BERTINI'S THEOREMS OVER DISCRETE VALUATION RINGS

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Abstract. — This notes is prepared for a talk in the joint seminar "Chow group of zero-cycles on varieties over local fields" in Capital Normal University and Southern University of Science and Technology during 2017-2018. In this note, we give a self-contained proof of a Bertini's theorem over discrete valuation rings. For a semi-stable regular projective flat scheme over a discrete valuation ring, we can always find a projective embedding and a hyperplane with respect to this embedding, such that the intersection of this hyperplane and the original scheme also keeps these properties.

Résumé. — Cette notes est preparée pour un exposé dans le séminaire joint "le groupe de Chow de zéro-cycles sur les variétés sur corps locaux" à l'Université normal de la capitale et l'Université de technologie du Sud pendant 2017-2018. Dans cette note, on donne une démonstration autonome d'un théorème de Bertini sur anneaux de valuations discrètes. Pour un schéma semi-stable régulier projectif plat sur un anneau de valuation discrète, on peut toujour trouver un prolongement projectif et un hyperplan par rapport à cette immersion, tels que l'intersection de cet hyperplan et le schéma original aurait ces propriétés aussi.

Contents

1. Introduction

1.1. Brief History. — Let $X \hookrightarrow \mathbb{P}^n_k$ be a quasi projective scheme, where k is the base scheme, which is often a ring or a field. A theorem is said to be a Bertini's theorem means that if X has some properties, then there exists a hyperplane H in $\mathbb{P}^n_k$, such that the intersection product $H \cdot X := H \times_{\mathbb{P}^n_K} X \hookrightarrow \mathbb{P}^n_k$ also has these properties. In 1880s, Eugenio Bertini proved that for the case that k is algebraically closed, if X is smooth over $\operatorname{Spec} k$, then there exists a hyperplane H such that $H \cdot X$ is also smooth over $\operatorname{Spec} k$. This result has no restriction on the characteristic of the underlying field, while the extensions require characteristic 0. In [**9**, Théorème 8.18], J.-P. Jouanolou proved that if the cardinal of k is infinity, then the Bertini's theorems are verified for the cases that "some properties" are of irreducible, smooth, geometrically reduced and geometrically irreducible.

In general, the Bertini's theorems are not verified any longer if k is a finite field, see [**12**, Theorem 3.1] for such a counter-example. But if we loosen the requirement of H, for example, if we allow that H can be a hypersurface section, then the Bertini's theorems can still be right, see [**12**, Theorem 1.1] and [**13**, Theorem 1.1] for the case of smoothness, and [**3**, Theorem 1.1] for the case of irreducibility.

1.2. Main target. — In order to prove [**14**, Theorem 9.7], we need a version of Bertini's theorems over discrete valuation rings, which is proved in [**14**, Theorem 4.2]. The aim of this notes is to give a self-contained proof of [**14**, Theorem 4.2] (Theorem 4.1 in this notes), which is a generalization of [**8**, Theorem 1.2].

1.3. Structure of this notes. — This notes is divided into three parts: in §2, we will prove the Bertini's theorems of smoothness over the fields which satisfy certain conditions (Theorem 2.1), where we will follow the approach of [**10**] for the case of infinite fields and the approach of [**13**] for the case of finite fields. In §3, we will prove that if we have the Bertini's theorem of a particular version over the residue fields, then we have the Bertini's theorem of the necessary version over discrete valuation rings, where we will follow the approach of [**8**, Lemma 1.3] (Theorem 3.2). In §4, we will prove [**14**, Theorem 4.2] (Theorem 4.1), where actually we will prove the assumption in Theorem 3.2 of §3 is verified.

2. Bertini's theorems over a field

In this section, we give a brief introduction to the proof of the Bertini's theorems of the following version of smoothness. Due to the limit of the space, we do not plan to give out every details of the proof.

Theorem 2.1. — *Let k be a field, and let $X \hookrightarrow \mathbb{P}^n_k$ be a smooth projective scheme of dimension d. Let Z be a closed subscheme of $\mathbb{P}^n_k$. Suppose that V is a smooth closed subscheme of X whose dimension is l. (If V is empty, take $l = -1$.) If $d > 2l$, then there exists a hypersurface H of $\mathbb{P}^n_k$ containing V, such that $X \cdot H$ is smooth of dimension $d-1$.*

In order to prove Theorem 2.1, we divide it into two cases: the field k is infinite and k is finite. The method of proving these two cases are very different.

2.1. The case of infinite field. — If the base field k mentioned in Theorem 2.1 is infinite, the result is a direct corollary of [**10**, Theorem 7].

2.2. The case of finite field. — If the base field k is finite, we follow the approach of B. Poonen in [**12**] and [**13**]. In fact, the main motivation of [**13**] is to prove Theorem 4.2 in [**14**], see [**13**, Introduction].

First, we recall the basic setting original from [**12**], which is also applied in [**13**] and [**3**] for finite field case, and in [**2**] for number field case.

Let $\mathbb{F}_q$ be the finite field with q elements, and $S = \mathbb{F}_q[T_0, \ldots, T_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n_{\mathbb{F}_q}$. Let $S_\delta \subseteq S$ be the $\mathbb{F}_q$-subspace of homogeneous polynomials of degree δ. For each $f \in S_\delta$, let $V(f)$ be the hypersurface $\mathrm{Proj}(S/(f)) \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$. For the rest of this subsection, we fix a closed subscheme $Z \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$. For $\delta \in \mathbb{N}^+$, let I_δ be the $\mathbb{F}_q$-subspace of $f \in S_\delta$ that vanish on Z. Let $I = \bigcup\limits_{\delta \geqslant 0} I_\delta$. Let $\mathcal{P}$ be a subset of I, we define the *density* of $\mathcal{P}$ relative to I as

$$\mu_Z(\mathcal{P}) = \lim_{\delta \to \infty} \frac{\#(\mathcal{P} \cap I_\delta)}{\# I_\delta}$$

if the limit exists. For a scheme X of finite type over $\mathbb{F}_q$, we define the zeta function [**15**]

$$(1) \qquad \zeta_X(s) = Z_X(q^{-s}) = \prod_{P \in X(\overline{\mathbb{F}}_q)} \left(1 - q^{-s\deg(P)}\right)^{-1} = \exp\left(\sum_{r=1}^{\infty} \frac{\# X(\mathbb{F}_{q^r})}{r} q^{-rs}\right),$$

which is converges when $\mathfrak{Re}(s) > \dim(X)$.

With the above notation, we will prove the following result.

Theorem 2.2 **(Theorem 1.1, [13]).** — *Let $X \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$ be a smooth quasi-projective subscheme over the finite field $\mathbb{F}_q$, whose dimension is $d > 0$, and $Z \hookrightarrow \mathbb{P}^n_{\mathbb{F}_q}$ be a closed subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$. Suppose that the scheme-theoretic intersection $V = Z \cap X$ is smooth of dimension l. (If V is empty, take $l = -1$.). Define*

$$\mathcal{P} = \{f \in I \mid V(f) \cdot X \text{ is smooth of dimension } d-1\}.$$

Then we have:

1. *If $d > 2l$, then*
$$\mu_Z(\mathcal{P}) = \frac{1}{\zeta_V(d-l)\zeta_{X \smallsetminus V}(d+1)}.$$
2. *If $d \leqslant 2l$, then $\mu_Z(\mathcal{P}) = 0$.*

A direct corollary of Theorem 2.2 gives the finite field case of Theorem 2.1.

In order to prove Theorem 2.2, we divide the closed points of X into three parts by their degrees. The method of control the singularities of low degree ([**13**, §2]) and medium degree ([**13**, §3]) is more arithmetic, while that of control high degree ([**13**, §4]) is more geometric.

2.2.1. *Singular points of low degree.* — Let $\mathcal{I}_Z \subseteq \mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of Z in $\mathbb{P}^n$, so we have $I_\delta = H^0(\mathbb{P}^n, \mathcal{I}_Z(\delta))$. Tensoring the surjection

$$\begin{array}{rcl} \mathcal{O}^{\oplus(n+1)} & \to & \mathcal{O} \\ (f_0, \ldots, f_n) & \mapsto & x_0 f_0 + \cdots + x_n f_n \end{array}$$

with $\mathcal{I}_Z$, twisting by $\mathcal{O}(\delta)$ and taking global sections shows that $S_1 I_\delta = I_{\delta+1}$ for $\delta \gg 1$. Fix an integer

$$c \in \mathbb{N} \tag{2}$$

such that $S_1 I_\delta = I_{\delta+1}$ for all $\delta \geqslant c$.

With the above notation, we introduce the following lemmas.

Lemma 2.3 **(Lemma 2.1, [13]).** — *Let Y be a finite scheme of $\mathbb{P}^n$. Let*

$$\phi_\delta : I_\delta = H^0(\mathbb{P}^n, \mathcal{I}_Z(\delta)) \to H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(\delta))$$

be the map induced by the map of sheaves

$$\mathcal{I}_Z \to \mathcal{I}_Z \cdot \mathcal{O}_Y$$

on $\mathbb{P}^n$. Then ϕ_δ is surjective for $\delta \geqslant c + \dim(H^0(Y, \mathcal{O}_Y))$, where the constant c is defined at (2).

Lemma 2.4 **(Lemma 2.2, [13]).** — *Suppose $\mathfrak{m} \subseteq \mathcal{O}_X$ is the ideal sheaf of a closed point $P \in X$. Let $Y \hookrightarrow X$ be the closed subscheme whose ideal sheaf is $\mathfrak{m}^2 \subseteq \mathcal{O}_X$. Then for any $\delta \in \mathbb{N}$, we have*

$$\#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(\delta)) = \begin{cases} q^{(d-l)\deg(P)}, & \text{if } P \in V; \\ q^{(d+1)\deg(P)}, & \text{if } P \notin V. \end{cases}$$

Proof. — For all $P \in X$, let $\kappa(P)$ be the residue field of P in X. Then we have $H^0(Y, \mathcal{O}_Y)$ is a $\kappa(P)$-vector space of dimension $d+1$ since P is non-singular in X. So $H^0(Y, \mathcal{O}_Y)$ is a $\mathbb{F}_q$-vector space of dimension $(d+1)\cdot\deg(P)$. We get

$$\#H^0(Y, \mathcal{O}_Y) = q^{(d+1)\deg(P)}.$$

On the other hand, since Y is finite, we have $H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y) = H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y(\delta))$ for all $\delta \in \mathbb{N}$. We also have the following exact sequence of sheaves

$$0 \to \mathcal{I}_Z \cdot \mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{O}_{Z \cap Y} \to 0.$$

Combine it with the fact that $\dim(Y) = 0$, we have

$$\begin{aligned} \#H^0(Y, \mathcal{I}_Z \cdot \mathcal{O}_Y) &= \frac{\#H^0(Y, \mathcal{O}_Y)}{\#H^0(Y, \mathcal{O}_{Z\cap Y})} \\ &= \begin{cases} q^{(d+l)\deg(P)}/q^{(l+l)\deg(P)}, & \text{if } P \in V; \\ q^{(d+1)\deg(P)}, & \text{if } P \notin V. \end{cases} \end{aligned}$$

So we obtain the result. □

If U is a scheme of finite type over $\mathbb{F}_q$, let $U_{<r}$ be the set of closed points of U of degree $< r$. Similarly we define the sets $U_{>r}$, $U_{\leqslant r}$ and $U_{\geqslant r}$.

Proposition 2.5 **(Lemma 2.3, [13]).** — *We keep the notation and the hypothesis in Theorem 2.2, and we define*

$$\mathcal{P}_{<r} = \{f \in I \mid V(f) \cdot X \text{ is smooth of dimension } d-1 \text{ at all } P \in X_{<r}\}.$$

Then

$$\mu_Z(\mathcal{P}_{<r}) = \prod_{P\in V_{<r}} \left(1-q^{-(d-l)\deg(P)}\right)\cdot \prod_{P\in (X\smallsetminus V)_{<r}} \left(1-q^{-(d+l)\deg(P)}\right).$$

Proof. — Let $X_{<r}=\{P_1,\dots,P_m\}$. For each $i\in\mathbb{N}\cap[1,m]$, let $\mathfrak{m}_i$ be the ideal sheaf of the point P_i in X, let Y_i be the closed sub scheme of X with the ideal sheaf $\mathfrak{m}_i^2\subseteq\mathcal{O}_X$, and let $Y=\bigcup_{i=1}^{m} Y_i$.

By the Jacobian criterion (cf. [**11**, Theorem 4.2.19]), for every $i\in\mathbb{N}\cap[1,m]$, the scheme $V(f)\cdot X$ is singular at P_i iff f restricts to a section of $\mathcal{O}_{Y_i}(\delta)$ is zero. By Lemma 2.3, $\mu_Z(P_r)$ equals to the fraction of elements in $H^0(Y,\mathcal{I}_Z\cdot\mathcal{O}_Y(\delta))$ whose restriction to a section of $\mathcal{O}_{Y_i}$ is non-zero for each $i\in\mathbb{N}\cap[1,m]$.

Then by Lemma 2.4, we obtain

$$\begin{aligned}\mu_Z(\mathcal{P}_r) &= \prod_{i=1}^{m}\frac{\#H^0(Y_i,\mathcal{Z}\cdot\mathcal{O}_{Y_i})-1}{\#H^0(Y_i,\mathcal{Z}\cdot\mathcal{O}_{Y_i})}\\ &= \prod_{P\in V_{<r}} \left(1-q^{-(d-l)\deg(P)}\right)\cdot \prod_{P\in (X\smallsetminus V)_{<r}} \left(1-q^{-(d+l)\deg(P)}\right).\end{aligned}$$

□

Corollary 2.6 **(Corollary 2.4, [13]).** — *If $d>2l$, then*

$$\lim_{r\to\infty}\mu_Z(\mathcal{P}_r)=\frac{1}{\zeta_V(d-l)\zeta_{X\smallsetminus V}(d+1)},$$

where $\zeta_V(s)$ is defined at (1).

Proof. — For the convergence, we need $d-l>\dim(V)=l$. Then we have the assertion by definition directly by Proposition 2.5. □

Proof of Item 2 of Theorem 2.2. — By definition, we have $\mathcal{P}\subseteq\mathcal{P}_r$. By Proposition 2.5, we have

$$\mu_Z(\mathcal{P}_r)\leqslant\prod_{P\in V_{<r}}\left(1-q^{-(d-l)\deg(P)}\right),$$

which tends to 0 as $r\to\infty$ if $d\leqslant 2l$. Thus $\mu_Z(\mathcal{P})=0$ in this case. □

From now on, we assume $d>2l$.

2.2.2. *Singularities of medium degree.* — We keep all the notation above. In this part, we deal with the closed points whose degrees are in the interval $\left[r,\frac{\delta-c}{d+1}\right]$, where the constants δ and c will be explained below.

Lemma 2.7 **(Lemma 3.1, [13]).** — *Let $P\in X$ be a closed point of degree e and $\delta\in\mathbb{N}^+$, where $e\leqslant\frac{\delta-c}{d+1}$ and the constant c is defined at* (2)*. Then the fraction of $f\in I_\delta$ such that $V(f)\cdot X$ is not smooth of dimension $d-1$ at P equals*

$$\begin{cases} q^{-(d-l)e}, & \textit{if } P\in V;\\ q^{-(d+1)e}, & \textit{if } P\notin V.\end{cases}$$

Proof. — By applying Lemma 2.3 to Lemma 2.4, we obtain the result. □

We define the $\overline{\mu}_Z(\mathcal{P})$ and $\underline{\mu}_Z(\mathcal{P})$ as the densities of the set $\mathcal{P} \subseteq I$ as $\mu_Z(\mathcal{P})$ was defined, but we use $\overline{\lim}$ and $\underline{\lim}$ in place of lim.

Proposition 2.8. — *Let $\mathcal{Q}^{\mathrm{mid}}_{r,\delta}$ be the subset of $f \in I_\delta$ such that there exists $P \in X(\overline{\mathbb{F}_q})$ with $r \leqslant \deg P \leqslant \frac{\delta - c}{d+1}$ such that $V(f) \cdot X$ is not smooth of dimension $d-1$ at P, and*

$$\mathcal{Q}^{\mathrm{mid}}_{r} = \bigcup_{\delta \geqslant 0} \mathcal{Q}^{\mathrm{mid}}_{r,\delta}.$$

Then we have

$$\lim_{r \to \infty} \overline{\mu}_Z(\mathcal{Q}^{\mathrm{mid}}_{r}) = 0.$$

Proof. — By Lemma 2.7, we have

$$\begin{aligned} \frac{\#\left(\mathcal{Q}^{\mathrm{mid}}_{r} \cap I_\delta\right)}{\# I_\delta} &\leqslant \sum_{\substack{P \in Z \\ r \leqslant \deg P \leqslant \frac{\delta - c}{d+1}}} q^{-(d-l)\deg(P)} + \sum_{\substack{P \in X \smallsetminus Z \\ r \leqslant \deg P \leqslant \frac{\delta - c}{d+1}}} q^{-(d+1)\deg(P)} \\ &\leqslant \sum_{P \in Z_{\geqslant r}} q^{-(d-l)\deg(P)} + \sum_{P \in (X \smallsetminus Z)_{\geqslant r}} q^{-(d+1)\deg(P)}. \end{aligned}$$

Using the trivial bound that an d-dimensional variety has at most $O(q^{ed})$ closed points of degree e, we show that each of the two sums converges to a value that is $O(q^{-r})$ as $r \to \infty$, under our assumption of $d \geqslant 2l$. □

2.2.3. *Singularities of high degree.* — Because of the place of this notes, in this part we only state the main result about the control of high degree points. We refer to [**13**, §4] as the proofs. The choice of the bound $\frac{\delta - c}{d+1}$ dues to the technique of the proof of Proposition 2.9 and Proposition 2.10 below.

Proposition 2.9 **(Lemma 4.2, [13]).** — *Let $\mathcal{Q}^{\mathrm{high}}_{X \smallsetminus V,\delta}$ be the subset of $f \in I_\delta$ which satisfies that there exists $P \in (X \smallsetminus V)(\overline{\mathbb{F}_q})$ with $\deg(P) > \frac{\delta - c}{d+1}$ such that $V(f) \cdot X$ is not smooth of dimension $d-1$ at P, and*

$$\mathcal{Q}^{\mathrm{high}}_{X \smallsetminus V} = \bigcup_{\delta \geqslant 0} \mathcal{Q}^{\mathrm{high}}_{X \smallsetminus V,\delta}.$$

Then we have

$$\overline{\mu}_Z(\mathcal{Q}^{\mathrm{high}}_{X \smallsetminus V}) = 0.$$

Proposition 2.10 **(Lemma 4.3, [13]).** — *Let $\mathcal{Q}^{\mathrm{high}}_{V,\delta}$ be the subset of $f \in I_\delta$ which satisfies that there exists $P \in V(\overline{\mathbb{F}_q})$ with $\deg(P) > \frac{\delta - c}{d+1}$ such that $V(f) \cdot X$ is not smooth of dimension $d-1$ at P, and*

$$\mathcal{Q}^{\mathrm{high}}_{V} = \bigcup_{\delta \geqslant 0} \mathcal{Q}^{\mathrm{high}}_{V,\delta}.$$

Then we have

$$\overline{\mu}_Z(\mathcal{Q}^{\mathrm{high}}_{V}) = 0.$$

2.2.4. *Conclusion.* — Finally, we will prove the main result of this section.

Proof of Item 1 of Theorem 2.2. — By definition, we have the relation

$$\mathcal{P} \subseteq \mathcal{P}_r \subseteq \mathcal{P} \cup \mathcal{Q}_r^{\mathrm{mid}} \cup \mathcal{Q}_{X \smallsetminus V}^{\mathrm{high}} \cup \mathcal{Q}_V^{\mathrm{high}}.$$

Then we have

$$|\overline{\mu}_Z(\mathcal{P}) - \mu_Z(\mathcal{P}_r)| \leqslant \overline{\mu}_Z(\mathcal{Q}_r^{\mathrm{mid}}) + \overline{\mu}_Z(\mathcal{Q}_{X \smallsetminus V}^{\mathrm{high}}) + \overline{\mu}_Z(\mathcal{Q}_V^{\mathrm{high}})$$

and

$$|\underline{\mu}_Z(\mathcal{P}) - \mu_Z(\mathcal{P}_r)| \leqslant \overline{\mu}_Z(\mathcal{Q}_r^{\mathrm{mid}}) + \overline{\mu}_Z(\mathcal{Q}_{X \smallsetminus V}^{\mathrm{high}}) + \overline{\mu}_Z(\mathcal{Q}_V^{\mathrm{high}}).$$

By Proposition 2.8, Proposition 2.9 and Proposition 2.10, when $r \to \infty$, we have

$$\overline{\mu}_Z(\mathcal{Q}_r^{\mathrm{mid}}) + \overline{\mu}_Z(\mathcal{Q}_{X \smallsetminus V}^{\mathrm{high}}) + \overline{\mu}_Z(\mathcal{Q}_V^{\mathrm{high}}) = o(1).$$

So we obtain

$$\mu_Z(\mathcal{P}) = \lim_{r \to \infty} (\mathcal{P}_r) = \frac{1}{\zeta_V(d-l)\zeta_{X \smallsetminus V}(d+1)},$$

which is the end of the proof. □

3. From the residue field to the original discrete valuation ring

In this section, we introduce the proof of [**8**, Lemma 1.3]. First, we introduce some notation, which follows that of [**8**] and [**14**].

Let R be a discrete valuation ring, k be the fraction field of R, F be the residue field of R, and π be a uniformizor of R. Then we suppose that $\operatorname{Spec} R = \{\eta, s\}$, where $s = \operatorname{Spec} F$ is the closed point and $\eta = \operatorname{Spec} k$ is the generic point. The definition below follows [**8**, Definition 1.1].

Definition 3.1 **(Quasi-semi-stable).** — Let $X \to \operatorname{Spec} R$ be a quasi-projective scheme. We say that X is *quasi-semi-stable* if the following conditions hold:

1. $X \to \operatorname{Spec} R$ is a regular flat scheme;
2. Zariski locally, X is smooth over a ring of type
$$R[x_1, \ldots, x_r]\left[\frac{1}{u}\right] / (\pi - u x_1^{e_1} \cdots x_r^{e_r}),$$
where $e_1, \ldots, e_r \in \mathbb{N}_{\geqslant 1}$ and $u \in R[x_1, \ldots, x_r] \smallsetminus \{0\}$.

In addition, we say that X is of *semi-stable* if $e_1 = \cdots = e_r = 1$ and $u = 1$ in the above Item 2.

We denote by $\mathcal{QS}$ the category of quasi-semi-stable schemes, and by $\mathcal{S}$ the category of semi-stable schemes. In addition, we denote by $s\mathcal{QS} \subset \mathcal{QS}$ (*resp.* $s\mathcal{S} \subset \mathcal{S}$) the subcategory of *strictly quasi-semi-stable* (*resp. strictly semi-stable*) schemes if for all object X in them, $X_{s,\mathrm{red}}$ has regular irreducible components.

The main result of this section is stated as follows.

***Theorem 3.2* (Lemma 1.3, [8]).** — *Let X be an object of $s\mathcal{QS}$ (resp. $s\mathcal{S}$), and $H \hookrightarrow \mathbb{P}^N_R$ be a hyperplane over R whose special fiber is $H_s \hookrightarrow \mathbb{P}^N_F$ and whose generic fiber is $H_\eta \hookrightarrow \mathbb{P}^N_k$. Let $Y_1, \ldots, Y_M$ be the irreducible components of $X_{s,\mathrm{red}}$, which are smooth varieties intersecting transversally in $\mathbb{P}^N_F$ by definition. Assume that:*

1. *H_s and $Y_{i_1,\ldots,i_p} := Y_{i_1} \cap \cdots \cap Y_{i_p}$ intersect transversally in $\mathbb{P}^N_F$ for any $i_1, \ldots, i_p$.*
2. *H_η and X_η intersect transversally in $\mathbb{P}^N_k$.*

Then X and H intersect transversally in $\mathbb{P}^N_R$ and $X \cdot H := X \times_{\mathbb{P}^N_R} H$ is an object of $s\mathcal{QS}$ (resp. $s\mathcal{S}$) and $(X \cdot H) \cup X_{s,\mathrm{red}}$ is a simple normal crossing divisor on X. If X is proper over R, then Assumption 2 is implied by Assumption 1.

Proof. — The scheme $Y_{i_1,\ldots,i_p}$ is smooth by Item 2 of Definition 3.1, and we have

$$(X \cdot H) \times_X Y_{i_1,\ldots,i_p} = (X \times_{\mathbb{P}^N_R} H) \times_X Y_{i_1,\ldots,i_p} = H \times_{\mathbb{P}^N_R} Y_{i_1,\ldots,i_p} = H_s \times_{\mathbb{P}^N_F} Y_{i_1,\ldots,i_p}.$$

So it suffices to show that $X \cdot H$ is an object of $s\mathcal{QS}$ (*resp.* $s\mathcal{S}$).

By the étale descent (cf. [**1**, IX.3])and the strictly Henselization (cf. [**6**, §18.8] or [**4**, §2.8]), we may assume that the field F is separably closed.

Pick an F-rational point $x \in X_s(F)$ which is contained in exactly r irreducible components of X_s. Then Item 2 of Definition 3.1 holds for a neighborhood of x iff $\widehat{\mathcal{O}}_{X,x}$ is isomorphic to

$$B = R[[x_1, \ldots, x_r, y_1, \ldots, y_m]]/\langle \pi - u x_1^{e_1} \cdots x_r^{e_r} \rangle$$

with a unit $u \in R[[x_1, \ldots, x_r, y_1, \ldots, y_m]]$.

Let $f \in B$ be the image of the local equation for H at x, and let $\mathfrak{n} \subseteq B$ be the maximal ideal of B. If $x \in H$, then $B/\langle f \rangle$ is the completion of the local ring of $X \cdot H$ at x. And in addition, the irreducible components of $(X \cdot H)_{s,\mathrm{red}} = X_{s,\mathrm{red}} \cap H_s$ are the connected components of $Y_i \cap H_s$, where $i = 1, \ldots, M$, so if we have the Lemma 3.4 below, then we prove that for every $x \in (X \cdot H)_s$, x has an open neighborhood in $X \cdot H$ which is an object of $s\mathcal{QS}$ (*resp.* $s\mathcal{S}$).

If X is proper over R, these neighborhoods cover $X \cdot H$ by the valuative criterion of properness (cf. [**7**, Chap. II, Theorem 4.7]). □

***Lemma 3.3* (Claim 1.3.1, [8]).** — *With all the notation and conditions in Theorem 3.2. Assumption 1 in Theorem 3.2 implies that*

(a) : *either f is a unit in B,*
(b) : *or $m \geqslant 1$, $f \in \mathfrak{n}$ and f has non-zero image in $\mathfrak{n}/(\mathfrak{n}^2 + \langle x_1, \ldots, x_r \rangle)$.*

***Lemma 3.4* (Claim 1.3.2, [8]).** — *Assume that Condition (b) in Lemma 3.3 holds. Then*

$$B/\langle f \rangle \cong R[[x_1, \ldots, x_r, y_1, \ldots, y_{m-1}]]/\langle \pi - \overline{u} \cdot x_1^{e_1} \cdots x_r^{e_r} \rangle,$$

where $\overline{u}$ is a unit of $R[[x_1, \ldots, x_r, y_1, \ldots, y_{m-1}]]$.

Proof of Lemma 3.4. — The elements $\{x_i \mod \mathfrak{n}^2\}_{i=1}^r$ and $\{y_j \mod \mathfrak{n}^2\}_{j=1}^m$ form an F-basis of $\mathfrak{n}/\mathfrak{n}^2$. Then we have

$$f = \sum_{i=1}^r a_i x_i + \sum_{j=1}^m a_{j+r} y_j \mod \mathfrak{n}^2,$$

where $a_i, a_{j+r} \in R$ which are determined modulo $\langle \pi \rangle$. If Condition (b) in Lemma 3.3 holds, then $a_{j+r} \in R^\times$ for some j. We may assume that $a_{r+m} = 1$ by possibly renumbering and multiplying f by a unit. Then we have

$$B/\langle f \rangle \cong R[[x_1, \dots, x_r, y_1, \dots, y_{m-1}]]/\langle \pi - \overline{u} \cdot x_1^{e_1} \cdots x_r^{e_r} \rangle,$$

where $\overline{u}$ is a unit of $R[[x_1, \dots, x_r, y_1, \dots, y_{m-1}]]$. $\square$

Proof of Lemma 3.3. — The elements $x_1, \dots, x_r$ are the images of the local equations for $Y_{i_1}, \dots, Y_{i_r}$ for suitable $1 \leqslant i_1 < \cdots < i_r \leqslant M$. Then the image of $Y_{i_1} \cap \cdots \cap Y_{i_r}$ in $\widehat{\mathcal{O}}_{X,x} \cong B$ is given by the ideal $\langle x_1, \dots, x_r \rangle$, i.e., by the quotient

$$B' = B/\langle x_1, \dots, x_r \rangle \cong F[[y_1, \dots, y_m]].$$

The ring B' is of dimension 0 iff $m = 0$, and in this case $Y_{i_1} \cap \cdots \cap Y_{i_r}$ is of dimension 0 as well. Then, by assumption on H, H does not intersect $Y_{i_1} \cap \cdots \cap Y_{i_r}$, and so f is a unit in B' and hence a unit in B.

If $m \geqslant 1$, H intersects $Y_{i_1,\dots,i_r}$ transversally at x iff the image of f in B' lies in $\mathfrak{n}' \smallsetminus (\mathfrak{n}')^2$, where $\mathfrak{n}'$ is the maximal ideal of B'. Now Lemma 3.3 follows the isomorphism

$$\mathfrak{n}'/(\mathfrak{n}')^2 \cong \mathfrak{n}/(\mathfrak{n}^2 + \langle x_1, \dots, x_r \rangle),$$

from which we obtain the result. $\square$

4. Bertini's theorem over the residue field

We keep all the notation in §3, and we would like to remind that R is a discrete valuation ring and $\operatorname{Spec} R = \{\eta, s\}$, where $\eta = \operatorname{Spec} k$ is the generic point and $s = \operatorname{Spec} F$ is the closed point. Let $\mathscr{Q}SP$ be the category of regular projective flat schemes X over $\operatorname{Spec} R$ on which the reduced divisor $X_{s,\mathrm{red}}$ has simple normal crossings. Let E be a free R-module of finite rank, and let

$$\mathbb{P}_R(E) \to \operatorname{Spec} R$$

be the associated projective bundle (cf. [**5**, 4.1.1]). Put $\mathbb{P}_s(E) = \mathbb{P}_R(E) \times_{\operatorname{Spec} R} s$ which is the projective bundle over F associated to $E_s = E \otimes_R F$. Let $G_R(E)$ be the set of invertible (i.e., rank 1) R-submodule $N \subset E$ such that E/N is free. Such a $N \in G_R(E)$ induces a closed immersion

$$H(N) := \mathbb{P}_R(E/N) \hookrightarrow \mathbb{P}_R(E),$$

which we call a hyperplane in $\mathbb{P}_R(E)$ with respect to N.

Next, there is a specialization map

$$\mathrm{sp}_E : G_R(E) \to G_s(E),\ N \mapsto N \otimes_R F,$$

where $G_s(E)$ is the set of 1-dimensional F-subspace of $E \otimes_R F$. This map is surjective. In terms of hyperplanes in projective bundles, sp_E assigns to a hyperplane $H(N) \hookrightarrow \mathbb{P}_R(E)$ over $\operatorname{Spec} R$, a hyperplane $H_s(N) := H(N) \times_{\operatorname{Spec} R} \operatorname{Spec} F \hookrightarrow \mathbb{P}_s(E)$ over s.

Let X be an object of $\mathscr{Q}SP$ over $\operatorname{Spec} R$, and $Y \hookrightarrow X$ be a hypersurface in X which is also an object of $\mathscr{Q}SP$, whose complement $X \smallsetminus Y$ is affine and for which $Y \cup X_{s,\mathrm{red}}$ is a reduced divisor with simple normal crossings on X. Such a pair (X, Y) is called *an ample $\mathscr{Q}SP$-pair.*

The main theorem of this section is stated as follows.

***Theorem 4.1* (Theorem 4.2, [14]).** — *Let X be an object in $\mathscr{2}SP$ with $\dim(X) = d+1 \geqslant 2$. Let $Y_1, \ldots, Y_r$ be the irreducible components of $Y := X_{s,\mathrm{red}}$, which are smooth of dimension d over F. For integers $1 \leqslant a \leqslant r$ and $1 \leqslant i_1 < \cdots < i_a \leqslant r$, we put $Y_{i_1,\ldots,i_a} := Y_{i_1} \cap \cdots \cap Y_{i_a}$. Let $W \hookrightarrow X$ be a closed subscheme satisfying the following three conditions:*

- **(i) :** *W is the disjoint union of integral regular schemes $W_1, \ldots, W_m$.*
- **(ii) :** *For integers $1 \leqslant a \leqslant d$, $1 \leqslant i_1 < \cdots < i_a \leqslant r$ and $1 \leqslant v \leqslant m$, if $W_v \not\subset Y_{i_1,\ldots,i_a}$, then $W_v \times_X Y_{i_1,\ldots,i_a}$ is empty or regular of dimension strictly smaller than $\frac{1}{2}\dim(Y_{i_1,\ldots,i_a})$.*
- **(iii) :** *For integers $1 \leqslant a \leqslant d$, $1 \leqslant i_1 < \cdots < i_a \leqslant r$ and $1 \leqslant v \leqslant m$, if $W_v \subset Y_{i_1,\ldots,i_a}$, then $\dim(W_v) < \frac{1}{2}\dim(Y_{i_1,\ldots,i_a})$.*

Then there exist a free R-module E of finite rank, an embedding $X \hookrightarrow \mathbb{P}^R(E)$ and an invertible R-module $N \in G_R(E)$ satisfying the conditions:

- **(1) :** *$X \cdot H(N) := X \times_{\mathbb{P}_R(E)} H(N)$ lies in $\mathscr{2}SP$ and $(X, X\cdot H(N))$ is a $\mathscr{2}SP$-pair.*
- **(2) :** *$H(N)$ contains W.*

The key point of the proof of Theorem 4.1 is to find the R-module E, the invertible R-module $N \in G_R(E)$, and the embedding morphism $X \hookrightarrow \mathbb{P}_R(E)$. We want to emphasize that Theorem 4.1 is NOT verified for every projective embedding $X \hookrightarrow \mathbb{P}_R(E)$.

In [**8**, Theorem 1.2], U. Jannsen and S. Saito proved Theorem 4.1 for the case of $W = \emptyset$, whose main ingredient is the application of [**8**, Lemma 1] (Theorem 3.2 in this notes).

Proof. — For $v = 1, \ldots, m$, let $W^f \subset W$ (*resp.* $W^{nf} \subset W$) be the (disjoint) union of those W_v's which are not contained (*resp.* contained) in Y. Since X is projective, we take a finite rank R-module E_0 and a closed embedding $i : X \to \mathbb{P}_R(E_0)$. Let $n \in \mathbb{N}^+$, and we put $\mathcal{O}_X(n) = i^*\mathcal{O}_{\mathbb{P}_R(E_0)}(n)$. By the Serre vanishing theorem (cf. [**7**, Chap. III, Theorem 5.2]), we have

$$H^1(X, \mathcal{O}_X(n)) = H^1(X, \mathcal{O}_X(n) \otimes \mathcal{I}_X(W^f)) = 0$$

for a sufficiently large $n > 0$, where $\mathcal{I}_X(W^f)$ is the ideal sheaf for $W^f \hookrightarrow X$.

We fix such a sufficiently large $n \in \mathbb{N}^+$ above, then we have

$$E_n := H^0(X, \mathcal{O}_X(n)) \supset \widetilde{E}_n := H^0(X, \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{I}_X(W^f))$$

are free of finite rank, since they are torsion free.

By definition, the schemes X and W^f are flat over $\operatorname{Spec} R$, then we have

$$E_n \otimes_R F = H^0(X_s, \mathcal{O}_{X_s}(n))$$

and

$$\widetilde{E}_n \otimes_R F = H^0\left(X_s, \mathcal{O}_{X_s} \otimes_{\mathcal{O}_X} \mathcal{I}_{X_s}(W^f_s)\right).$$

The flatness of $W^f \to \operatorname{Spec} R$ also implies $\mathcal{I}_{X_s}(W^f_s) = \mathcal{I}_X(W^f) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s}$. So we have the following short exact sequence of sheaves

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{I}_X(W^f) &\xrightarrow{\times\pi} \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{I}_X(W^f) \\ &\longrightarrow \mathcal{O}_{X_s}(n) \otimes_{\mathcal{O}_{X_s}} \mathcal{I}_{X_s}(W^f_s) \longrightarrow 0, \end{aligned}$$

where π is the uniformizor of R.

By the above exact sequence, the quotient $E_n/\widetilde{E}_n$ is free. So we have that for an invertible R-submodule $N \subset \widetilde{E}_N$ satisfying $\widetilde{E}_n/N$ is free, the quotient E_n/N is again free. Then we obtain the following commutative diagram

$$\begin{tikzcd} G_R(\widetilde{E}_n) \arrow[r, hook, "\iota"] \arrow[d, "\mathrm{sp}_{\widetilde{E}_n}"'] & G_R(E_n) \arrow[d, "\mathrm{sp}_{E_n}"] \\ G_s(\widetilde{E}_n) \arrow[r, hook, "\iota"] & G_s(E_n). \end{tikzcd}$$

The images $\iota(G_R(\widetilde{E}_n))$ and $\iota(G_s(\widetilde{E}_n))$ are identified with the sets

$$\Phi := \{N \in G_R(E_n) | W^f \subset H(N)\}$$

and

$$\Phi_s := \{M \in G_R(E_n) | W^f_s \subset H(M)\}$$

respectively. Here, $H(N)$ denotes the hyperplane $\mathbb{P}_R(E_n/N) \hookrightarrow \mathbb{P}_R(E_n)$, and $H(M)$ denotes the hyperplane $\mathbb{P}_s((E_n \otimes_R F)/M) \hookrightarrow \mathbb{P}_s(E_n \otimes_R F)$. By the relation $W^f \subset H(N)$, we can defined an embedding $X \hookrightarrow \mathbb{P}_R(E_n)$ associated to $\mathcal{O}_X(n)$.

Consider the set

$$\begin{aligned} \Phi_{s,\mathrm{red}} &:= \{M \in G_s(E_n) | W^f_{s,\mathrm{red}} = W^f \cap X_{s,\mathrm{red}} \subset H_s(M)\} \\ &= \{M \in G_s(E_n) | W^f \cap Y_{i_1,\ldots,i_a} \subset H_s(M) \text{ for any } i_1, \ldots, i_a\}. \end{aligned}$$

By definition, we have $\Phi_s \subset \Phi_{s,\mathrm{red}}$.

The lemma below will be useful in the following steps of this proof. We will give the proof of it after we accomplish the proof of this theorem.

Lemma 4.2. — *Let*

$$\mathscr{F} := \ker\left(\mathcal{I}_{X_s}(X_{s,\mathrm{red}}) \to \mathcal{I}_{X_s}(X_{s,\mathrm{red}}) \cdot \mathcal{O}_{W^f_s}\right) \cong \mathcal{I}_{X_s}(X_{s,\mathrm{red}}) \otimes_{\mathcal{O}_{X_s}} \mathcal{I}_{X_s}(W^f_s).$$

We choose a sufficiently large enough such that $H^1(X_s, \mathcal{O}_{X_s}(n) \otimes \mathscr{F}) = 0$. Then for a given $M \in \Phi_{s,\mathrm{red}}$, there exists an $M' \in \Phi_s$ such that

$$H_s(M) \cap X_{s,\mathrm{red}} = H_s(M') \cap X_{s,\mathrm{red}} \subset \mathbb{P}_s(E_n)$$

is verified.

We go back to the proof of Theorem 4.1. Put

$$\begin{aligned} \Phi_{s,\mathrm{red}} \subset \Psi &:= \{M \in G_s(E_n) |\, W^{nf} \cup W^f_{s,\mathrm{red}} \subset H_s(M)\} \\ &= \{M \in G_s(E_n) |\, W \cap Y_{i_1,\ldots,i_a} \subset H_s(M),\ \forall\, i_1, \ldots, i_a \in \mathbb{N}\}, \end{aligned}$$

and let $\Psi_{\mathrm{reg}} \subset \Psi$ be the subset of such M that $H_s(M)$ and $Y_{i_1,\ldots,i_a}$ intersect transversally on $\mathbb{P}_s(E_n)$ for any $i_1, \ldots, i_a$.

By Theorem 2.1 and the assumptions of Theorem 4.1, we may suppose that Ψ_{reg} is nonempty, where we can choose a sufficiently large $n \in \mathbb{N}$ if necessary. By Lemma 4.2 and the fact that $\mathrm{sp}_{\widetilde{E}_n}$ is surjective, there exists an $N \in G_R(E_n)$ satisfying $\mathrm{sp}_{E_n}(N) \in \Psi_{\mathrm{reg}}$. Then by Theorem 3.2, we finish the proof of this result.

□

Now we prove the technical lemma (Lemma 4.2) used in the proof of Theorem 4.1.

Proof of Lemma 4.2. — We keep all the notation in Theorem 4.1. By the canonical morphisms of sheaves, we have the following induced map

$$H^0\left(X_s, \mathcal{O}_{X_s}(n)\right) \xrightarrow{\tau} H^0\left(X_s, \mathcal{O}_{X_s}(n) \otimes \mathcal{O}_{W_s^f}\right) \xrightarrow{\sigma} H^0\left(X_s, \mathcal{O}_{X_s}(n) \otimes \mathcal{O}_{W^f_{s,\mathrm{red}}}\right).$$

By definition, we have

$$\Phi_s = \left(\ker(\tau) \smallsetminus \{0\}\right)/F^{\times},\ \Phi_{s,\mathrm{red}} = \left(\ker(\sigma\tau) \smallsetminus \{0\}\right)/F^{\times}. \tag{3}$$

Meanwhile, we have the following short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{X_s}(n) \otimes_{\mathcal{O}_{X_s}} \mathscr{F} \longrightarrow \mathcal{O}_{X_s}(n) \otimes \mathcal{I}_{X_s}(X_{s,\mathrm{red}}) \longrightarrow \mathcal{O}_{X_s}(n) \otimes \left(\mathcal{I}_{X_s}(X_{s,\mathrm{red}}) \cdot \mathcal{O}_{W_s^f}\right) \longrightarrow 0.$$

By the fact $H^1(X_s, \mathcal{O}_{X_s}(n) \otimes \mathscr{F}) = 0$, the induced homomorphism

$$H^0\left(X_s, \mathcal{O}_{X_s}(n) \otimes \mathcal{I}_{X_s}(X_{s,\mathrm{red}})\right) \to H^0\left(X_s, \mathcal{O}_{X_s}(n) \otimes \left(\mathcal{I}_{X_s}(X_{s,\mathrm{red}}) \cdot \mathcal{O}_{W_s^f}\right)\right) = \ker(\sigma)$$

is surjective. Hence we have

$$\ker(\sigma\tau) = \ker(\tau) + H^0\left(X_s, \mathcal{O}_{X_s}(n) \otimes \mathcal{I}_{X_s}(X_{s,\mathrm{red}})\right).$$

Combine the above equality with (3), we obtain that for each element in $\Phi_{s,\mathrm{red}}$, and it corresponds to an element in Φ_s by the above equality. So we prove Lemma 4.2. □

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