0-cycles over local fields Seminar

Talk 7 by 鄭維喆

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P. 1

§ The vanishing theorem

§1. Statement

$R =$ henselian DVR

$$\eta \hookrightarrow B = \mathrm{Spec}(R) \hookleftarrow s = \mathrm{Spec}(F)$$

$\ell \neq \mathrm{char}(F)$. $\quad \Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ or $\mathbb{Q}_\ell/\mathbb{Z}_\ell = \lim \mathbb{Z}/\ell^n\mathbb{Z}$

$$\begin{tikzcd} X \arrow[d, "f"', "\text{separated, finite type}"] \\ B \end{tikzcd} \qquad H_q(X, \Lambda) := H^{2-q}(X, Rf^! \Lambda_B)$$

Notation $\quad \delta_X := \max\{\dim X_\eta + 1, \dim X_s\}$

$$\begin{tikzcd} X \arrow[r, hook, "\text{dense}"] \arrow[dr] & X' \arrow[d, "\text{proper}"] \\ & B \end{tikzcd} \qquad \text{Then } \delta_X = \dim X'$$

Trivial vanishing.

- $R^q f^! \Lambda = 0$ if $q < -2(\delta_X - 1)$

  ($\Rightarrow$ $H_q(X, \Lambda) = 0$ for $q > 2\delta_X$)

- $H_q(X, \Lambda) = 0$ for $q < 0$ & $F = F^{sep}$
  or $q < -1$ & F finite

<u>Def</u> $X/B \in QS$ if

- X regular, X/B flat & quasi-projective
- $X_{s, red}$ SNC

Example: $X = \mathrm{Spec}\left(R[t_1, \ldots, t_d] / (t_1^{a_1} \cdots t_r^{a_r} - \pi)\right)$

$a_i \in \mathbb{N}^*$, $r \leq d$

$(X, Y)/B$ is a <u>QS-pair</u> if

- X/B is QS.
- Y divisor on X, and Y/B is QS
- $Y + X_{s, red}$ is SNC

<u>Example</u> $r < d$, $Y =$ Divisor of t_{r+1} in X.

<u>Thm</u> $(X, Y)/B$ QS pair, X/B projective equidimensional

$U = X - Y$ affine $\quad \dim X = d+1 \geq 2$

Then

(1) $F = F^{sep} \Rightarrow H_q(U, \Lambda) = 0$ for $q \leq d+1$

(2) $F = \mathbb{F}_q \Rightarrow H_q(U, \Lambda) = 0$ for $q \leq d$

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(3) $F = \mathbb{F}_q \Rightarrow H_{d+1}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$

Ingredients.
- absolute purity
- affine Lefschetz
- (3) uses weight theory

§2 Parity

- relative purity

$$\begin{array}{l} X \\ f \Big\downarrow \text{ smooth, separated finite type} \\ S \to \text{quasi-compact, quasi-separated} \end{array}$$

$Rf^! = f^*(d)[2d]$

"Poincaré duality" $d = \dim(f)$

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr, "g"', "\text{smooth}"] & X \arrow[d, "f", "\text{smooth}"'] \\ & S \end{tikzcd}$$

$\Rightarrow Ri^! Rf^! \Lambda \cong Rg^! \Lambda = \Lambda(\dim(g))[2\dim(g)]$

$\qquad \| $

$Ri^! \Lambda(\dim(f))[2\dim(f)]$

$\Rightarrow Ri^! \Lambda = \Lambda(-c)[-2c]$

where $c = \mathrm{codim} = \dim(f) - \dim(g)$

In other words

$$R^q i^! \Lambda = \begin{cases} \Lambda(-c) & q = 2c \\ 0 & q \neq 2c \end{cases}$$

$\boxed{\text{P. 4}}$

- Absolute purity

Thm: (Gabber) $Y \overset{i}{\hookrightarrow} X$ noetherian, X, Y both regular, purely of codim c

Then $Ri^! \Lambda_X \cong \Lambda_Y(-c)[-2c]$

Conjectured by Grothendieck [SGA 5 I]

First proof → written up by Fujiwara 2002

Second proof → written up by Riou 2014 (* 363-364)

Cor $f: X \to S$ quasi-projective X, S regular noetherian

Then $Rf^! \Lambda_S \cong \Lambda_X(d)[2d]$

where $d = \dim(f)$ = relative dimension

Pf:
$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & \mathbb{P}^N_S \arrow[d, "\pi"] \\ & S \end{tikzcd}$$

$$Rf^! \Lambda_S \cong Ri^! R\pi^! \Lambda_S \underset{\text{Poincaré duality}}{\simeq} Ri^! \Lambda_{\mathbb{P}^n}[2n]$$

$$\underset{\text{absolute purity}}{\cong} \Lambda_X(n-c)[2(n-c)]$$

$d = n - c$

$\square$

Cycle classes (All schemes are noetherian)

$D \subseteq X$ effective Cartier divisor $U := X - D$

$\to$ natural trivialization

$1_U : \mathcal{O}_U \xrightarrow{\sim} \mathcal{O}(D)|_U$

$$\begin{array}{ccccc}
c(\mathcal{O}(D), 1_U) & \in & H^1_D(X, \mathbb{G}_m) & \xrightarrow{\delta} & H^2_D(X, \Lambda(1)) \ni cl(D) \\
\downarrow & & \downarrow & & \downarrow \\
c(\mathcal{O}(D)) & \in & H^1(X, \mathbb{G}_m) & \xrightarrow{\delta} & H^2(X, \Lambda(1)) \ni c_1(\mathcal{O}(D)) \\
 & & & & \downarrow \\
 & & c_1(\mathcal{O}(D)|_D) & \in & H^2(D, \Lambda(1))
\end{array}$$

$c(\mathcal{O}(D), 1_U) \mapsto cl(D)$, $\quad c(\mathcal{O}(D)) \mapsto c_1(\mathcal{O}(D))$

(δ comes from Kummer theory)

$\boxed{\text{P. 5}}$

$Y \overset{i}{\hookrightarrow} X$ regular embedding ($\Leftrightarrow$ I_Y/I_Y^2 locally free $\mathcal{O}_Y$-sheaf)

$$\begin{array}{ccccc}
H^{2c}_Y(X, \Lambda(c)) & \longrightarrow & H^{2c}(X, \Lambda(c)) & \xrightarrow{i^*} & H^{2c}(Y, \Lambda(c)) \\
\Downarrow & & & & \\
cl(Y) & & \longmapsto & & c_c\left((I_Y/I_Y^2)^\vee\right)
\end{array}$$

Definition of $c_c\left((I_Y/I_Y^2)^\vee\right)$:

$$\begin{array}{ccc}
\mathbb{P}(I_Y/I_Y^2) = E & \longrightarrow & \tilde{X} \\
\pi_Y \downarrow & & \downarrow \pi \text{ blowup} \\
Y & \hookrightarrow & X
\end{array}$$

$c_c = c_1(\mathcal{O}_E(1))$

projective bundle formula let $\xi = c_1(\mathcal{O}(E)|_E) \in H^2(E, \Lambda(1))$

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$$\bigoplus_{i=1}^{c} H^{2i}(Y, \Lambda(i)) \xrightarrow{\sim} H^{2c}(E, \Lambda_E(c))$$

$$\gamma_i \longmapsto \pi_Y^*(\gamma_i)\xi^{c-i}$$

$\Rightarrow \exists!\ c_i((I_Y/I_Y^2)^\vee)$ s.t.

$$\xi^c + \sum_{i=1}^{c} c_i((I_Y/I_Y^2)^\vee)\,\xi^{c-i} = 0.$$

Lemma (refined projective bundle formula)

$$\bigoplus_{i=1}^{c-1} H^{2i}(Y, \Lambda(i)) \oplus H^{2c}_Y(X, \Lambda(c)) \xrightarrow{\sim} H^{2c}_E(\widetilde{X}, \Lambda(c))$$

$$\gamma_i \longmapsto \pi_Y^*(\gamma_i)\cdot \eta^{c-i}$$

where $\eta = cl(E) \in H^2_E(\widetilde{X}, \Lambda(1))$

$\Rightarrow \exists!\ cl(Y) \in H^{2c}_Y(X, \Lambda(c))$ s.t.

$$\eta^c + \sum_{i=1}^{c-1} c_i((I_Y/I_Y^2)^\vee)\,\eta^{c-i} + \pi_Y^*(cl(Y)) = 0$$

§3 Affine Lefschetz

S regular dim ≤ 1.

X/S finite type

$$\delta_X = \sup_{s \in S} \left(\dim \overline{\{s\}} + \dim X_s \right)$$

Thm

$$\begin{tikzcd} X \arrow[rr, "\text{affine}", "f"'] \arrow[dr, "\text{finite type}"'] & & Y \arrow[dl, "\text{finite type}"] \\ & S & \end{tikzcd}$$

$\mathcal{F} \in$ Sheaf of Λ-modules on X

Then $\delta_{\mathrm{Supp}(R^q f_* \mathcal{F})} \leq \delta_{\mathrm{supp}(\mathcal{F})} - q, \ \forall q.$

Cor. $R^q f_* \mathcal{F} = 0$ for $q > \delta_X$

We only use

- $S = \mathrm{Spec}(\text{field})$, SGA 4 XIV Artin
- $S =$ reg dim 1 Gabber see Illusie (2002)

(see also more generally S quasi-excellent, * 363-364)

$B = \mathrm{Spec}(R)$, U regular $/B$.

$$\begin{tikzcd} U \arrow[d, "a \text{ quasi-projective}"] \\ B \end{tikzcd}$$

$$H_q(U, \Lambda) := H^{2-q}(U, Ra^! \Lambda_B)$$

$$= H^{2d+2-q}(U, \Lambda_U(d))$$

$$= H^{2d+2-q}(B, Ra_* \Lambda_U(d))$$

by purity

$Ra^! \Lambda_B = \Lambda_U(d)[2d]$

若 $F = F^{sep}$, $= \left(R^{2d+2-q} a_* \Lambda_U(d)\right)_s$ (stalk)

hence $= 0$ for $q \le d$

只用到 $cd = 1$

若 $F = \mathbb{F}_q$, $\overset{?}{=} 0$ for $q \le d-1$

by affine Lefschetz for $a : U \to B$

($\cancel{G_F = \hat{\mathbb{Z}}}$ $\bar{U} = U \times_B \bar{B}$, $\bar{B} = \mathrm{Spec}(R^{ur})$

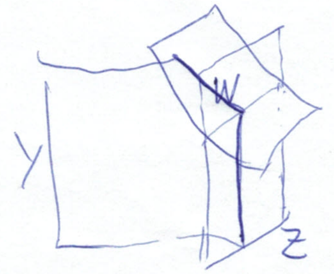
$$0 \to H^{r-1}(\bar{U}, \Lambda(d))_{G_F} \to H^r(U, \Lambda(d)) \to H^r(\bar{U}, \Lambda(d))^{G_F} \to 0$$)

<u>Key Lemma.</u> (X, Y) QS pair $/B$

$$\begin{tikzcd} \underset{Y_{s,red}}{W} \arrow[r, hook, "\kappa"] \arrow[d, "i_Y"'] \arrow[dr, phantom, "\square"] & Z = X_{s,red} \arrow[d, "i_X"] \arrow[dr, phantom, "\square"] & V \arrow[l, hook'] \arrow[d, "i_U"] \\ Y \arrow[r, hook, "i"] & X & U \arrow[l, hook'] \end{tikzcd}$$

(1) $W \subseteq Z$ effective Cartier divisor

$Cl(W)$ induces $\Lambda_W \xrightarrow{\sim} R\kappa^! \Lambda_Z(1)[2]$



(2) Assume X/B projective

Then $U = X \setminus Y$, $V = U_{s,red}$

$$H^q(U, \Lambda_U(j)) \xrightarrow[\sim]{i_U^*} H^q(V, \Lambda_V(j))$$

Proof (1) $Z = \sum_{i \in I} Z_i$, Z_i irreducible components

$$Z^{(q)} = \coprod_{\substack{J \subseteq I \\ |J| = q}} \bigcap_{j \in J} Z_j$$

$$\begin{array}{c} Z^{(q)} \\ \big\downarrow{\scriptstyle z_q} \\ Z \end{array}$$

exact sequence (dual Koszul complex)

$$0 \to \Lambda_Z \to z_{1*}\Lambda_{Z^{(1)}} \to z_{2*}\Lambda_{Z^{(2)}} \to \cdots$$

$$\cdots \to z_{q*}\Lambda_{Z^{(q)}}$$

$$\Rightarrow \quad E_1^{pq} = R^q k^! z_{p*}\Lambda_{Z^{(p)}} \Longrightarrow R^{p+q-1} k^! \Lambda_Z$$

$$\| \qquad\qquad\qquad$$

$$w_{p*} R^q k^{(p)!} \Lambda_{Z^{(p)}}.$$

Define $W^{(p)}$ by the cartesian square

$$\begin{array}{ccc} W^{(p)} & \xrightarrow{k^{(p)}} & Z^{(p)} \\ \big\downarrow{\scriptstyle w_p} & & \big\downarrow{\scriptstyle z_p} \\ W & \xrightarrow{k} & Z \end{array}$$

⚠ 文中$W^{(p)}$的定义不太精确

Now $W^{(p)}$, $Z^{(p)}$ are regular,

(by QS-pair condition)

$\Rightarrow$ by purity

$$E_1^{pq} = \begin{cases} w_{p*}\Lambda_{W^{(p)}}(-1) & \text{if } q = 1,\ p \geq 1 \quad \text{(by absolute purity)} \\ 0 & \text{otherwise} \end{cases}$$

The nonvanishing row is

$$0 \to \Lambda_W(-1) \to W_{1*}\Lambda_{W^{(1)}}(-1) \to W_{2*}\Lambda_{W^{(2)}}(-1) \to \dots$$

is exact!

$\boxed{P. 10}$

This shows that $R^q k^! \Lambda_Z = \begin{cases} \Lambda_W(-1) & q=2 \\ 0 & q\neq 2 \end{cases}$

(2) $\boxed{\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "i_W"] & Z \arrow[d, "i_X"] & V \arrow[l, "v"'] \arrow[d, "i_U"] \\ Y \arrow[r, "i"] & X & U \arrow[l, "u"'] \end{tikzcd}}$

$$\begin{tikzcd} i_X^* i_* R i^! \Lambda_Y \arrow[r] \arrow[d, "\simeq"] & i_X^* \Lambda_X \arrow[r] \arrow[d, equal] & i_X^* Ru_* \Lambda_U \arrow[r, "+1"] \arrow[d, "\alpha"] & {} \\ k_* R k^! \Lambda_Z \arrow[r] & \Lambda_Z \arrow[r] & Rv_* \Lambda_V \arrow[r, "+1"] & {} \end{tikzcd}$$

$i_X^* i_* R i^! \Lambda_Y = k_* i_Y^* R i^! \Lambda_Y$ // purity $k_* i_Y^* \Lambda_Y(-1)[-2] = k_* R k^! \Lambda_Z$

$k_* R k^! \Lambda_Z = k_* \Lambda_W(-1)[-2]$ (1)

$$\Rightarrow \quad i_X^* Ru_* \Lambda_U \xrightarrow[\simeq]{\alpha} Rv_* \Lambda_V$$

$$\Rightarrow \quad H^q(Z, i_X^* Ru_* \Lambda_U(j)) \xrightarrow{\sim} H^q(Z, Rv_* \Lambda_V(j))$$

// proper base change $\| \simeq$

$H^q(X, Ru_* \Lambda_U(j)) = H^q(U, \Lambda_U(j)) \xrightarrow{\beta} H^q(V, \Lambda_V(j)) \Rightarrow \beta$ is an iso.

Proof of Thm.

$\boxed{P.\ 11}$

$$H_q(U,\Lambda) \overset{\text{purity}}{\cong} H^{2d+2-q}(U, \Lambda_U(d))$$

$$\overset{\text{Lemma}}{=\!=} H^{2d+2-q}(V, \Lambda_V(d)) \quad \dim V = d$$

affine Lefschetz for V/F

$$= 0 \quad \begin{cases} \text{if } F = F^{sep} \ \& \ q \le d+1 & \underline{\text{Case (1)}} \\ \text{if } \cancel{F = \mathbb{F}_q} \ \& \ q \le d \\ \quad cd(F) = 1 & \underline{\text{Case (2)}} \end{cases}$$

~~For F finite~~ For $cd(F) = 1$ use

$$0 \to H^{r-1}(V_{\bar F}, \Lambda_{\bar V}(d))_{G_F} \to H^r(V, \Lambda_V(d)) \to H^r(\bar V, \Lambda_{\bar V}(d))^{G_F} \to 0$$

$V_{\bar F} = \bar V$

$(3) \Leftrightarrow \ H^{d+1}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) = 0.$

Use $\quad 0 \to \mathbb{Z}_\ell \to \mathbb{Q}_\ell \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0$

to get

$$H^{d+1}(V, \mathbb{Q}_\ell(d)) \longrightarrow H^{d+1}(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \longrightarrow \underbrace{H^{d+2}(V, \mathbb{Z}_\ell(d))}_{\substack{\parallel \\ 0}}.$$

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Claim $H^{d+1}(V, \mathbb{Q}_\ell(d)) = 0.$

Proof of Claim

$$0 \to H^d(\bar{V}, \mathbb{Q}_\ell(d))_{G_F} \to H^{d+1}(V, \mathbb{Q}_\ell(d)) \to \underbrace{H^{d+1}(\bar{V}, \mathbb{Q}_\ell(d))^{G_F}}_{\substack{\parallel \\ 0}} \to 0$$

So it's sufficient to prove

$$H^d(\bar{V}, \mathbb{Q}_\ell(d))_{G_F} = 0.$$

Proof of ~~Lemma~~ part (1) uses the distinguished $\triangle$:

$$i_* \Lambda_W(-1)[-2] \longrightarrow \Lambda_Z \longrightarrow Rv_* \Lambda_V \xrightarrow{+1}$$

G_F-equivariant.

$$\Rightarrow H^d(\bar{Z}, \mathbb{Q}_\ell(d)) \longrightarrow H^d(\bar{V}, \mathbb{Q}_\ell(d)) \xrightarrow{\delta} H^{d-1}(\bar{W}, \mathbb{Q}_\ell(d-1))$$

Reduces to prove

$$H^d(\bar{Z}, \mathbb{Q}_\ell(d))_{G_F} = 0 = H^{d-1}(\bar{W}, \mathbb{Q}_\ell(d-1))_{G_F}.$$

Thm (Deligne, Weil II) T separated finite type over $F = \mathbb{F}_q$.

Fr = geometric Frobenius, $a \mapsto a^{\frac{1}{q}}$

$\circlearrowleft$

$H_c^i(\overline{T}, \mathbb{Q}_\ell)$

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Then $\forall$ eigenvalue α of this action has weights $\leq i$.

i.e., $\forall \iota: \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$, $|\alpha| \leq q^{i/2}$

$\Rightarrow$ weight of $H^d(\overline{Z}, \mathbb{Q}_\ell(d))$ $\leq -d < 0$ (since $d \geq 2$)

weight of $H^{d-1}(\overline{W}, \mathbb{Q}_\ell(d-1))$ $\leq -(d-1) < 0$

$\Rightarrow$ 1 is not an eigenvalue

$\Rightarrow$ has no coinvariants. $\square$