delta: E_{p,q}^1(U) \longrightarrow E_{p-1,q}^1(Y).$$

Proof. (a) follows by $X_p \cap Y = Y_p$, $X_p \cap U = U_p$.

(b). For any $(Z, Z') \in \mathcal{Z}_p/\mathcal{Z}_{p-1}(U)$, let $\delta(Z, Z') = (\overline{Z} \cap Y, \overline{Z'} \cap Y) \in \mathcal{Z}_{p-1}/\mathcal{Z}_{p-2}(Y)$,

and commutative diagram

$$\begin{array}{ccc} H_{a+1}(Z \setminus Z') & \xrightarrow{-\delta} & H_a(Z') \\ \downarrow \delta & & \downarrow \delta \\ H_a(\delta(Z) \setminus \delta(Z')) & \xrightarrow{\delta} & H_{a-1}(\delta(Z')) \end{array}$$

The map $\delta: E_{p,q}^1(U) = \bigoplus_{x \in U_p} H_{p+q}(x) \longrightarrow \bigoplus_{y \in Y_p} H_{p+q-1}(y) = E_{p-1,q}^1(Y)$ is induced by

$\delta_{xy}^X: H_{p+q}(x) \longrightarrow H_{p+q-1}(y)$ as above

(c) For each $q \in \mathbb{Z}$, the family of functors $(E_{p,q}^2)_{p \in \mathbb{Z}}$ define a homology theory on $\mathcal{C}$.

The case of discrete valuation ring

Proposition: Let $S = \operatorname{Spec} A$ for a discrete valuation ring A with generic point η and closed point s. ~~For any~~ Let X be a separated and of finite type scheme over S. and let H be a homology theory on $\operatorname{Sub}(X)$.

(a) The connection map $\delta: H_a(Z_\eta) \longrightarrow H_a(Z_s)$ for any $Z \in \operatorname{Sub}(X)$ induces a morphism of spectral sequence

$$\Delta_X: E^1_{p,q}(X_\eta)^{(-)} \longrightarrow E^1_{p,q-1}(X_s)$$

This morphism is functorial with ~~proper~~ closed morphism and open immersion. so we get

$$\begin{array}{ccccccccc} 0 \to & E^1_{p,q}(Y_\eta)^{(-)} & \longrightarrow & E^1_{p,q}(X_\eta)^{(-)} & \longrightarrow & E^1_{p,q}(U_\eta)^{(-)} & \to 0 \\ & \downarrow \Delta_Y & & \downarrow \Delta_X & & \downarrow \Delta_U & \\ 0 \to & E^1_{p,q}(Y_s) & \longrightarrow & E^1_{p,q}(X_s) & \longrightarrow & E^1_{p,q}(U_s) & \to 0 \end{array}$$

for every closed subscheme Y of X with complement U.

(b) If X is proper over S. then $j: X_\eta \to X$ induces a morphism of spectral sequence

$$j^*: E^1_{p,q}(X) \longrightarrow E^1_{p+1,q}(X_\eta).$$

such that

$$0 \longrightarrow E^1_{p,q}(X_s) \xrightarrow{i_*} E^1_{p,q}(X) \xrightarrow{j^*} E^1_{p+1,q}(X_\eta) \to 0$$

is exact for all p, q where $i: X_s \to X$ is the special fiber of X.

Proof: a) holds by $\delta: H^{(X_\eta)}[1] \longrightarrow H^{(X_s)}$ and $\delta: Z_p(X_\eta) \longrightarrow Z_p(X_s)$, $\delta(Z) = \overline{Z}_\eta \cap X_s$.

b) If $X \to S$ is proper. $X_p \cap X_\eta = (X_\eta)_{p+1}$.

Two special examples

$S = \operatorname{Spec} F$ for a field F. $n, b \in \mathbb{Z}$. ~~Consider two cases~~

~~(i)~~ n is invertible in F.

$S = \operatorname{Spec} F$ for a field F. $n \in \mathbb{Z}$ invertible in F. $b \in \mathbb{Z}$

~~for~~ Any $X \in Sch_{ft}/S$. ~~define~~ associated to

$$H_a(X/F, \mathbb{Z}/n(b)) = H^{-a}(X_{et}, Rf_X^! \mathbb{Z}/n(b))$$

defines a homology theory on Sch_{ft}/S. We have a spectral sequence

$$E^1_{p,q}(X/F, \mathbb{Z}/n(b)) = \bigoplus_{x \in X_p} H_{p+q}(x, \mathbb{Z}/n(b)) \Longrightarrow H_{p+q}(X, \mathbb{Z}/n(b))$$

Theorem. Let X be separated of finite type over F

(a) We have canonical isomorphisms

$$H_a(x/F, \mathbb{Z}/n(b)) \simeq H^{2p-a}(k(x), \mathbb{Z}/n(p+b)) \quad \text{for } x \in X_p$$

(b) If the cohomological ℓ-dimension $cd_\ell(F) \le c$ for all prime $\ell | n$. then $E^1_{p,q}(X, \mathbb{Z}/n(b)) = 0$ for all $q < -c$. In particular, we get a canonical edge

$$\varepsilon(X): H_{a-c}(X/F, \mathbb{Z}/n(b)) \longrightarrow E^2_{a,-c}(X/F, \mathbb{Z}/n(b))$$

(c) If X is smooth of pure dimension over F, then we have canonical isomorphisms

$$H_a(X/F, \mathbb{Z}/n(b)) \simeq H^{2d-a}(X_{et}, \mathbb{Z}/n(d+b))$$

Proof: (c) ~~If~~ $f_X: X \to \operatorname{Spec} F$ is smooth of pure dimension d. we have

$$\alpha_X: \quad Rf_X^! \mathbb{Z}/n(b) = \mathbb{Z}/n(d+b)[2d]$$

(a) We may assume F is perfect. Any $x \in X_p$ has an open neighborhood V of $\overline{\{x\}}$ which is smooth of pure dimension p over F. Then

$$H_a(x, \mathbb{Z}/n(b)) = \varinjlim_{V \subset \overline{\{x\}}} H_a(V, \mathbb{Z}/n(b)) \simeq \varinjlim_{V \subset \overline{\{x\}}} H^{2p-a}(V_{et}, \mathbb{Z}/n(p+b)) = H^{2p-a}(x, \mathbb{Z}/n(p+b))$$

(b) For $x \in X_p$. tr.deg$(k(x)/F) = p$. then for prime $\ell | n$. $cd_\ell(k(x)) \le c+p$.

$$H_{p+q}(x/F, \mathbb{Z}/n(b)) = H^{p-q}(k(x), \mathbb{Z}/n(p+b)) = 0 \text{ for } p-q > c+p. \text{ i.e. } q < -c.$$

So $E^1_{p,q}(X, \mathbb{Z}/n(b)) = 0$ for $q < -c$. Thus $E^r_{p,q}(X, \mathbb{Z}/n(b)) = 0$ for $q < -c$. $r \ge 1$

Thus by $E^r_{p+r,-c-r+1} \xrightarrow{d_{p+r,-c-r+1}} E^r_{p,-c} \xrightarrow{d_{p,q}} E^r_{p-r,-c+r-1}$ we have $E^{r+1}_{p,-c} \subset E^r_{p,-c}$. $r \ge 2$ so $E^\infty_{p,-c} \subseteq E^2_{p,-c}$.

$F_i H_{p-c}/F_{i-1}H_{p-c} \simeq E^\infty_{i,p-c-i} = 0$ for $i > p$. Then $F_p H_{p-c} = H_{p-c}$. and $H_{p-c}/F_{p-1}H_{p-c} \simeq E^\infty_{p,-c}$

The edge morphism $\varepsilon(X)$ is the composition

$$H_{p-c}(X/F, \mathbb{Z}/n(b)) \twoheadrightarrow F_pH_{p-c}/F_{p-1}H_{p-c} \simeq E^\infty_{p,-c}(X/F, \mathbb{Z}/n(b)) \hookrightarrow E^2_{p,-c}(X/F, \mathbb{Z}/n(b))$$

Moreover. ~~$\varepsilon(X)$ is a morp~~ $\varepsilon(X): H_{\cdot-c}(X/F, \mathbb{Z}/n(b)) \to E^2_{\cdot,-c}(X/F, \mathbb{Z}/n(b))$

is a morphism of homology theories.

Let $S = \operatorname{spec} A$ for a discrete valuation ring with residue field k and fraction field K.
Let $j: \eta = \operatorname{spec} K \to S$ and $i: s = \operatorname{spec} k \to S$ be the generic point and closed point of S.
For any S-scheme T, let T_s, T_η be its special fiber and generic fiber.

Consider two cases.

(i) n is invertible on S and b is arbitrary

(ii) K is a field of char 0, k is perfect of char $p>0$, $n = p^m$, $m \geq 1$, $b \geq 1$

In case (i), $\Lambda_S = \mathbb{Z}/n(b)$

In case (ii), $\Lambda_S = \operatorname{Cone}\left(Rj_* \mathbb{Z}/n(1)_\eta \xrightarrow{\sigma} i_* \mathbb{Z}/n_s[-1]\right)[-1]$ where σ is defined by its adjoint: $i^* Rj_* \mathbb{Z}/n(1)_\eta \longrightarrow \mathbb{Z}/n_s[-1]$, defined as follows

$(\mathbb{Z}/n)_s[-1]$ is concentrated at degree zero with stalk $\mathbb{Z}/n$.

$i^* Rj_* \mathbb{Z}/n(1)_\eta = R\Gamma(K^{ur}, \mu_n)$ where K^{ur} is the maximal unramified exten. of K.

So $\mathcal{H}^{\bullet}$ $i^* Rj_* \mathbb{Z}/n(1)_\eta$ is concentrated at degree 0 and 1, and

$i^* Rj_* \mathbb{Z}/n(1)_\eta \longrightarrow (\mathbb{Z}/n)_s[-1]$ is defined by $\quad H^1(K^{ur}, \mu_n) = K^{ur\times}/(K^{ur})^{\times n} \xrightarrow{\text{valuat}} \mathbb{Z}/n$.

For any $f: X \to S \in \mathrm{Sch}_{ft}/S$ associated to $H_a(X, \Lambda_S) = H^{-a}(X_{et}, Rf^! \Lambda_S)$ defines a homology theory on Sch_{ft}/S.

Lemma: There are isomorphism of spectral sequence

$$E^1_{p,q}(X_\eta/S, \mathbb{Z}/n(b)_S) \simeq E^1_{p,q}(X_\eta/\eta, \mathbb{Z}/n(b))$$

$$E^1_{p,q}(X_s/S, \mathbb{Z}/n(b)_S) \simeq E^1_{p,q+2}(X_s/s, \mathbb{Z}/n(b-1))$$

Proof: $j^* \mathbb{Z}/n(b)_S = \mathbb{Z}/n(b)_\eta$, $\quad Ri^! \mathbb{Z}/n(b)_S = \mathbb{Z}/n(b-1)_s[-2]$.

R discrete valuation ring K, k

$S = \mathrm{Spec}\, R$, $\eta = \mathrm{Spec}\, K$, $s = \mathrm{Spec}\, k$

Sch_S^{ft}:

$X \in Sch_S^{ft}$, X_η, X_s. $\dim'(X) = \dim(\overline{X})$ for $\overline{X}$ ~~?~~ a compactification of X over S.

$X_p = \{x \in X \mid \dim'(\overline{\{x\}}) = p\}$

Y subscheme of $X \Rightarrow Y \cap X_p = Y_p$.

If $X \to S$ doesn't factor through η, then $\dim(X) = \dim'(X)$

Def: Let $Z_p(X)$ be the free abelian group ~~over~~ the set X_p. There is a division map

$$\mathrm{div}: \bigoplus_{x \in X_{p+1}} k(x)^\times \longrightarrow \bigoplus_{x \in X_p} \mathbb{Z}$$

whose cokernel is called the Chow group of p-cycles modulo rational equivalence and is denoted by $CH_p(X)$.

Let $\Lambda \in D^b(S_{et})$ be a bounded complex of étale sheaves on S. Then we have a homology theory H on Sch_S^{ft} defined by

$$H_a(X/S, \Lambda) := H^{2-a}(X_{et}, Rf_X^! \Lambda)$$

for $f_X: X \to S$ in Sch_S^{ft} and $a \in \mathbb{Z}$.

From now on, let ℓ be a prime different from char k.

Let $\Lambda_n = \mathbb{Z}/\ell^n\mathbb{Z}$ for $n \geq 1$ and $\Lambda_\infty = \mathbb{Q}_\ell/\mathbb{Z}_\ell$. Let $\Lambda = \Lambda_n$ or Λ_∞

Lem: Let $X \in Sch_S^{ft}$ and $d = \dim(X)$ ~~-1~~. Assume X is integral and regular

(1) there exists a canonical isomorphism

$$T_X: \Lambda(d)_X[2d] \cong Rf_X^! \Lambda_S \in D(X_{et})$$

(2) for any closed subscheme Y of X, $\exists$ a canonical isomorphism

$$\tau_{Y,X}: H_Y^{2d-a+2}(X, \Lambda(d)) \cong H_a(Y, \Lambda)$$

satisfying:

(a) Let $f': X' \to X$ be a proper (étale) morph in Sch_S^{ft} such that X' is regular integral and let Y' be a closed subscheme of $f^{-1}Y = X' \times_X Y$. ($Y' = f^{-1}Y$)

Let g be the composite $Y' \to f^{-1}Y \to Y$. Then we have commutative diag

$$\begin{tikzcd}
H_{Y'}^{2d-a+2}(X', \Lambda(d)) \arrow[r, "\tau_{Y',X'}", "\sim"'] \arrow[d, "f_*"] & H_a(Y', \Lambda) \arrow[d, "f_*"] \\
H_Y^{2d-a+2}(X, \Lambda(d)) \arrow[r, "\tau_{Y,X}"] & H_a(Y, \Lambda)
\end{tikzcd}
\quad \Bigg(\text{resp} \quad
\begin{tikzcd}
H_{Y'}^{2d-a+2}(X', \Lambda(d)) \arrow[r, "\tau_{Y',X'}", "\sim"'] & H_a(Y', \Lambda) \\
H_Y^{2d-a+2}(X, \Lambda(d)) \arrow[r] \arrow[u, "f^*"] & H_a(Y, \Lambda) \arrow[u, "g^*"]
\end{tikzcd}$$

(b) Let $i: Y \to X$ be a closed subscheme with complement $j: U \to X$. There is a commutative diagram with exact arrows

$$\begin{tikzcd}
\cdots \arrow[r] & H^r_Y(X, \Lambda(d)) \arrow[r, "i_*"] \arrow[d, "\tau_{Y/X}"] & H^r(X, \Lambda(d)) \arrow[r, "j^*"] \arrow[d, "\tau_{X/X}"] & H^r(U, \Lambda(d)) \arrow[r, "\delta"] \arrow[d] & H^{r+1}_Y(X, \Lambda(d)) \arrow[r] \arrow[d] & \cdots \\
& H_a(Y, \Lambda) \arrow[r, "i_*"] & H_a(X, \Lambda) \arrow[r, "j^*"] & H_a(U, \Lambda) \arrow[r, "\delta"] & H_{a-1}(Y, \Lambda) &
\end{tikzcd}$$

Proof: first prove (1) $\Rightarrow$ (2)

~~Applying~~ By $\tau_X: \Lambda(d)_X[2d] \xrightarrow{\sim} Rf_X^! \Lambda_S$. we have

$$H^{2-a+2d}_Y(X, \Lambda(d)) \simeq H^{2-a}(Y, Ri^! \Lambda(d)_X[2d]) \simeq H^{2-a}(Y, Ri^! Rf_X^! \Lambda_S) = H^{2-a}(Y, Rf_Y^! \Lambda_S) = H_a(Y, \Lambda)$$

(b) follows by applying $H^{2-a}(X, \cdot)$ to the commutative diagrams.

$$\begin{tikzcd}
i_* Ri^! \Lambda(d)_X[2d] \arrow[r] \arrow[d] & \Lambda(d)_X[2d] \arrow[r] \arrow[d] & Rj_* Rj^* \Lambda(d)_X[2d] \arrow[d] \\
i_* Ri^! Rf_X^! \Lambda_S \arrow[r] & Rf_X^! \Lambda_S \arrow[r] & Rj_* Rj^! Rf_X^! \Lambda_S
\end{tikzcd}$$

(a) For a proper morph $f: X' \to X$, define the morph

$$Tr_f: Rf_* \Lambda(d')_{X'}[2d'] \longrightarrow \Lambda(d)_X[2d] \quad \text{in } D(X_{et})$$

by the commutative diagram

$$\begin{tikzcd}
Rf_* \Lambda(d')_{X'}[2d'] \arrow[r, "T_{X'}"] \arrow[d, "Tr_f"] & Rf_* Rf_{X'}^! \Lambda_S = Rf_* Rf^! Rf_X^! \Lambda_S \arrow[dl, "adj"] \\
\Lambda(d)_X[2d] \arrow[r] & Rf_X^! \Lambda_S
\end{tikzcd}$$

then the left map f_* is induced by $H^r_Y(X', \cdot) \longrightarrow H^r_Y(X, Rf_* \cdot)$

It remains to prove (1). We define $\tau_X: \Lambda(d)_X[2d] \xrightarrow{\sim} ~~R f~~ Rf_X^! \Lambda_S$

by the adjoint map of $Tr_{f_X}: Rf_{X!} \Lambda(d)_X[2d] \longrightarrow ~~\cdot~~ \Lambda_S$.

Since ~~R~~ is ~~degenerate~~ Dedekind ring, then $X \xrightarrow{f_X} S$ ~~factors through~~ ~~η~~ — is flat or fact as $X \xrightarrow{\beta} s \xrightarrow{i} S$. In the first case, the trace map

$Tr_{f_X}: Rf_{X!} \Lambda(d)_X[2d] \to \Lambda_S$ can be found in SGA XVIII 3.1.13.2

In the second case, def T_{f_X} as the composite

$$Rf_{X!} \Lambda(d)_X[2d] = Ri_* R\beta_! \Lambda(d)_X[2d] \xrightarrow{Tr_\beta} Ri_* \Lambda(-1)_s[-2] \xrightarrow{Gys_i} \Lambda_S$$

where the last arrow is the Gys map.

Let's prove τ_X is an isomph. Fix a immersion $X \xrightarrow{\gamma} \mathbb{P}^N_S =: \mathbb{P}$

such that $\pi \circ \gamma = f_X$ where $\pi: \mathbb{P}^N_S \to S$ is the project. Then τ_X fact as

$$\Lambda(d)_X[2d] \xrightarrow{Gys_\gamma} R\gamma^! \Lambda(N)_{\mathbb{P}}[2N] \xrightarrow{\tau_\pi} R\gamma^! R\pi^! \Lambda_S = Rf_X^! \Lambda_S$$

Cor: There is a homologically type spectral sequence

$$E^1_{p,q}(X) = \bigoplus_{x \in X_p} \cancel{H_{p+q}(x)} \, H^{p-q}(x, \Lambda(p-1)) \Longrightarrow H_{p+q}(X, \Lambda)$$

This spectral sequence is covariant for proper morph and contravariant to etale morph in Sch^{ft}_S.

Proof: $E^1_{p,q}(X, \Lambda) = \bigoplus_{x \in X_p} H_{p+q}(x, \Lambda) \Longrightarrow H_{p+q}(X, \Lambda)$

For $x \in X_p$, we only need to show $H_{p+q}(x, \Lambda) \simeq H^{p-q}(x, \Lambda(p-1))$

Recall $H_{p+q}(x, \Lambda) = \varinjlim_{x \in U \subset \overline{\{x\}}} H_{p+q}(U, \Lambda)$. By cohomology of pure insep ext. we may assume $k(x)$ is perfect. Then we can only take smooth U. Then $\dim U = d+1$. So

$$H_{p+q}(U, \Lambda) = H^{2-p-q}(U, Rf^!_U \Lambda) = H^{2-p-q}(U, \Lambda(p-1)[2p-2]) = H^{p-q}(U, \Lambda(p-1))$$

$$H_{p+q}(x, \Lambda) = H^{p-q}(x, \Lambda(p-1))$$

Then $E^1_{p,q} = 0$ unless $0 \le p \le d+1$ and $q \le p$.

$$\begin{array}{lccccc}
H_3(X,\Lambda) & & & & & E^1_{d+1,d+1} \\
 & & & & & E^1_{d,d} \\
H_2(X,\Lambda) & & & E^1_{2,2} & \leftarrow \cdots \leftarrow & E^1_{d+1,2} \\
 & & E^1_{1,1} & \leftarrow E^1_{2,1} & \leftarrow \cdots \leftarrow & E^1_{d+1,1} \\
H_1(X,\Lambda) & & & & & \\
H_0(X,\Lambda) & E^1_{00} & \leftarrow E^1_{10} & \leftarrow E^1_{20} & \leftarrow \cdots \leftarrow & E^1_{d+1,0} \\
 & E^1_{0,-1} & \leftarrow E^1_{1,-1} & \leftarrow E^1_{2,-1} & \leftarrow & E^1_{d+1,-1}
\end{array}$$

$$E^2_{11}(X) = \operatorname{coker}\Big(d^1_{21} : \bigoplus_{x \in X_2} k(x)^\times \otimes \Lambda \longrightarrow \bigoplus_{x \in X_1} \Lambda\Big) \simeq CH_1(X) \otimes \Lambda.$$

Define the cycle class map

$$\rho_X : CH_1(X) \otimes \Lambda \longrightarrow H_2(X, \Lambda)$$

to be the edge homomorphism $E^2_{11}(X) \longrightarrow H_2(X, \Lambda)$.

Prop: (1) The cycle class map is covariant w.r.t. proper morph and contravariant w.r.t. etale morphisms

(2) For a closed immersion $i: Y \longrightarrow X$ with complement $j: U \to X$

$$\begin{array}{ccccccc}
CH_1(Y)\otimes\Lambda & \xrightarrow{i_*} & CH_1(X)\otimes\Lambda & \longrightarrow & CH_1(U)\otimes\Lambda & \longrightarrow & 0 \\
\downarrow \rho_Y & & \downarrow \rho_X & & \downarrow \rho_U & & \\
H_2(Y,\Lambda) & \longrightarrow & H_2(X,\Lambda) & \longrightarrow & H_2(U,\Lambda) & &
\end{array}$$

(3)
$$\begin{array}{ccccc} CH_1(X)\otimes\Lambda_m & \xrightarrow{\times \ell^n} & CH_1(X)\otimes\Lambda_{m+n} & \xrightarrow{\mathrm{mod}\,\ell^n} & CH_1(X)\otimes\Lambda_n \\ \downarrow \rho_X & & \downarrow \rho_X & & \downarrow \rho_X \\ H_2(X,\Lambda_m) & \xrightarrow{\times \ell^n} & H_2(X,\Lambda_{m+n}) & \longrightarrow & H_2(X,\Lambda_n) \end{array}$$

proof: (2). For any X we have a short exact sequence $0 \to E^1_{p,q}(Y) \xrightarrow{i_*} E^1_{p,q}(X) \xrightarrow{j^*} E^1_{p,q}(U) \to 0$

(3) low row of (3) holds by applying $H_2(X,\cdot)$ to $0\to\Lambda_m\to\Lambda_{m+n}\to\Lambda_n\to 0$

(4) For any $q\in\mathbb{Z}$, $E^2_{p,q}$ is a homology theory on Sch_S^{op}.

Cor: Suppose $\dim(X)\le 2$. Then $\rho_X: CH_1(X)\otimes\mathbb{Z}/\ell^n \to H_2(X,\Lambda_n)$ is injective

proof: $E^1_{p,q}(X)=0$ for [illegible] $0\le a\le 2$. Then $E^2_{11} \xrightarrow{\sim} E^\infty_{11} \simeq F^1H_2(X,\Lambda_n)/F_0H_2(X,\Lambda_n)$ $CH_1(X,\Lambda_n)$

Def. $E^1_{p,q}(X,\Lambda) = \bigoplus_{x\in X_p} H^{p-q}(x,\Lambda(p-1)) \Rightarrow H_{p+q}(X,\Lambda)$

$KC(X,\Lambda)$: $E^1_{p0}(X,\Lambda) \to E^1_{p-1,0}(X,\Lambda) \to \cdots \to E^1_{10}(X,\Lambda) \to E^1_{00}(X,\Lambda)$

$$\begin{array}{ccccccc} \wr\!\!\mid & & \wr\!\!\mid & & & & \\ \bigoplus_{x\in X_p} H^p(x,\Lambda(p)) & - & \bigoplus_{x\in X_{p-1}} H^{p-1}(x,\Lambda(p-1)) & & \bigoplus_{x\in X_1} H^1(x,\Lambda) & \to & \bigoplus_{x\in X_0} H^1(x,\Lambda(-1)) \end{array}$$

$KH_a(X,\Lambda) = E^2_{a0}(X,\Lambda)$

$Y \longrightarrow X \hookleftarrow U$ $\quad 0 \to KC(Y,\Lambda) \xrightarrow{i_*} KC(X,\Lambda) \xrightarrow{j^*} KC(U,\Lambda) \to 0$

Lem: Assume R is henselian.

1) If k is sep. closed, then $E^1_{p,q}(X,\Lambda_n)=0$ for $q\le -1$

2) If k is finite, $E^1_{p,q}(X,\Lambda_n)=0$ for $q\le -2$, $E^1_{p,q}(X,\Lambda_\infty)=0$ for $q\le -1$

proof. 1) $x\in X_p$. $cd(x)\le cd(k)+p$. $H^i(x,\Lambda)=0$ for $i\ge p+1$.

$E^{p,q}_1(X,\Lambda) = \bigoplus_{x\in X_p} H^{p-q}(x,\Lambda(p-1)) = 0$ for $p-q\ge p+1$, i.e. $q\le -1$.

Lem. R henselian with residue field k finite. Let $r\ge 0$.

L a field either f.g. over K with $\mathrm{trdeg}_K(L)=r-1$ or f.g. over k. $\mathrm{trdeg}_k(L)=r$

Then $H^{r+1}(X,\Lambda_\infty(n))=0$ for any $n\ne r$.