Notes on moving lemmas

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This is my notes preparing for a talk in the joint reading seminar of arithmetical geometry to be held in 2018.01.13-01.14. The material is taken from [SS10, §7] and [GLL13, §2 and 8], with a few explanations added.

1 Notations

1. $R = \mathrm{DVR}$, $k = \mathrm{Frac}(R)$, $F = $ Residue field of R.
2. $B = \mathrm{Spec}(R)$, $s = \mathrm{Spec}(F)$, $\eta = \mathrm{Spec}(k)$.
3. $\mathrm{Sch}_B^{qp} = $ the category of quasi-projective B-schemes and B-morphisms.
4. For $X \in \mathrm{ob}(\mathrm{Sch}_B^{qp})$, $X_s = X \times_B s$, $X_\eta = X \times_B \eta$, $\dim'(X) = $ the Krull dimension of a compactification of X over B.
5. For $q \geq 0$,
$$X_q = \{x \in X : \dim'(\overline{\{x\}}) = q\},$$
where $\overline{\{x\}}$ is the closure of x in X.
6. $\mathscr{C} = $ the full subcategory of Sch_B^{qp} consisting of objects whose structural morphism do not factor through η.

2 A moving lemma

Theorem 2.1. (=[SS10, prop.7.1]) *Let $X \in \mathrm{ob}(\mathscr{C})$, integral and regular, Y a closed subscheme of X, $U = X - Y \neq \varnothing$. Suppose in addtion that R is excellent. Then for $q \geq 0$,*
$$\bigoplus_{x \in X_q \cap U} \mathbb{Z} \twoheadrightarrow \mathrm{CH}_q(X).$$

Proof: Take an arbitrary $y \in Y_q$, we need to show $[y]$ is in the above image. By lemma 2.2, there exists an integral closed subscheme Z of dimension $q+1$ such that $Z \cap U \neq \varnothing$, $y \in Z$ and Z is regular at y. Thus $\dim(Z \cap Y) \leq q$ and $Z_q \cap Y$ is finite. Let $\pi : \widetilde{Z} \to Z$ be the normalization; since R is excellent, so is Z, and thus π is finite. Since X is quasi-projective over B, Z and $\widetilde{Z}$ both are quasi-projective over B. By Lemma 2.3, there exists an open subset V of $\widetilde{Z}$ which contains $\pi^{-1}(Z_q \cap Y)$, and a dense open immersion $V \hookrightarrow U = \mathrm{Spec}(A)$ where A is a ring over B. Let A'

be the semi-localization of A at $\pi^{-1}(Z_q \cap Y)$. Thus A' is a regular semi-local domain of dimension one, and thus is a PID. Since Z is regular at y, there is a unique point $\tilde{y} \in \widetilde{Z}$ over y. Since $\tilde{y}$ is a principal divisor of $\mathrm{Spec}(A')$, we can take an element $f \in A'$ which generates the ideal corresponding of $\tilde{y}$. We regard f as a rational function of $\widetilde{Z}$. Then the divisor of f on $\widetilde{Z}$ takes the following form

$$\mathrm{div}_{\widetilde{Z}}(f) = [\tilde{y}] + \sum_i m_i[x_i]$$

for some points $x_i \in \widetilde{Z}\backslash\{\tilde{y}\} \cup \pi^{-1}(Z_q \cap Y)$. Pushing forward to Z, we obtain

$$\mathrm{div}_Z(f) = \pi_*\mathrm{div}_{\widetilde{Z}}(f) = [y] + \sum_i m'_i[x'_i]$$

and $x_i \in Z\backslash\{y\} \cup (Z_q \cap Y)$, $m'_i \in \mathbb{Z}$. Thus $-\sum_i m'_i[x'_i]$ lies in $\bigoplus_{x \in X_q \cap U} \mathbb{Z}$ and maps to $[y]$. $\square$

Lemma 2.2. *Keep the notations as in 2.1. Let $y \in Y$, $c = \mathrm{codim}_X(y) \geq 1$. Then there exists an integral closed subscheme $Z \subset X$ of codimension $c-1$ such that $Z \cap U \neq \varnothing$, $y \in Z$ and Z is regular at y.*

Proof : We can assume that Y is a divisor on X, and replace X by $\mathrm{Spec}(\mathcal{O}_{X,y})$, Y defined by a nonzero element $\pi \in \mathcal{O}_{X,y}$ and it suffices to show that for a regular local ring $(A, \mathfrak{m})$ of dimension $c \geq 1$ and a nonzero element $\pi \in \mathfrak{m}$, there exists a regular system of parameters $\{a_1, \cdots, a_{c-1}, a_c\}$ of A such that $\pi \notin (a_1, \cdots, a_{c-1})$; in fact we can choose $Z = \overline{\mathrm{Spec}(A/(a_1, \cdots, a_{c-1}))}$ where the closure is taken in the original X. If $c = 1$ we take $Z = \mathrm{Spec}(A)$. Suppose $c \geq 2$, then $\mathfrak{m}/(\mathfrak{m}^2 + (\pi)) \neq 0$. Since A is a UFD, we can write $\pi = ux_1 \cdots x_k$, where u is a unit and $x_1, \cdots, x_k$ are prime elements. Since (x_i) are prime ideals, by prime avoidance (allowing for at most two non-prime idelas),

$$\mathfrak{m} \neq (x_1) \cup \cdots \cup (x_k) \cup \mathfrak{m}^2.$$

So there exists a prime element a_1 such that $a_1 \nmid \pi$ and $a_1 \notin \mathfrak{m}^2$. Now let $A' = A/(a_1)$. Thus A' is a regular local ring of dimension $c-1$ and $\overline{\pi} = \pi \mod (a_1) \neq 0$. By induction hypothesis there exists a regular system $\{b_2, \cdots, b_c\}$ of A' such that $\overline{\pi} \notin (b_2, \cdots, b_{c-1})$. Lifting $\{b_2, \cdots, b_c\}$ to elements $\{a_2, \cdots, a_c\}$ of A, we are done. $\square$

Lemma 2.3. *Let Z be a quasi-projective scheme over B, $z_1, \cdots, z_r \in Z$. Then there exists a dense open $V \subset Z$ and a dense open immersion $V \hookrightarrow U$ such that V contains $z_1, \cdots, z_r$ and U is an affine scheme over B.*

Proof : It is easy to reduce to the case $Z = \mathbb{P}^N_B = \mathrm{Proj}(R[T_0, \cdots, T_N])$, and thus to the case $Z = \mathbb{P}^N_s = \mathrm{Proj}(F[T_0, \cdots, T_N])$. Let $\mathfrak{m}_i \subset A = F[T_0, \cdots, T_N]$ be the homogeneous ideal corrsponding to z_i. It sufficies to show that there exists a nonzero homogeneous polynomial $f \in A$ such that $f \notin \mathfrak{m}_i$ for $1 \leq i \leq r$; in fact we can then lift f to an element $f' \in R[T_0, \cdots, T_N]$ and take $U = \{f' \neq 0\}$. We separate this claim into two cases.

(i) F is infinite. The set of points in $(\mathbb{P}_F^N)^\vee$ that corresponding to hyperplanes in $\mathbb{P}_F^N$ that contains at least one z_i, forms a codimension 1 closed subscheme, thus its complements has at least one F-point, then take f to be the corresponding hyperplane.

(ii) F is finite. By the case (i), there is a finite Galois extension L of F, and a linear form g over L such that the correspoinding hyperplane does not contain z_i for $1 \leq i \leq r$, and so does g^σ for any $\sigma \in \mathrm{Gal}(L/F)$. So the polynomial $f = \prod_{\sigma \in \mathrm{Gal}(L/F)} g^\sigma$ does not contain z_i for $1 \leq i \leq r$.

□

3 Another moving lemma

3.1 Thorup's theory of cycles and rational equivalence

For general Noetherian schemes, Thorup introduced a notion of rational equivalence depending on a grading δ_X on X, which turns the quotient $A(X, \delta_X)$ of $Z(X)$ by this equivalence into a covariant functor for proper morphisms.

1. A grading on a non-empty scheme X is a map $\delta_X : X \to \mathbb{Z}$ such that if $x \in \overline{\{y\}}$, then $\mathrm{ht}(x/y) = \mathrm{codim}_{\overline{\{y\}}} \overline{\{x\}} \leq \delta_X(y) - \delta_X(x)$. It is called *catenary* if the equality holds for any such pair x, y.

2. Canonical grading $\delta_{\mathrm{can}}(x) := -\dim \mathcal{O}_{X,x}$. This grading is catenary if and only if X is catenary and every local ring is equidimensional.

3. Let Y be an integral closed subscheme of X with generic point η, and let $f \in k(Y)^*$. Denote by $[\mathrm{div}(f)]^{(1)}$ the cycle $[\mathrm{div}(f)]$ where we discount all components $\overline{\{x\}}$ such that $\delta_X(x) < \delta_X(\eta) - 1$. One defines the *graded rational equivalence* on $Z(X)$ using the subgroup generated by the cycles $[\mathrm{div}(f)]^{(1)}$, for all closed integral subschemes of X.

4. Important property : If δ_X is catenary, then the graded rational equivalence is the same as the usual ungraded one.

5. Let $f : Y \to X$ be a morphism essentially of finite type. Let δ_X be a grading on X. Then f induces a grading δ_f on Y by

$$\delta_f(y) = \delta_X(f(y)) + \mathrm{trdeg}(k(y)/k(f(y))).$$

   If f is proper, then f induces a homomorphism $f_* : A(Y, \delta_f) \to A(X, \delta_X)$.

6. If X is universally catenary and equidimensional at every point, and $\delta_X = \delta_{\mathrm{can}}$, then δ_f is a catenary grading on Y.

7. As a corollary, if X and Y are schemes of finite type over a Noetherian scheme S which is universally catenary and equidimensional at every point, and $f : Y \to X$ is a proper morphism of S-schemes. Let C and C' be two cycles on Y classically rationally equivalent. Then $f_*(C)$ and $f_*(C')$ are classically rationally equivalent on X.

3.2 Definition of horizontal 1-cycles

Let S be a separated integral Noetherian regular scheme of dimension at most 1. Let η denote its generic point. Endow S with the catenary grading $1 + \delta_{\text{can}}$. Thus if $S = \text{Spec}(R)$ where R is a DVR then $\delta(\eta) = 1$, $\delta(s) = 0$.

Let $f : X \to S$ be a morphism of finite type, and endow X with the grading δ_f, which is catenary. The irreducible 1-cycles on (X, δ_f) are of two types: the integral closed subschemes C of X of dimension 1 such that C meets at least one closed fiber, and the closed points of X contained in X_η (in which case S must be semi-local). We say that a 1-cycle is horizontal if its support is quasi-finite over S, and that it is vertical if its support is not dominant over S.

3.3 Moving lemma

Lemma 3.1. *Let $U = \text{Spec}(A)$ be a Noetherian affine scheme, $C = V(J)$ be a closed subset of U, $\Gamma_1, \cdots, \Gamma_n$ irreducible closed subsets of U, $f_1, \cdots, f_\delta \in J$. Then there exist $g_1, \cdots, g_\delta \in J$ such that $g_i \in f_i + J^2$ for all $i = 1, ..., \delta$, and for $1 \leq j \leq n$, $1 \leq i \leq \delta$, any irreducible component of $\Gamma_j \cap V(g_1, \cdots, g_i)$ not contained in C has codimension i in Γ_j and thus dimension at most $\dim \Gamma_j - i$.*

Proof : Let $\mathfrak{q}_j$ be the prime ideal of A corresponding to Γ_j. WLOG we assume that $\mathfrak{q}_j \not\supset J$ for $1 \leq j \leq n$. We proceed by induction on δ. Suppose $\delta = 1$. Since $f_1 A + J^2 \not\supset \mathfrak{q}_j$ for $1 \leq j \leq n$, by prime avoidance we can find $a_1 \in J^2$ such that $g_1 = f_1 + a_1 \notin \bigcup_{1 \leq j \leq n} \mathfrak{q}_j$. Let Θ be an irreducible component of $\Gamma_j \cap V(g_1)$. Then Θ has codimension 1 in Γ_j and $\dim \Theta \leq \dim \Gamma_j - 1$ (the equality holds if A is catenary). The induction step is similar to that of lemma 2.2. □

Proposition 3.2. *Let S be a semi-local affine Noetherian scheme, $U \to S$ a morphism of finite type with U affine, C an integral closed subscheme of U of codimension $d \geq 1$, and finite over S, and suppose $C \to U$ is a regular immersion. Let F be a closed subset of U such that for all closed points $s \in S$, the irreducible components of $F \cap U_s$ that intersect C all have dimension at most $d - 1$. Then there exists a cycle C' on U rationally equivalent to C and such that:*

(1) The support of C' is finite over S and does not meet $F \cup C$ and for any closed point $s \in S$, $\text{Supp}(C')$ does not contain any irreducible component of U_s.

(2) Suppose that S is universally catenary. Let $Y \to S$ be a separated morphism of finite type and let $h : U \to Y$ be a S-morphism. Then $h_*(C)$ is rationally equivalent to $h_*(C')$ on Y.

Proof : omitted.

Lemma 3.3. *Let A be a Dedekind domain, with field of fractions K, B be an integral domain containing A, and with field of fractions L. Assume that B is finite over A. Then there exists a domain C with $B \supset C \supset L$ such that C is finite over A, and a local complete intersection over A.*

Proof : omitted.

Theorem 3.4. (=[GLL13, theorem 2.3]) *Let S be the spectrum of a semi-local Dedekind domain R. Let $f : X \to S$ be a separated morphism of finite type, with X regular and FA. Let C be a horizontal 1-cycle on X with $\mathrm{Supp}(C)$ finite over S. Let F be a closed subset of X such that for every $s \in S$, any irreducible component of $F \cap X_s$ that meets C is not an irreducible component of X_s. Then there exists a horizontal 1-cycle C' on X with $f|C'$ finite, rationally equivalent to C, and such that $\mathrm{Supp}(C') \cap F = \varnothing$.*

In addition, since S is semi-local, C consists of finitely many points, and since X is FA, there exists an affine open subset V of X which contains C. Then, for any such open subset V, the horizontal 1-cycle C' can be chosen to be contained in V, and to be such that if $g : Y \to S$ is any separated morphism of finite type with an open embedding $V \to Y$ over S, then C and C' are closed and rationally equivalent on Y.

Proof : WLOG we assume C is irreducible and $\mathrm{Supp}(C) \cap F \neq \varnothing$. Since X is FA, we can find an affine open subset V containing C, and thus C is also affine. By lemma 3.3, there exists a finite birational morphism $D \to C$ such that the composition $D \to C \to S$ is a lci. Since C is affine and $D \to C$ is finite, there exists for a a closed immersion $D \to C \times_S \mathbb{A}^N_S \subset V \times_S \mathbb{A}^N_S$, which is a regular immersion since $D \to S$ is lci [EGA IV 19.3.2].

Consider the theorem replacing X by $U = V \times_S \mathbb{A}^N_S$, C by D, and F by $\mathbf{F} = F \times_S \mathbb{A}^N_S$, f by $f' : U \to S$. Let x be a closed point of D, $s = f'(x)$. Then $\dim \mathcal{O}_{D,x} = \dim S$, $\dim \mathcal{O}_{U,x} = \dim S + N$, $\dim \mathcal{O}_{U_s,x} = N$, and each irreducible component of $\mathbf{F} \cap U_s$ passing through x has dimension at most $N - 1$. Then by proposition 3.2 there exists D' satisfying the conclusions.

Return to the case (X, C, F, f). Let $V \to Y$ be any open immersion over S. Consider the associated open immersion $U \to Y \times_S \mathbb{P}^N_S$ and the projection $p : Y \times_S \mathbb{P}^N_S \to Y$. By the conclusions for $(U, D, \mathbf{F}, f')$, D and D' are closed and rationally equivalent in $Y \times_S \mathbb{P}^N_S$. Then $p_*(D) = C$ is rationally equivalent to $C' = p_*(D')$ on Y, and $\mathrm{Supp}(C') \cap F = \varnothing$. □

3.4 Application to the index of an algebraic variety

Theorem 3.5. *Let K be the field of fractions of a discrete valuation ring $\mathcal{O}_K$, with maximal ideal (π) and residue field k. Let $S = Spec(\mathcal{O}_K)$, $\mathcal{X}$ an integral regular scheme and suppose $\mathcal{X}$ is FA, and let $f : \mathcal{X} \to S$ be a proper flat surjective morphism. Since f is flat, $\mathrm{div}(\pi)$ is a Cartier divisor on X, and we denote its associated cycle by $[\mathrm{div}(\pi)] = \pi_i r_i \Gamma_i$. Each Γ_i is an integral variety over k, of multiplicity r_i in X_k. Let*

X/K denote the generic fiber of X/S. Then $\gcd\{r_i\delta(\Gamma^{\mathrm{reg}}/k)\}$ *divides* $\bar{\delta}(X/K)$*, where* $\bar{\delta}(X/K)$ *denotes the greatest common divisor of the integers* $\deg_K(P)$*, with $P \in X$ closed, and whose closure in X is finite over S.*

Proof : Let P be a closed point of X such that its closure C in $\mathcal{X}$ is finite and flat over S. Then

$$\deg_K(P) = \sum_{x \in \mathcal{X}_k \cap C} (\sum_{\Gamma_i \ni x} r_i (\Gamma_i \cdot C)_x \deg_k(x)). \tag{1}$$

It sufficies to consider the case that C intersects some Γ_i^{sing}. Theorem 3.4 shows that there exists an affine open subset V of X which contains the 1-cycle C and a 1-cycle C' rationally equivalent to C in V, and whose support is proper over S and does not intersect the singular locus F of $(X_k)_{\mathrm{red}}$. Then P is rationally equivalent on V_K to $C'|V_K$, whose support is a union of closed points of X. We claim that $\deg_K(P) = \deg_K C'|X$. In fact, since V is affine, there is an open immersion $V \to \mathcal{Y}$ over S where $\mathcal{Y}$ is a projective S-scheme. Thus we are in the situation in the assumption of the theorem 3.4, which shows that C and C' are closed and rationally equivalent in $\mathcal{Y}$. Then $\deg_K(P) = \deg_K C'|\mathcal{Y}_K$. Since $\deg_K C'|\mathcal{Y}_K = \deg_K C'|V_X = \deg_K C'|X$, we have $\deg_K(P) = \deg_K C'|X$. The equation (1) shows that the degree of each point in $\mathrm{Supp}(C'|X)$ is divisible by $\gcd\{r_i\delta(\Gamma^{\deg}/k)\}$, so $\gcd\{r_i\delta(\Gamma^{\deg}/k)\}$ divides $\deg_K P$. $\square$

References

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