Motivic (co)homology and Milnor K-theory

Aim: Introduce the audience the notion of motivic (co)homology attached to smooth algebraic varieties over a field, and its relation with Milnor K-theory.

k = field. perfect

Sm_k = Cat of smooth seperated schemes of finite type over k.

For $\{(p,q) \in \mathbb{Z}^2\}$ Voevodsky (Suslin, and etc) defines

$$H^{p,q}(-,\mathbb{Z}) : Sm_k \longrightarrow Ab$$

Contravariant functors, satisfying

(1) ($\mathbb{A}^1$- Homotopy invariance)

$X \underset{g}{\overset{f}{\rightrightarrows}} Y$ $\quad f \simeq g$. (defined later).

Then $f^* = g^* : H^{p,q}(Y,\mathbb{Z}) \to H^{p,q}(X,\mathbb{Z})$

(2) (MV- sequence)

$X = U \cup V$, open covering. Then LES

$$H^{p+1,q}(X,\mathbb{Z}) \xleftarrow{\delta} H^{p,q}(U\cap V,\mathbb{Z}) \leftarrow H^{p,q}(U,\mathbb{Z}) \oplus H^{p,q}(V,\mathbb{Z}) \leftarrow H^{p,q}(X,\mathbb{Z}) \xleftarrow{\delta}$$

(3) (Gysin-sequence)

$Z \subset X$ smooth closed subscheme, $\mathrm{cod}_X(Z) = c$

$\Rightarrow$ LES.

$$H^{p-1,q}(\underset{\underset{X \setminus Z}{\shortparallel}}{U}, \mathbb{Z}) \xrightarrow{\delta} H^{p-2c,q-c}(Z, \mathbb{Z}) \longrightarrow H^{p,q}(X, \mathbb{Z}) \to H^{p,q}(U, \mathbb{Z})$$

(4) (disjoint union)

$X = \coprod X_\alpha$

$$H^{p,q}(X, \mathbb{Z}) \simeq \bigoplus_\alpha H^{p,q}(X_\alpha, \mathbb{Z})$$

(5) (ring structure)

$$H^{p,q}(X, \mathbb{Z}) \otimes H^{p',q'}(X, \mathbb{Z}) \xrightarrow{\cdot} H^{p+p', q+q'}(X, \mathbb{Z})$$

$\left(\bigoplus_p \left(\bigoplus_q H^{p,q}(X, \mathbb{Z}) \right), \cdot \right)$ is graded commutative.

rk: (1)–(4) holds for arbitrary abelian group A

(5) - - - - - comm. ring R

Connection with Hodge theory:

k: perfect. $\quad H^{p,q}(X, A) \simeq CH^q(X, 2q-p; A)$

A: abelian group.

In particular

$$H^{p,2p}(X,\mathbb{Z}) = CH^p(X).$$

For $k \overset{\sigma}{\hookrightarrow} \mathbb{C}$. one constructs naturally the regulator map
$A = \mathbb{Q}$,

$$H^{p,q}(X,\mathbb{Q}) \xrightarrow{R_\sigma} H^p_{\mathcal{D}}(X_{an}, \mathbb{Q}(q))$$

Outline : (I) Basic construction

(II) $q = 0, 1$

(III) Milnor K-theory.

Part I : Basic construction.

Recall : $\Delta^p_{\mathbb{R}} = \{ \sum_{i=0}^{p} t_i = 1 \} \subset \mathbb{R}^{p+1}$, $\quad p \geq 0$

$$\begin{array}{ccc} \Delta^p_{\mathbb{R}} & \xrightarrow{\partial_i} & \Delta^{p+1}_{\mathbb{R}} \\ \psi & & \psi \\ (t_0 \ldots t_p) & \mapsto & (t_0 \ldots \underset{\underset{i}{\uparrow}}{0} \ldots t_p) \end{array}$$

i-th face map

$S^q = \{ \sum_{i=0}^{q} t_i^2 = 1 \} \subset \mathbb{R}^{q+1}$, $\quad q \geq 0$

Topologically, $S^q = \underbrace{S^1 \wedge \ldots \wedge S^1}_{q} = \dfrac{S^1 \times \ldots \times S^1}{\{1\} \times S^1 \times \ldots \times S^1 \cup \ldots \cup S^1 \times \ldots \times S^1 \times \{1\}}$

X: Smooth manifold / CW-complex

$$C_\cdot(X)_\cdot : \quad \to C_p(X) \xrightarrow{\partial = \sum_{i=0}^{n} (-1)^i \partial_i} C_{p-1}(X) \to$$

$$C_p(X) = \mathrm{Mor}(\Delta^p, X), \qquad C_{p-1}(X) = \mathrm{Mor}(\Delta^{p-1}, X)$$

$$C_\cdot(X) \xrightarrow{\text{Cohomology}} \{H_p(X)\}_p$$

dual $\Big\}$ complex

$$C^\cdot(X) = \mathrm{Hom}(C_\cdot(X), \mathbb{Z}) \xrightarrow{\text{c-hu.}} \{H^p(X)\}_p$$

Choose a base point $x_\cdot \in X$

$$\pi_q(X, x) = [(S^q, 1), (X, x)]$$ ∼ homotopy classes of ~~pointed~~ pointed maps.

Analogue: $\circ$ $\Delta^p = \mathrm{Spec}\left(\dfrac{k[t_0 \ldots t_p]}{\sum t_i = 1}\right) \subset \mathbb{A}^{p+1}_k$

$\partial_i : \Delta^{p-1} \longrightarrow \Delta^{p+1}$ is defined as above.

- $\mathbb{G}_m = \mathrm{Spec}\ k[t, t^{-1}]$ (Analogue of S^1)

- Naive $\mathbb{A}^1$- Homotopy.

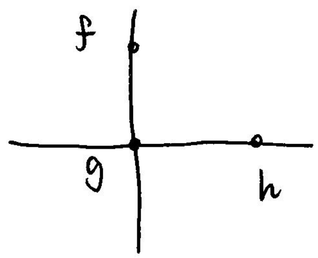
$X \underset{g}{\overset{f}{\rightrightarrows}} Y \qquad f \simeq g$ if $\exists\, F: X \times \mathbb{A}^1 \longrightarrow Y$, s.t.

$$\begin{cases} F|_{X\times\{0\}} = f \\ F|_{X\times\{1\}} = g \end{cases}$$

However, this is not a relation, as the transitivity fails.

e.g. (1) Y: singular.

$X = \mathrm{Spec}\, k$, $\quad Y = \mathrm{Spec}\, \dfrac{k[x,y]}{x\cdot y}$.

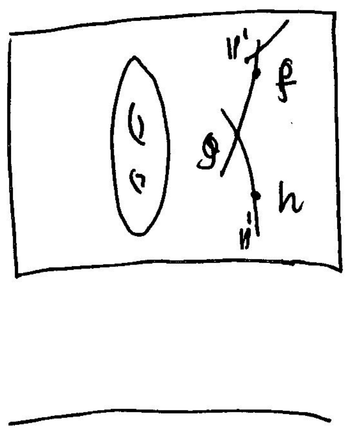


$\longleftarrow$ Spec k

$f \simeq g,\ g \simeq h \not\Rightarrow f \simeq h$

(2) Y smooth

$X = \mathrm{Spec}\, k$



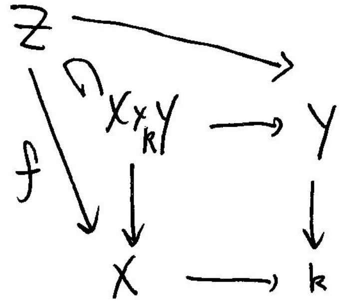
$$\begin{array}{c} Y \\ \downarrow \\ C \end{array} \quad g(C) \geq 1$$

Step 1 : $Sm_k \hookrightarrow Cor_k$

$ob(Cor_k) = ob(Sm_k)$

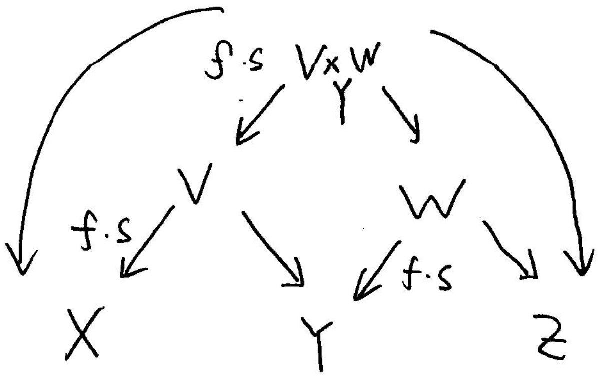
$X, Y \in ob(Sm_k)$

$\cancel{M}Cor_k(X,Y) = \bigoplus_{Z} \mathbb{Z}[Z]$



- $Z \subset X \times_k Y$ irred; closed
- $f : Z \to X$ finite, surjective morphism (as schemes)

define $Cor(X,Y) \times Cor(Y,Z) \xrightarrow{\circ} Cor(X,Z)$ by



Cor_k : additive category

$$Sm_k \hookrightarrow Cor_k, \qquad \cancel{f}\ Mor(X,Y) \hookrightarrow Cor(X,Y)$$
$$f \longmapsto \Gamma_f$$

Notation:

Element in $Cor(X,Y)$ is denoted by

$$f : X \longmapsto Y$$

Definition: $f, g \in Cor(X, Y)$, f is $\mathbb{A}^1$-homotopy to g if $\exists F \in Cor(X \times \mathbb{A}^1, Y)$, s.t

$$\begin{cases} F|_{X \times \{0\}} = f \\ F|_{X \times \{1\}} = g \end{cases}$$

Lemma: $\mathbb{A}^1$-homotopy is an equ. rel. on $Cor(X,Y)$.

i.e

(i) $f \overset{\sim}{\to} f$

(ii) $f \overset{\sim}{\to} g \Rightarrow g \overset{\sim}{\to} f$

(iii) $f \overset{\sim}{\to} g,\ g \overset{\sim}{\to} h \Rightarrow f \overset{\sim}{\to} h$

Moreover, the relation is closed under composition.

i.e $\left.\begin{array}{l} f \overset{\sim}{\to} g \text{ in } Cor(X, Y) \\ f' \overset{\sim}{\to} g' \text{ in } Cor(Y, Z) \end{array}\right\} \Rightarrow f' \circ f \overset{\sim}{\to} g' \circ g$ in $Cor(X, Z)$.

Pf: prove only (iii).

Let $H_1 \in Cor(X \times \mathbb{A}^1, Y)$ be the homotopy from f to g

H_2 - - - - - - - - - g to h.

define: $H = H_1 + H_2 - g\circ\pi$ $\quad \pi : X \times \mathbb{A}^1 \to X$.

Then $H|_{X\times\{0\}} = H_1|_{X\times\{0\}} + H_2|_{X\times\{0\}} - g\cdot\pi|_{X\times\{0\}}$

$$= f + g - g = f$$

$H_2|_{X\times\{0\}} = g + h - g = h$

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Rmk: The argument works for any fixed curves and two fixed pts.

Variant: Let $A = \sum_{i\in I} A_i$, $A_i \subset X$ (P-equivalent of Reitmann). smooth, ~~irred.~~ closed red. subscheme.

Define $Cor((X, A), Y)$

$$= \mathrm{Ker}\left(Cor(X, Y) \to \bigoplus_{i\in I} Cor(A_i, Y)\right)$$

and.

$Cor(Y, (X, A))$

$$= \mathrm{coker}\left(\bigoplus_{i\in I} Cor(Y, A_i) \to Cor(Y, X)\right)$$

Define $\{x_i \in X_i\}_{i\in I}$ ~~R~~ rat. pt, ~~de~~

$Cor(X, \bigwedge_{i\in I}(X_i; x_i))$ ~~$\wedge X_n$~~

$= Cor(X, (\times_{i\in I} X_i, A))$, where

$A = \sum A_i$, $\quad A_i = X_1 \times \cdots \times \{x_i\} \times \cdots \times X_n \times \cdots$

Similarly def $Cor(\bigwedge_{i\in I}(X_i, x_i), X)$..

Step 2. Motivic homology groups.

$X \in Sm_k$,

$$H_{p,q}(X,\mathbb{Z}) = \begin{cases} H_{p+q}(Cor(\Delta^{\cdot}, X\times \mathbb{G}_m^{\wedge q})) & q \geq 0 \\ H_{p+q}(Cor(\mathbb{G}_m^{\wedge -q}\times\Delta^{\cdot}, X)) & q \leq 0 \end{cases}$$

Here $\mathbb{G}_m^{\wedge q} = \underbrace{(\mathbb{G}_m, 1)\wedge\cdots\wedge(\mathbb{G}_m,1)}_{q}$ (alg analogue of S^q).

Rmk: (i) $q=0$. $H_p(X,\mathbb{Z}) = H_p(Cor(\Delta^{\cdot}, X))$ is called $\overset{\text{Suslin}}{=} CH^n(X,p)$, $n=\dim X$.

Suslin's homology groups (cf Suslin-Voevodsky Algebraic singular homology Inv. Math 96)

(ii) For, $q\leq 0$, replace cor by Mor, and X ~~smooth~~ CW-complex.

Then $Mor(\mathbb{G}_m^{\wedge -q}\times\Delta^n, X) = Mor(\Delta^{\cdot}, Mor(S^{-q}, X))$.

$Mor(S^{-q}, X) = \Omega^{-q}(X)$. So it is related to the homology groups of loop spaces.

(iii) One may wonder if ~~thee~~ get a new coh. theory for X a CW-complex.

$\mathcal{C}$ = category of CW pairs

define

$$\overset{n}{C}_n(X) = \mathrm{Cor}(\Delta^n, X)$$

$$= \bigoplus_{Z} \mathbb{Z}[Z], \qquad \begin{tikzcd} & Z \subset \Delta^n \times X \arrow[dl, "f"'] \arrow[dl] \arrow[dr] & \\ \Delta^n & & X \end{tikzcd}$$

Z irred. closed. and f is finite and proper

For $(X, A) \in \mathrm{ob}(\mathcal{C})$, define

$$\overset{n}{C}_n(X, A) = \mathrm{Cor}(\Delta^n, (X, A))$$

$$= \mathrm{Coker}(\mathrm{Cor}(\Delta^n, A) \to \mathrm{Cor}(\Delta^n, X))$$

define $\overset{n}{h}_* : \mathcal{C} \longrightarrow Ab.$ by

$$\tilde{h}_*(X, A) = H_*(\overset{n}{C}_\cdot(X, A)).$$

Note: $C_\cdot(X, A) \hookrightarrow \overset{n}{C}_\cdot(X, A)$ Thus, it induces

natural transformation $\mu : h_* \to \tilde{h}_*$.

rep: $h_* \overset{\mu}{\underset{\sim}{\to}} \overset{n}{h}_*$.

Pf: Step 1: $\tilde{h}_*$ is a cohomology theory.

It remains to show:

(i) Homotopy invariance:

$f \simeq g$ homotopy $(X,A) \to (Y,B)$ $\Rightarrow$ Chain homotopy $f^\# \simeq g^\# : \tilde{C}_\cdot(X,A) \to \tilde{C}_\cdot(Y,B)$

Let $F : X \times \Delta^1 \to Y$ be the homotopy.

Use the prism operator,

$$\begin{array}{ccc} \mathrm{Cor}(\Delta^n, (X,A)) & \xrightarrow{P_F} & \mathrm{Cor}(\Delta^{n+1}, (Y,B)) \\ \in & & \in \\ \sigma & \longmapsto & P_F(\sigma) \end{array}$$

$$P_F(\sigma) = \sum_i (-1)^i F \circ \sigma \circ \pi \big|_{[v_0 \cdots v_i, w_i, \cdots w_n]}$$

where $\pi : \Delta^n \times \Delta^1 \to \Delta^n$

Then $\partial P_F + P_F \partial = g^\# - f^\#$

(ii) Excision Theorem:

(X,A) pair. $\Rightarrow$ $\tilde{h}_*(X,A) \leftrightarrow \tilde{h}_*(X/A, pt)$

The key is to show $X = U \cup V$, U, V open subset of X

define $\tilde{C}{}^{\mathcal{U}}_\cdot(X)$ by the image of a correspondence lying either in $\mathcal{U}$ or V.

show that the natural inclusion

$$\overset{\text{?}}{C}^{\mathcal{U}}_{\cdot}(X) \hookrightarrow \overset{\text{?}}{C}_{\cdot}(X)$$ is a chain homotopy

Here we use the technique of ~~b~~iterated barycentric subdivision.

In the process, the properness of any elt in $\mathrm{Cor}(\Delta^n, X)$ is used. The old proof for the corresponding statement for $C^{\mathcal{U}}_{\cdot}(X) \hookrightarrow C_{\cdot}(X)$ works verbatim.

Step 2. Since $h_*, \overset{?}{h}_*$ are both coho. theory and $\mu : h_* \to \overset{?}{h}_*$ is obviously natural transformation of two coh. theories, it suffices to show

$$\mu : h_0(pt, \phi) \xrightarrow{\sim} \overset{?}{h}_0(pt, \phi).$$

This is clear.

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Prop: $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, $f \simeq g$ ($\mathbb{A}^1$-homotopy). Then

$$f^* = g^* : H_{p,q}(X, \mathbb{Z}) \xrightarrow{\sim} H_{p,q}(Y, \mathbb{Z}). \qquad \forall p, q$$

Pf: use the prism operator, to construct the homotopy operator

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Step 3. $Cor_k \hookrightarrow PST_k$

A presheaf with transfers is a contravariant additive functor

$$F: Cor_k \longrightarrow Ab$$

Note:

$$\begin{tikzcd} Sm_k \arrow[r, hook] \arrow[rr, bend right, "\text{presheaf on } Sm_k"'] & Cor_k \arrow[r, "F"] & Ab \end{tikzcd}$$

For $k=\bar{k}$, to lift a presheaf on Sm_k to Cor_k, it is equivalent to define a family of transfers.

$$\left\{ \check{F}(f): F(Y) \to F(X), \quad f: Y \to X \text{ finite surjective} \right\}$$

Satisfying

$$\begin{tikzcd} Z \arrow[dr, "\check{F}(g)"] \arrow[dd, "\check{F}(f\circ g)"'] & \\ & Y \arrow[dl, "\check{F}(f)"] \\ X & \end{tikzcd}$$

$X \times Y$ ← use a k-rational pt of X.

$\cup$

$$\Gamma \quad \begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[dr, "id"] & \\ X & & \cancel{\operatorname{Spec} k}\, Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(Y) \arrow[r, "id"] \arrow[dr, "F(f)"'] & F(Y) \arrow[d, dotted, "\check{F}(f) \triangleq F(f)."] \\ & F(X) \end{tikzcd}$$

Conversely,

$$\begin{tikzcd} & Z \subset X\times Y \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

f. f. surj.

$$\begin{tikzcd} F(Y) \arrow[r, "F(g)"] \arrow[rr, bend right, "\cancel{F(g.f^t)}"'] & F(Z) \arrow[r, "\check{F}(f)"] & F(X) \end{tikzcd}$$

Fact: PST_k is an abelian category, with enough injectives and projectives.

Examples: (1) Yoneda embedding

$$\begin{array}{ccc} Cor_k & \hookrightarrow & PST_k \\ \Psi & & \Psi \\ X & \longmapsto & \mathbb{Z}_{tr}(X) \end{array}, \text{ defined by } \mathbb{Z}_{tr}(X)(Y) \overset{\circ}{=} Cor(Y, X).$$

Fact: $Cor(X,Y) = \mathrm{Hom}_{PST}(\mathbb{Z}_{tr}(X), \mathbb{Z}_{tr}(Y))$.

(More generally, $\mathrm{Hom}_{PST}(\mathbb{Z}_{tr}(X), F) = F(X)$, $\forall F \in PST$)

This is good: e.g $Z \hookrightarrow X$ closed embedding.

it is impossible in general to make sense of X/Z

But $\mathrm{Coker}(\mathbb{Z}_{tr}(Z) \longrightarrow \mathbb{Z}_{tr}(X))$ is kind of $\mathbb{Z}_{tr}(``X/Z")$

(2) (Lecomte-Wach). char $k = 0$.

$$\begin{array}{rccc} \Omega^p : & Sm_k & \longrightarrow & Ab \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & X & \longmapsto & H^0(X, \Omega^p_{X/k}) \end{array}$$

Can be lifted a presheaf with transfer.

For $p=0$, this is classical, ie given by the trace map.

(3) $\mathcal{O}^*$

(4) CH^i :

$$\begin{tikzcd} & \cancel{W} \subset X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

finite correspondence

$$CH^i(Y) \longrightarrow CH^i(X)$$

$$\alpha \longmapsto p_*(\cancel{W} \cdot q^*\alpha)$$

Tricky pt: no matter ~~W~~ is proper over k,

the support $\cancel{W} \cdot q^*\alpha$ is finite over X.

(5) Constant presheaf

$\mathbb{Z} = \mathbb{Z}_{tr}(\mathrm{Spec}\, k)$.

X smooth.

$\mathbb{Z}(X) = \mathrm{Cor}(X, k) = \bigoplus_i \mathbb{Z}[X_i]$ $\quad X = \coprod X_i$ irreducible = connected compont.

$$\begin{tikzcd} \cancel{Z} \subset X\times Y \arrow[d, "f"'] \arrow[d, "p_X"] \arrow[dr, "p_Y"] & \\ X & Y \end{tikzcd}$$

with X, Y connected, and Z irred.

then
$$\begin{tikzcd} \mathbb{Z}(Y) \arrow[r, "\cdot n"] \arrow[d, equal] & \mathbb{Z}(X) \arrow[d, equal] \\ \mathbb{Z} & \mathbb{Z} \end{tikzcd}, \quad n = \deg(f).$$

Pf: $\mathrm{Cor}(Y, \operatorname{Spec} k) \xrightarrow{\circ Z} \mathrm{Cor}(X, \operatorname{Spec} k)$

Take $Y \xrightarrow{\alpha} \operatorname{Spec} k$ Then

$$Z \circ \alpha = (\mathrm{id}\times\alpha)_*([Z])$$

$$\begin{tikzcd} X\times Y \arrow[r, "1\times\alpha"] & X\times \operatorname{Spec} k \arrow[d, equal] \\ Z \arrow[u, phantom, "\cup"] \arrow[r, "f"] & X \end{tikzcd} \qquad (p_X : X\times Y \to X)$$

$$= \deg(f) \cdot [X] \quad \#$$

Warning: there is another "Constant presheaf"

$\mathbb{Z}(\phi)$ defined by, $\forall X \in Sm_k$
(may not be connected)

$$\mathbb{Z}(\phi)(X) = \mathbb{Z}.$$

and $\forall X \xrightarrow{f} Y$,
$$\begin{tikzcd} \mathbb{Z}(\phi)(Y) \arrow[r] \arrow[d, equal] & \mathbb{Z}(\phi)(X) \arrow[d, equal] \\ \mathbb{Z} \arrow[r, "\mathrm{id}"] & \mathbb{Z} \end{tikzcd}$$

Step 4. Simplicial method.

$C(PST_k)$ = category of complex of objects in PST_k.

We have an exact functor

$$\begin{array}{cccc} C_* : & PST_k & \longrightarrow & C(PST_k) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & F & \longmapsto & C_*(F) \end{array}$$

C_*F:

$$\begin{array}{ccccc} \to C_2(F) & \xrightarrow{\partial} & C_1(F) & \xrightarrow{\partial} & C_0(F) \\ \| & & \| & & \| \\ F^{\Delta^2} & & F^{\Delta^1} & & F^{\Delta^0} = F \end{array}$$

For $X \in Sm_k$, $F \in PST_k$

$F^X \in PST_k$ defined by $U \longmapsto F(U \times X)$

$\partial : F^{\Delta^n} \to F^{\Delta^{n-1}}$ defined by

$$\begin{array}{ccc} F^{\Delta^n}(U) & \to & F^{\Delta^{n-1}}(U) \\ \| & & \| \\ F(U \times \Delta^n) & \xrightarrow{\Sigma(-1)^i \partial_i} & F(U \times \Delta^{n-1}) \end{array}$$

Note. Yoneda ~~embedding~~ $\cancel{\Rightarrow \forall X \in Sm_k}$, $C_*(X) = C_*\mathbb{Z}_{tr}(X)(\mathrm{Spec}\,k)$

$$\cancel{C_*(\mathbb{Z}_{tr}(X)) = \mathbb{Z}_{tr} C_*(X)}$$

i.e.

$C_* \mathbb{Z}_{tr}(X)(\mathbb{G}_m)$:

$$\xrightarrow{\partial} Cor(\Delta^1 \times \mathbb{G}_m, X) \xrightarrow{\partial} Cor(\mathbb{G}_m, X) \to 0$$

Suitably defined, $q \leq 0$

$C_* \mathbb{Z}_{tr}(X)(\mathbb{G}_m^{\wedge q})$:

$$\xrightarrow{\partial} Cor(\Delta^1 \times \mathbb{G}_m^{-q}, X) \xrightarrow{\partial} Cor(\mathbb{G}_m^{-q}, X) \to 0$$

This is the complex defined for $H_{p,q}(X, \mathbb{Z})$, $q \leq 0$.

Lemma: $\mathbb{Z}_{tr}(X)$ is a sheaf in Zar top. Consequently, $C_* \mathbb{Z}_{tr}(X)$ is a chain complex of sheaves

Pf: $U \in Sm_k$, $U_i \overset{\text{open}}{\subset} U$, $U = U_1 \cup U_2$

$$0 \to Cor(U, X) \xrightarrow{diag} Cor(U_1, X) \oplus Cor(U_2, X) \xrightarrow{(+,-)} Cor(U_1 \cap U_2, X).$$

(i) injective.

In fact, $Cor(U, X) \longrightarrow Cor(U_i, X)$ is injective

$$f \longmapsto f|_{U_i}$$

- (ii) $Z_1 = \sum_{i \in I} m_i Z_{1i} \subset U_1 \times Y$

$$Z_2 = \sum_{j \in J} n_j Z_{2j} \subset U_2 \times Y$$

$$Z_1|_{U_1 \cap U_2} = Z_2|_{U_1 \cap U_2} \quad \in Cor(U_1 \cap U_2, X).$$

$\Rightarrow$ ~~pair~~ $I = J$, and arrange the index i with j

$$m_i = n_j$$

We may assume Z_1, Z_2 both irreducible.

Thus, $Z_1|_{U_1 \cap U_2} = Z_2|_{U_1 \cap U_2}$

$\Rightarrow$ $Z_1 \cup Z_2 \subset U \times X$ well-defined, closed subset.

The exactness at middle is shown.

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Step 5. Motivic cohomology

$\forall q \geq 0$. $\mathbb{Z}(q) \overset{\Delta}{=} C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q]$

Moreover we regard this as a cochain complex;

$$\mathbb{Z}(q)^i = C_{q-i} \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}).$$

- Concretely, $\forall\, U \in Sm_k$.

$\mathbb{Z}(q)(U)$ is a complex of abelian groups :

$$\begin{array}{ccccccc}
\cdots & \to & Cor(U\times\Delta^1, \mathbb{G}_m^{\wedge q}) & \xrightarrow{\partial} & Cor(U, \mathbb{G}_m^{\wedge q}) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & q-1 & & q & & q+1
\end{array}$$

Now, fix $X \in Sm_k$. Then $X_{Zar} \subset Sm_k \subset Cor_k \subset PST_k$

Denote $\mathbb{Z}(q)_X$ for the restriction of $\mathbb{Z}(q)$ $\left(\in C(ST_k) \subset C(PST_k)\right)$

to X_{Zar}.

$\mathbb{Z}(q)_X$: $\underbrace{\text{Zariski sheaves}}_{\text{Cochain complex of}}$ (with transfers) on X.

Def.: $\forall\, (p,q) \in \mathbb{Z}^2$, define

$$H^{p,q}(X,\mathbb{Z}) = \mathbb{H}^p(X_{Zar}, \mathbb{Z}(q)_X)$$

By ~~defin.~~ convention, for $q<0$, $\underbrace{H^{p,q}(X,\mathbb{Z})}_{\mathbb{Z}(q)=0 \text{ and then}} = 0$.

<u>Functoriality:</u> $f : X \to Y$

$$H^{p,q}(Y) \xrightarrow{f^*} H^{p,q}(X) \quad \text{is seen as follows.}$$

$$\forall\ U \overset{\text{open}}{\subset} Y,\quad f^{-1}(U) \overset{\text{open}}{\subset} X$$

$$\mathrm{Cor}(U, \cancel{\mathbb{Z}(q)_X}\ \mathbb{G}_m^{\wedge q}) \xrightarrow{f} \mathrm{Cor}(f^{-1}(U), \mathbb{G}_m^{\wedge q})$$

$$\rightsquigarrow \quad \mathbb{Z}(q)_Y \longrightarrow f_* \mathbb{Z}(q)_X.$$

$$\begin{tikzcd}
\rightsquigarrow\ \mathbb{H}^p(Y, \mathbb{Z}(q)_Y) \arrow[r] \arrow[ddr, bend right, "f^*"'] & \mathbb{H}^p(Y, f_* \mathbb{Z}(q)_X) \arrow[d] \\
 & \mathbb{H}^p(Y, Rf_* \mathbb{Z}(q)_X) \arrow[d, equal] \\
 & \mathbb{H}^p(X, \mathbb{Z}(q)_X).
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[dl, "\beta"] \\
\mathrm{Spec}\, k &
\end{tikzcd}$$

<u>Ring structure:</u>

(i) $X, Y \in Sm(k)$, $\exists$ ~~natrl~~ natural transformation

$$\mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(Y) \xrightarrow{\Delta} \mathbb{Z}_{tr}(X \times_k Y)$$

~~[illegible]~~.

Indeed, $\forall\, U \in Sm_k$, we define

$$\begin{tikzcd}
\mathbb{Z}_{tr}(X)(U) \otimes \mathbb{Z}_{tr}(Y)(U) \arrow[d, equal] & \\
Cor(U, X) \otimes Cor(U, Y) \arrow[r] \arrow[dr] & Cor(U \times U, X \times Y) \arrow[d, "\Delta"] \\
& Cor(U, X \times Y) = \mathbb{Z}_{tr}(X \times Y)(U)
\end{tikzcd}$$

Easy to see that; this induces a natural map.

$$\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}) \otimes \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q'}) \longrightarrow \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q} \wedge \mathbb{G}_m^{\wedge q'})$$

$$\mathbb{G}_m^{\wedge q} \wedge \mathbb{G}_m^{\wedge q'} = \mathbb{G}_m^{\wedge (q+q')}$$

and ~~therefore~~

$$\cancel{\mathbb{Z}_{tr}(q) \otimes \mathbb{Z}_{tr}(q') \longrightarrow}$$

(ii) Eilenberg-Zilber Theorem:

Topological version: X, Y top space.

$$C_*(X) \otimes C_*(Y) \xrightarrow[q]{\sim} C_*(X \times Y)$$

can. quasi-iso

(The proof uses acyclic carriers).

Algebraic version:

$$C_*(\mathbb{Z}_{tr}(X)) \otimes C_*(\mathbb{Z}_{tr}(Y)) \xrightarrow[q]{\sim} C_*(\mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(Y))$$

can. quasi-iso

Prop: $\forall\, q, q' \geq 0$, $\exists$ natural product map

$$\mathbb{Z}(q) \otimes \mathbb{Z}(q') \longrightarrow \mathbb{Z}(q+q')$$

pf:

$$\begin{array}{c}
C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q] \otimes C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q'})[-q'] \\
\downarrow \wr \ (ii) \\
C_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}) \otimes \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q'}))[-q-q'] \\
\downarrow (i) \\
C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q+q'})[-q-q']
\end{array}$$

#

Remark (Künneth).

For top spaces X, Y, from

$$C^*(X) \otimes C^*(Y) \simeq C^*(X \times Y),$$

we get:

$$\begin{array}{c}
0 \\
\uparrow \\
\bigoplus_i \mathrm{Tor}(H^i(C^*(X)), H^{k-i}(C^*(Y))) \\
\uparrow \\
H^k(C^*(X) \otimes C^*(Y)) \cong H^k(C^*(X \times Y)) = H^k(X \times Y) \\
\uparrow \\
\bigoplus_i H^i(C^*(X)) \otimes H^{k-i}(C^*(Y)) = \bigoplus_i H^i(X) \otimes H^{k-i}(Y) \\
\uparrow \\
0
\end{array}$$

Consequently, $H^k(X \times Y, \mathbb{Q}) \cong \bigoplus H^i(X, \mathbb{Q}) \otimes H^{k-i}(Y, \mathbb{Q})$.

However, we ~~do~~ NOT get single Künneth decomposition for the motivic cohomology. (But can we analyze the relation (i.e. Chow-Künneth decomp for Chow groups of product) from the above discussion?)

$\mathbb{A}^1$-Homotopy Invariance:

This is a $\boxed{\text{difficult}}$ result:

Step 1: Cohomology sheaves of $C_*\mathbb{Z}_{tr}(X)$ is homotopy-invariant.

(Recall that: $F \in PST(k)$ is $\mathbb{A}^1$-homotopy inv, if, for $p: X \times \mathbb{A}^1 \to X$, p^* induces iso.

$$F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$$

We remark that Ω^p is NOT homotopy invariant)

Pf: Step 1: F is Homotopy invariant $\iff$

$$i_0^* = i_1^* : F(X \times \mathbb{A}^1) \longrightarrow F(X), \quad \text{when}$$

$$i_0 : X \longrightarrow X \times \mathbb{A}^1, \quad i_1 : X \longrightarrow X \times \mathbb{A}^1$$
$$x \longmapsto (x, 0) \qquad\qquad x \longmapsto (x, 1).$$

If: Use the multiplication map

$$\mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$$
$$(x, y) \longmapsto x \cdot y$$

$$\begin{tikzcd}
 & F(X\times\mathbb{A}^1) \arrow[r, "i_0^*"] \arrow[dl, "\mathbb{1}"'] \arrow[d, "(1\times m)^*"] \arrow[dr] & F(X) \arrow[d, "p^*"] \\
F(X\times\mathbb{A}^1) & F(X\times\mathbb{A}^1\times\mathbb{A}^1) \arrow[l, "(i_1\times\mathbb{1})^*"'] \arrow[r, "(i_0\times\mathbb{1})^*"] & F(X\times\mathbb{A}^1)
\end{tikzcd}$$

$$\Uparrow$$

$$\begin{tikzcd}
 & X\times\mathbb{A}^1 & X \arrow[l, "i_0"'] \\
X\times\mathbb{A}^1 \arrow[ur, "\mathbb{1}"] \arrow[r, "i_1\times 1"'] & X\times\mathbb{A}^1\times\mathbb{A}^1 \arrow[u, "1\times m"] \arrow[ur, phantom, "\square"] & X\times\mathbb{A}^1 \arrow[l, "i_0\times 1"] \arrow[u, "p"']
\end{tikzcd}$$

p^* is injective (always)! Then.
it suffices to show: $p^*\circ i^*$ surjective. $\Rightarrow p^*$ is surjective.

$$p^*\circ i^* = (i_0\times\mathbb{1}^*)\circ(1\times m)^* = (i_1\times\mathbb{1})^*\circ(1\times m)^* = \mathbb{1}$$

\#

Step 2: Homology sheaves $H_n C_* F$ is homotopy invariant for any $F\in PST$.

pf: By step 1 it suffices to show

$$H_n C_* F(X\times\mathbb{A}^1) \underset{i_1^*}{\overset{i_0^*}{\rightrightarrows}} H_n C_* F(X). \qquad i_0^* = i_1^*$$

But this is clear, as

$$C_* F(X\times \mathbb{A}^1) \underset{i_1^\#}{\overset{i_0^\#}{\rightrightarrows}} C_* F(X)$$

is chain homotopy, using prism operator.

#

Consequently, $H_n((C_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))$ is homotopy invariant. $\forall n$.

Step 2. $\forall X \in Sm(k)$. $\mathbb{Z}_{tr}(X)$ is a sheaf on X_{et}. Hence also a sheaf on X_{nis}.

Step 3. (Prop 13.10. [MVW], Example 13.11, [MVW])

$$\mathbb{H}^{p,q}(X) = \mathbb{H}^p(X_{zar}, \mathbb{Z}(q)_X)$$

$$= \mathbb{H}^p(X_{nis}, \mathbb{Z}(q)_X).$$

Step 4 (Theorem 13.8 [MVW]).

k perfect field. F a homotopy inv. presheaf with transfer.

Then $H^n_{nis}(-, F_{nis})$ is homotopy invariant.

Step 5 Use the spectral sequence for hypercohomology

$$\underset{\Leftarrow\ \text{Homotopy invariant}}{\text{step 1}}$$

$$E_2^{i,j} = H^i(X_{nis}, \overset{\text{step 2}\ (H^j\mathbb{Z}(q)_X)_{nis}}{H^j\mathbb{Z}(q)_X}) \Rightarrow \mathbb{H}^{i+j=p}(X_{nis}, \mathbb{Z}(q)_X)$$

$$\overset{\text{Step 3}}{=} H^{p,q}(X)$$

Homotopy ~~invariant by Step 5~~.

$\Rightarrow$ $H^{p,q}(X)$ is also Homotopy invariant.

\#

(II). Theorem. $X \in Sm_k$. Then

$$H^{p,q}(X,\mathbb{Z}) = \begin{cases} 0, & q \le 1 \text{ and } (p,q) \ne (0,0),(1,1),(2,1) \\ \bigoplus_i \mathbb{Z}(X_i) \text{ "}\mathbb{Z}(X)\text{"} & (0,0) \\ \mathcal{O}^*(X) & (1,1) \\ \mathrm{Pic}(X) & (2,1) \end{cases}$$

$X = \coprod X_i$ connected compts

$H^{-2,2}$	$H^{-1,2}$	$H^{0,2}$	$H^{1,2}$	$H^{2,2}$	$H^{3,2}$
0	0	0	$\mathcal{O}^*(X)$	$\mathrm{Pic}(X)$	0
0	0	$\mathbb{Z}(X)$	0	0	0
0	0	0	0	0	0

Rmk: (1) $H^{p,q}(X,\mathbb{Z})=0$, $p > q + \dim X$.

Reason: $(\mathbb{Z}(q))^{j} = 0$, $i > q$. Grothendieck vanishing $H^{j}(X, F) = 0$, $j > \dim X$.

$$\Rightarrow E_2^{i,j} = H^i(X_{Zar}, \mathcal{H}^j \mathbb{Z}(q))$$

$$= 0 \qquad i+j \overset{\Delta}{=} p > q + \dim X$$

$$\Rightarrow \mathbb{H}^{p,q}(X, \mathbb{Z})$$

(2) We $H^{p,q}(X) = CH^q(X, 2q-p)$

$$= 0, \quad \text{for } p > 2q$$

(3) Beilinson–Soulé vanishing conj

$$H^{p,q}(X, \mathbb{Z}) = 0, \quad \text{for } p < 0.$$

(4) $q = 0$, $\quad C_* \mathbb{Z}_{tr}(\mathrm{Spec}\, k) = C_* \mathbb{Z} \cong \mathbb{Z}$

$$\begin{array}{ccccccc} \xrightarrow{0} \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \to & 0 \\ & & \uparrow & & \uparrow & & \\ & & -1 & & 0 & & \end{array}$$

constant sheaf

Thus $H^{p,0}(X, \mathbb{Z}) = H^p(X_{Zar}, \mathbb{Z}) = \begin{cases} 0 & p > 0 \\ \mathbb{Z}(X) & p = 0 \end{cases}$

Theorem: $\exists : \mathbb{Z}(1) \xrightarrow{\simeq} \mathcal{O}^*[-1]$

Quasi-isomorphism.

Pf: recall: $\mathbb{Z}(1) = C_*\mathbb{Z}_{tr}(\mathbb{G}_m)[-1]$, where $\mathbb{G}_m = (\mathbb{A}^1 - \{0\}, 1)$

observe that $C_*\mathcal{O}^*(X)$:

$$\cdots \xrightarrow{\partial} \underset{1}{\mathcal{O}^*(X\times\Delta^1)} \xrightarrow{\partial} \underset{0}{\mathcal{O}^*(X)}$$

$$\mathcal{O}^*(X\times\Delta^n) = H^0(X\times\Delta^n, \mathcal{O}^*(X\times\Delta^n))$$

⌜For any affine $X = \operatorname{spec} R$

$$\begin{aligned}\mathcal{O}^*(X\times\Delta^n) &= (\mathcal{O}(X\times\Delta^n))^* \\ &\cong (R[t_1, \dots t_n])^* \\ &= R^* \lrcorner\end{aligned}$$

$$= \mathcal{O}^*(X)$$

Thus: $C_*\mathcal{O}^* =$

$$\xrightarrow{0} \underset{2}{\mathcal{O}^*} \xrightarrow{id} \underset{1}{\mathcal{O}^*} \xrightarrow{0} \underset{0}{\mathcal{O}^*} \to 0$$

$$\simeq \mathcal{O}^*$$

define a natural transformation

$$\lambda : \mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\}) \longrightarrow \mathbb{Z} \oplus \mathcal{O}^*$$

as follows: $X \in Sm(k)$.

$$\mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\})(X) = Cor(X, \mathbb{A}^1 - \{0\})$$

$$\subset Cor(X, \mathbb{A}^1) \quad \subset Cor(X, \mathbb{P}^1)$$

Let t be the coordinate of $\mathbb{A}^1$, and $Z \subset X \times \mathbb{A}^1$ irreducible.

$$\begin{tikzcd} Z \arrow[r, phantom, "\subset"] \arrow[dr, "f.s."'] & X \times \mathbb{A}^1 \arrow[d] \\ & X \end{tikzcd}$$

$K \triangleq k(X)$, $\xi : \operatorname{Spec} K \hookrightarrow X$

$Z_\xi \subset \operatorname{Spec} K[t]$, prime ideal.

So, we get $\tilde{f}(t) = \check{a}_n t^n + \cdots + \check{a}_0$, $\check{a}_i \in K$, $\check{a}_n \neq 0$

s.t $\quad Z_\xi = Z(\tilde{f})$

Note, if we normalize $\check{f}$ by $f = \check{f}/\check{a}_n$

and regard $f \in k(X \times \mathbb{A}^1)$, then

$$\operatorname{div}(f) = Z + nX_\infty, \qquad X_\infty = X \times \{\infty\}.$$

and $f \in \mathcal{O}[t]$

Why ?

Set $f(t) = t^n + a_{n-1} t^{n-1} + \cdots + a_0$,

A priori, $a_i \in K_0$. Set $B \subsetneq X$ to be the closed proper locus, where a_i is not defined for some i.

Then over $X \setminus B$, $Z(f)$ is finite and surjective.

$$Z(f)_\xi = Z_\xi \Rightarrow Z(f) = Z \quad \text{on } X \setminus B \times \mathbb{A}^1$$
$$\Rightarrow Z(f) = Z \quad \text{on } X \times \mathbb{A}^1$$

~~and~~ $a_i \in \mathcal{O}_X(X)$..

Now. for $Z \in \mathrm{Cor}(X, \mathbb{A}^1 - \{0\})$

$$Z \cap X_0 = \phi, \quad X_0 = X \times \{0\}$$

$$\Rightarrow f(0) = a_0 \in \mathcal{O}^*(X)$$

Now, define λ for X (connected, and Z irred, as above)

$$\mathrm{Cor}(X, \mathbb{A}^1 - \{0\}) \xrightarrow{\lambda(X)} \mathbb{Z}(X) \oplus \mathcal{O}^*(X)$$
$$\begin{matrix} \Psi & & \Psi \\ Z & \longmapsto & (\deg f, (-1)^{\deg f} f(0)) \end{matrix}$$

(Then extend λ to a homo of abelian groups)

The ~~diagram (-1)~~ $\deg(f)$ is ~~used for showing that~~

Rmk: (1)
λ respects the transfers. i.e $\forall f \in Cor(X, Y)$

$$\begin{array}{ccc} \mathbb{Z}_{tr}(\mathbb{A}^1-\{0\})(X) & \xrightarrow{\lambda(X)} & (\mathbb{Z}\oplus\mathcal{O}^*)(X) \\ f^* \uparrow & \square & \uparrow f^* \\ \mathbb{Z}_{tr}(\mathbb{A}^1-\{0\})(Y) & \xrightarrow{\lambda(Y)} & (\mathbb{Z}\oplus\mathcal{O}^*)(Y) \end{array}$$

(2) λ is surjective.

~~For~~ $u \in \mathcal{O}^*(X) = \mathrm{Hom}_k(k[t,t^{-1}], \mathcal{O}(X)) = \mathrm{Mor}_k(X, \mathbb{A}^1-\{0\})$

$$\subset Cor_k(X, \mathbb{A}^1-\{0\})$$

$$u \longmapsto Z_u = t-u \in Cor_k(X, \mathbb{A}^1-\{0\})$$

$$\left.\begin{array}{l} \lambda(Z_u) = (1, u) \\ \lambda(Z_1) = (1, 1) \end{array}\right\} \Rightarrow \cancel{\lambda(Z_u - Z_1) = (0, u)}$$

Thus, $\forall (n, u) \in \mathbb{Z}\oplus\mathcal{O}^*$, $\lambda((n-1)Z_1 + Z_u) = (n, u)$.

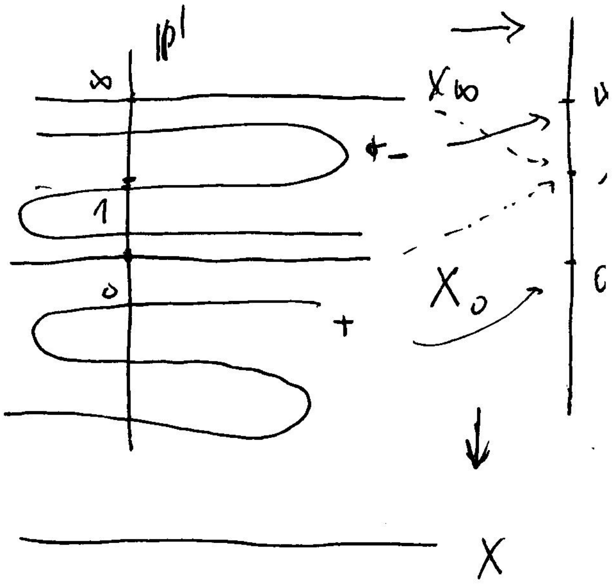
(3) $\mathrm{Ker}(\lambda) = M^*(\mathbb{P}^1; 0, \infty) \in PST(k)$ which can be described as follows:

$\forall X \in Sm(k)$,

$$\mathcal{M}^*(\mathbb{P}^1; 0, \infty)(X) = \left\{ f \in k(X \times \mathbb{P}^1)^* \;\middle|\; \begin{array}{l} f \text{ regular in a} \\ \text{nbd of } X_0 \dot\cup X_\infty, \\ \text{and } f|_{X_0} = f|_{X_\infty} = 1 \end{array} \right\}$$

and the map

$$\begin{array}{ccc} \mathcal{M}^*(\mathbb{P}^1; 0, \infty)(X) & \longrightarrow & Cor(X, \mathbb{A}^1 - \{0\}) \\ \Downarrow & & \Downarrow \\ f & \longmapsto & div(f) \end{array}$$



Thus, we get a short exact sequence in $PST(k)$

$$0 \to \mathcal{M}^* \longrightarrow \mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\}) \longrightarrow \mathbb{Z} \oplus \mathcal{O}^* \to 0$$

and then

$$\begin{array}{ccccccccc} 0 \to & 0 & \longrightarrow & C_*\mathbb{Z}_{tr}(Spec\, k) & \xrightarrow{\cong} & C_*\mathbb{Z} & \to 0 \\ & \downarrow & & \downarrow i_1 & & \downarrow & \\ 0 \to & C_*(\mathcal{M}^*) & \longrightarrow & C_*\mathbb{Z}_{tr}(\mathbb{A}^1 - \{0\}) & \longrightarrow & C_*\mathbb{Z} \oplus C_*\mathcal{O}^* & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & C_*(\mathcal{M}^*) & \longrightarrow & C_*\mathbb{Z}_{tr}(\mathbb{G}_m) & \longrightarrow & C_*\mathcal{O}^* \overset{\cong}{\leftarrow} \mathcal{O}^* & \to 0 \\ & \downarrow & & \downarrow & & & \end{array}$$

It remains to show $C_*(\mathcal{M}^*)$ is acyclic (i.e ~~exact~~ $H_n(C_*(\mathcal{M}^*)) = 0 \ \forall n$)

This will imply

$$\mathbb{Z}(1)[1] \xrightarrow[\sim]{\text{quasi-iso}} \mathcal{O}^* \quad \text{and hence } \mathbb{Z}(1) \xrightarrow{\sim} \mathcal{O}^*[-1].$$

For that, we consider

$$\underset{\substack{\shortparallel \\ \text{normalized chain complex}}}{N_*(\mathcal{M}^*)} \subset C_*(\mathcal{M}^*) \quad \text{(which is a quasi-isomorphism)}$$

$$\begin{tikzcd}
N_n(\mathcal{M}^*) = \bigcap_{i=0}^{n-1} \operatorname{Ker}(\partial_i : C_n(\mathcal{M}^*) \to C_{n-1}(\mathcal{M}^*)) \arrow[d, "(-1)^n \partial_n"] \arrow[r, phantom, "\subset"] & C_n(\mathcal{M}^*) \arrow[d, "\partial"] \\
N_{n-1}(\mathcal{M}^*) \arrow[r, phantom, "\subset"] & C_{n-1}(\mathcal{M}^*)
\end{tikzcd}$$

Take $f \in N_n(\mathcal{M}^*)(X)$, $\partial_n(f) = 1$.

i.e $f \in k(X \times \Delta^n \times \mathbb{P}^1)^*$, regular on $U \underset{X \times \Delta^n \times \mathbb{P}^1}{\subset} \supset Z = X \times \Delta^n \times \{0, \infty\}$.

s.t (1) $f|_Z = 1$

(2) $f|_{X \times \Delta^{n-1}_i \times \mathbb{P}^1} = 1$, $\forall i = 0, 1 .. n.$

Δ^{n-1}_i: i-th face of Δ^n

define $h(f) = 1 - t(1-f)) \in k(\mathbb{A}^1 \times X \times \Delta^n \times \mathbb{P}^1)$, with t the coordinate on $\mathbb{A}^1$

Satisfying

(0) $h(f)$ regular on $\mathbb{A}^1 \times U \supset \mathbb{A}^1 \times Z$

(1) $h(f)\big|_{\mathbb{A}^1 \times Z} = 1$, (2) $h(f)\big|_{\mathbb{A}^1 \times X \times \Delta^n_i \times \mathbb{P}^1} = 1$ $\forall i = 0, \dots n$.

$\mathbb{A}^1 \times Z = (\mathbb{A}^1 \times X \times \Delta^n) \times \{0, \infty\}$

$\Rightarrow h(f) \in N_n(\mathcal{M}^*)(\mathbb{A}^1 \times X)$, and $\partial h(f) = 0$

Now:

$$N_n(\mathcal{M}^*)(\mathbb{A}^1 \times X) \underset{i_1^*}{\overset{i_0^*}{\rightrightarrows}} N_n(\mathcal{M}^*)(X)$$

$i_0^* \simeq i_1^*$ chain homotopy

i.e $i_1^* - i_0^* = \partial D + D \partial$

$$\Rightarrow \underbrace{i_1^*(h(f))}_{= f} - \underbrace{i_0^*(h(f))}_{= 1} = \partial(D(h(f)) + D\partial(\overset{0}{\overbrace{h(f)}})$$

$\Rightarrow f = \partial(D(h(f)) + 1 = \partial_{n+1}(D(h(f)) + (-1)^{n+1})$

we're done! #

(III). Milnor's K-theory

R . comm. ring

$$T(R^*) = \underset{\uparrow \atop 0}{\mathbb{Z}} \oplus \underset{\uparrow \atop 1}{R^*} \oplus \underset{\uparrow \atop 2}{R^* \otimes_{\mathbb{Z}} R^*} \oplus \cdots$$ graded ring (non-commutative)

$K^M_*(R) = T(R^*)/I$, I = homogenous ideal, generated by

$a \otimes 1-a$, for $a, 1-a \in R^*$.

Thus $K^M_0(R) = \mathbb{Z}$, $K^M_1(R) = R^*$. $K^M_2(R)$ becomes mysterious.

For X, scheme, define

$\mathcal{K}^M_*$ to be the Zariski sheaf of the presheaf

$$U \longmapsto K^M_*(\mathcal{O}_X(U))$$

Thus $\mathcal{K}^M_0 = \mathbb{Z}$ constant sheaf

$\mathcal{K}^M_1 = \mathcal{O}^*$. (over integral scheme, $U \mapsto \mathcal{O}(U)^*$ the presheaf is a sheaf)

$\mathcal{K}^M_2 = ?$

Theorem (M. Kerz. 2009)

$|k| = +\infty$ $X \in Sm(k)$ $H^n(X_{Zar}, \mathcal{K}^M_n) \cong CH^n(X)$

The key to the above theorem is the following

<u>Gersten resolution</u> (Constructed by Kato 1986
proved by Kerz 2009)

(flasque).

$$0 \to \mathcal{K}_n^M \to \bigoplus_{x \in X^{(0)}} i_{x*}(K_n^M(x)) \to \cdots$$

$$\to \bigoplus_{x \in X^{(n-1)}} i_{x*}(K_1^M(x)) \xrightarrow{div} \bigoplus_{x \in X^{(n)}} i_{x*}(K_0^M(x)) \to 0$$

$X^{(i)}$ = set of cod i pts in X.

E.g: $n = 1$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}^* \arrow[r] & \mathcal{K}^* \arrow[r, "div"] \arrow[dr] & \bigoplus_{x \in X^{(1)}} i_{x*}(\mathbb{Z}) \arrow[r] & 0 \\
& & & \mathcal{K}^*/\mathcal{O}^* \arrow[u, "\wr", "div"'] \arrow[ur] &
\end{tikzcd}$$

<u>flasque resolution</u>.

$\Rightarrow$ $H^n(X, \mathcal{O}^*) = 0$, $n \geq 2$. e.g