1. Beilinson's conjecture and the motivic pro-complex

We firstly recall the definition of the motivic complex.

Definition 1.1. For every integer $r \geq 0$, the motivic complex $\mathbb{Z}(r)$ is given by the following complex of presheaves with transfers:

$$\mathbb{Z}(r) = \mathcal{C}_*\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge r})[-r]$$

[MVW], chapter 5 shows the following relationship between motivic cohomology and Milnor K-theory on a point, that is

Theorem 1.2. *For any field F and integer n we have $H^{n,n}(\mathrm{Spec}F, \mathbb{Z}) \cong K_n^M(F)$.*

We will omit the detail of the proof, only point out that we have $H^{n,n}(\mathrm{Spec}F, \mathbb{Z}) = \mathrm{coker}(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1) \overset{\partial_0 - \partial_1}{\longrightarrow} \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})(\mathrm{Spec}F))$ and the isomorphic is given by $[a_1, \cdots, a_n] \mapsto a_1 \otimes \cdots \otimes a_n$, where $a_i \in F \backslash \{0, 1\}$, and $[a_1, \cdots, a_n]$ is the coordinate in $\mathbb{G}_m^{\wedge n}$.

The generalization of this result is the Beilinson's conjecture: $\mathcal{H}^r(\mathbb{Z}(r)) \cong \mathcal{K}_r^M$ where $\mathcal{K}_*^M$ is the (Zariski in the proof, but we can restrict it to Nisnevich) Milnor K-sheaf. This conjecture is proven by Elbaz-Vincent/Müller-Stach[EM], Gabber and Kerz[Ke].

Sketch of the proof: We have the Gersten resolution of Milnor K theory of a scheme X as follow:

$$0 \to \mathcal{K}_n^M|_X \to \oplus_{x \in X^{(0)}} i_{x^*}(K_n^M(k(x))) \to \oplus_{x \in X^{(1)}} i_{x^*}(K_{n-1}^M(k(x))) \to \cdots$$

where the maps are given by Kato. We know that this complex is exact if X is regular, contains a field, and all residue fields of X contain "enough" elements (see [Ke], Remark 5.8 for the precise meaning of "enough"). For the proof of this statement, see [Pa].

By this resolution, we have the following result:

Theorem 1.3. *(Bloch formula) There is a canonical isomorphism $H^n(X, \mathcal{K}_n^M) \cong CH^n(X)$ for every $n \geq 0$ and X is as mentioned.*

Proof. $H^n(X, \mathcal{K}_n^M) \cong \mathrm{coker}(\oplus_{x \in X^{(n-1)}} K_1(k(x)) \to \oplus_{x \in X^{(n)}} K_0(k(x))$
$\cong CH^n(X)$. ☐

Furthermore, we have

Theorem 1.4. *(Beilinson's conjecture) $\mathcal{H}^n(\mathbb{Z}(n)) \cong \mathcal{K}_n^M$*

Proof. Consider the morphism of exact Gersten complexes of sheaves from Milnor K-theory to motivic cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_n^M|_X & \longrightarrow & \oplus_{x\in X^{(0)}} i_{x^*}(K_n^M(k(x))) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H}^n(\mathbb{Z}(n))|_X & \longrightarrow & \oplus_{x\in X^{(0)}} i_{x^*}(H^n(x,\mathbb{Z}(n))) & \longrightarrow & \cdots \end{array}$$

where the top row is given by 1.2 and the bottom row is given by the contraction map, see [MVW] §23.

□

According to Voevodsky ([MVW], Theorem 19.1 and Corollary 19.2), we have the following equation for $X \in \mathrm{Sm}_k$:

$$H^r(X, \mathcal{K}_r^M) = CH^r(X) = H^{2r}(X, \mathbb{Z}(r))$$

(We have a question: There should exist a nature map from $H^r(X, \mathcal{K}_r^M)$ to $H^{2r}(X, \mathbb{Z}(r))$. Does this map commute with the two "="s here (given by Bloch's formula and Voedodsky's result respectively)?)

Now we are ready to define the motivic pro-complex of $X_\bullet$. From now on we follow the notations in [BEK]. In particular $X_\bullet/W_\bullet$ is in $\mathrm{Sm}_{W_\bullet}$ and $X_1 = X \otimes_W k$. We assume $r < p$.

Beilinson's conjecture allows us to define the following map:

$$\log : \mathbb{Z}_{X_1}(r) \to \mathcal{H}^r(\mathbb{Z}_{X_1}(r))[-r] = \mathcal{K}^M_{X_1,r}[-r] \stackrel{d\log[]}{\longrightarrow} W_\bullet\Omega^r_{X_1,\log}[-r]$$

where [ ] is the Teichmüller lift. Also recall the simplicial complex $\mathfrak{S}_{X_\bullet}(r)$ and the map $\Phi^J : \mathfrak{S}_{X_\bullet}(r) \to W_\bullet\Omega^r_{X_1,\log}[-r]$. The motivic pro-complex $\mathbb{Z}_{X_\bullet}(r)$ is defined by

$$\mathbb{Z}_{X_\bullet}(r) = \mathrm{cone}(\mathfrak{S}_{X_\bullet}(r) \oplus \mathbb{Z}_{X_1}(r) \stackrel{\Phi^J\oplus -\log}{\longrightarrow} W_\bullet\Omega^r_{X_1,\log}[-r])[-1]$$

Note that $\mathcal{H}^r(\mathbb{Z}_{X_1}(r)) = \mathcal{K}^M_{X_1,r} \to W_\bullet\Omega^r_{X_1,\log}$ is an epimorphism, since $W_\bullet\Omega^r_{X_1,\log}$ is generated by symbols (by definition).

2. Properties of the motivic pro-complex

Followings are some properties of the motivic pro-complex, compared with motivic complex we introduced yesterday:

Proposition 2.1. *$\mathbb{Z}_{X_\bullet}(0) = \mathbb{Z}$, the constant sheaf $\mathbb{Z}$ at degree 0.*

Proof. We have $W_\bullet\Omega^0_{X_1,\log} = \mathbb{Z}/p^\bullet$, $\mathbb{Z}_{X_1}(0) = \mathbb{Z}$ and $\mathfrak{S}_{X_\bullet}(0) = \mathbb{Z}/p^\bullet$ according to Theorem 5.4. Thus $\mathbb{Z}_{X_\bullet}(0) = \mathbb{Z}$ by definition. □

Proposition 2.2. *$\mathbb{Z}_{X_\bullet}(r)$ has support in cohomological degrees $\leq r$. For $r \geq 1$, if the Beilinson-Soulé conjecture is true, it has support in cohomological degrees $[1, r]$.*

Proof. We have the following long exact sequence

$$(2.1) \quad \cdots \to \mathcal{H}^i(\mathbb{Z}_{X_\bullet}(r)) \to \mathcal{H}^i(\mathfrak{S}_{X_\bullet}(r)) \oplus \mathcal{H}^i(\mathbb{Z}_{X_1}(r)) \to \mathcal{H}^i(W_\bullet\Omega^r_{X_1,\log}[-r]) \to \cdots$$

and we know that $\mathfrak{S}_{X_\bullet}(r)$ has support in cohomological degrees $[1, r]$ for $r > 1$, and $\mathbb{Z}_{X_1}(r)$ has support in cohomological degrees $\leq r$ (resp. $[1, r]$ if Beilinson-Soulé conjecture holds). Consider $\mathcal{H}^r(\mathbb{Z}_{X_1}(r)) \stackrel{-\log}{\to} \mathcal{H}^r(W_\bullet\Omega^r_{X_1,\log}[-r]) \stackrel{[1]}{\to} \mathcal{H}^{r+1}(\mathbb{Z}_{X_\bullet}(r))$. Since $[1] \circ (-\log) = 0 = 0 \circ (-\log)$, by log is an epimorphism we know that $[1] = 0$, thus $\mathcal{H}^{r+1}(\mathbb{Z}_{X_\bullet}(r)) = 0$. ☐

Proposition 2.3. $\mathcal{H}^r(\mathbb{Z}_{X_\bullet}(r)) = \mathcal{K}^M_{X_\bullet,r}$

Proof. We have the following exact sequence:

$$0 \to \mathcal{H}^r(\mathbb{Z}_{X_\bullet}(r)) \to \mathcal{H}^r(\mathfrak{S}_{X_\bullet}(r)) \oplus \mathcal{H}^r(\mathbb{Z}_{X_1}(r)) \stackrel{\Phi^J \oplus -\log}{\longrightarrow} W_\bullet\Omega^r_{X_1,\log} \to 0$$

and take the nth cohomology of Theorem 5.4 in [BEK], we have the following exact sequence:

$$0 \to p\Omega^{r-1}_{X_\bullet}/p^2 d\Omega^{r-2}_{X_\bullet} \to \mathcal{H}^r(\mathfrak{S}_{X_\bullet}(r)) \stackrel{\Phi^J}{\to} W_\bullet\Omega^r_{X_1,-\log} \to 0$$

which induces the upper row in the diagram below. The bottom row is given by Theorem 12.3 in [BEK]:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p\Omega^{r-1}_{X_\bullet}/p^2 d\Omega^{r-2}_{X_\bullet} & \longrightarrow & \mathcal{H}^r(\mathbb{Z}_{X_\bullet}(r)) & \longrightarrow & \mathcal{H}^r(\mathbb{Z}_{X_1}(r)) & \longrightarrow & 0 \\
 & & \| & & (*)\uparrow & & \uparrow & & \\
0 & \longrightarrow & p\Omega^{r-1}_{X_\bullet}/p^2 d\Omega^{r-2}_{X_\bullet} & \longrightarrow & \mathcal{K}^M_{X_\bullet,r} & \longrightarrow & \mathcal{K}^M_{X_1,r} & \longrightarrow & 0
\end{array}$$

where $(*)$ is induced by Kato's syntomic regulator map, and the rightmost vertical arrow is an isomorphism by Beilinson's conjecture, so by 5-lemma, $(*)$ is also an isomorphism.

☐

Proposition 2.4. $\mathbb{Z}_{X_\bullet}(1) = \mathbb{G}_{m,X_\bullet}[-1]$

Proof. This is a corollary from the previous two properties since the Beilinson-Soulé vanishing conjecture is true for $r = 1$. ☐

Proposition 2.5. $\mathbb{Z}_{X_\bullet}(r) \otimes^L_{\mathbb{Z}} \mathbb{Z}/p^\bullet = \mathfrak{S}_{X_\bullet}(r)$

Proof. The sheaf $W_n\Omega^r_{X_1,\log}$ is a sheaf of flat $\mathbb{Z}/p^n$-modules, so $W_\bullet\Omega^r_{X_1,\log}\otimes^L_{\mathbb{Z}}\mathbb{Z}/p^\bullet = W_\bullet\Omega^r_{X_1,\log}$. By Theorem 5.4, this implies $\mathfrak{S}_{X_\bullet}(r)\otimes^L_{\mathbb{Z}}\mathbb{Z}/p^\bullet = \mathfrak{S}_{X_\bullet}(r)$. According to Theorem 8.3 in [GL], we have $\mathbb{Z}_{X_1}(r)\otimes^L_{\mathbb{Z}}\mathbb{Z}/p^n = W_n\Omega^r_{X_1,\log}[-r]$, so from the definition of $\mathbb{Z}_{X_\bullet}(r)$, we have $\mathbb{Z}_{X_\bullet}(r)\otimes^L_{\mathbb{Z}}\mathbb{Z}/p^\bullet = \mathfrak{S}_{X_\bullet}(r)$. □

Sketch of proof for Thm 8.3 in [GL]:

Proposition 2.6. *There is a canonical product structure*

$$\mathbb{Z}_{X_\bullet}(r)\otimes^L_{\mathbb{Z}}\mathbb{Z}_{X_\bullet}(r') \to \mathbb{Z}_{X_\bullet}(r+r')$$

compatible with the products on $\mathbb{Z}_{X_1}(r)$ *and* $\mathfrak{S}_{X_\bullet}(r)$.

Proof. We let the two following morphisms

$$\mathbb{Z}_{X_\bullet}(r)\otimes^L_{\mathbb{Z}}\mathbb{Z}_{X_\bullet}(r') \to \mathbb{Z}_{X_1}(r+r')$$

$$\mathbb{Z}_{X_\bullet}(r)\otimes^L_{\mathbb{Z}}\mathbb{Z}_{X_\bullet}(r') \to \mathfrak{S}_{X_\bullet}(r+r')$$

be induced by the product of motivic complex and the product on the syntomic complex. Then we have the following map

$$\begin{aligned}(2.2)\quad \mathbb{Z}_{X_\bullet}(r)\otimes^L_{\mathbb{Z}}\mathbb{Z}_{X_\bullet}(r') &\to \mathrm{cone}(\mathbb{Z}_{X_1}(r+r')\oplus\mathfrak{S}_{X_\bullet}(r+r') \\ &\to W_\bullet\Omega^{r+r'}_{X_1,\log}[-r-r'])[-1] \cong \mathbb{Z}_{X_\bullet}(r+r').\end{aligned}$$

□

3. Motivic fundamental triangle

Proposition 3.1. *We have a unique commutative diagram of exact triangles in* $\mathrm{D}_{\mathrm{pro}}(X_1)$

$$\begin{array}{ccccccc} p(r)\Omega^{<r}_{X_\bullet}[-1] & \longrightarrow & \mathbb{Z}_{X_\bullet}(r) & \longrightarrow & \mathbb{Z}_{X_1}(r) & \longrightarrow & \cdots \\ \| & & \downarrow & & \downarrow{\scriptstyle d\log} & & \\ p(r)\Omega^{<r}_{X_\bullet}[-1] & \longrightarrow & \mathfrak{S}_{X_\bullet}(r) & \longrightarrow & W_\bullet\Omega^r_{X_1,\log}[-r] & \longrightarrow & \cdots \end{array}$$

where the bottom exact triangle comes from Theorem 5.4 and the maps in the right square are the canonical maps.

Proof. The existence of the top triangle can be shown by the octahedral axiom from the following three triangles: $p(r)\Omega^{<r}_{X_\bullet}[-1]\to\mathfrak{S}_{X_\bullet}(r)\to W_\bullet\Omega^r_{X_1,\log}[-r]\to\cdots$ (fundamental triangle), $\mathbb{Z}_{X_\bullet}(r)\to\mathbb{Z}_{X_1}(r)\oplus\mathfrak{S}_{X_\bullet}(r)\to W_\bullet\Omega^r_{X_1,\log}[-r]\to\cdots$ (definition of $\mathbb{Z}_{X_\bullet}$), and $\mathbb{Z}_{X_1}(r)\to\mathbb{Z}_{X_1}(r)\oplus$

$\mathfrak{S}_{X_\bullet}(r) \to \mathfrak{S}_{X_\bullet} \to \cdots$. We can also apply the dual of [Ne], Lemma 1.4.4 and note that the square

$$\begin{array}{ccc} \mathbb{Z}_{X_\bullet}(r) & \longrightarrow & \mathbb{Z}_{X_1}(r) \\ \downarrow & & \downarrow d\log \\ \mathfrak{S}_{X_\bullet}(r) & \longrightarrow & W_\bullet\Omega^r_{X_1,\log}[-r] \end{array}$$

is a homotopy cartesian square, and a cocartesian square is still cartesian. Following is [Ne], Lemma 1.4.4:

Lemma 3.2. *Let*

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow g & & \downarrow h \\ Y' & \longrightarrow & Z' \end{array}$$

be a homotopy cartesian square.

If $Y \xrightarrow{g} Y' \longrightarrow Y'' \longrightarrow Y[1]$ *is a triangle, then there is a triangle* $Z \xrightarrow{h} Z' \longrightarrow Y'' \longrightarrow Z[1]$ *which completes the homotopy cartesian square to a map of triangles*

$$\begin{array}{ccccccc} Y & \longrightarrow & Y' & \longrightarrow & Y'' & \longrightarrow & Y[1] \\ \downarrow & & \downarrow & & \| & & \downarrow \\ Z & \xrightarrow{h} & Z' & \longrightarrow & Y'' & \longrightarrow & Z[1] \end{array}$$

For uniqueness, we need to show that the morphism $p(r)\Omega^{<r}_{X_\bullet}[-1] \to \mathbb{Z}_{X_\bullet}(r)$ is uniquely defined by the requirements of the proposition, which follows from Lemma A.2 in [BEK].

□

Corollary 3.3. *For* $Y_\bullet = X_\bullet \times \mathbb{P}^m$ *one has a projective bundle isomorphism*

$$\bigoplus_{s=0}^{m} H^{r'-2s}_{\mathrm{cont}}(X_1, \mathbb{Z}_{X_\bullet}(r-s)) \xrightarrow{\oplus_s c_1(\mathcal{O}(1))^s} H^{r'}_{\mathrm{cont}}(Y_1, \mathbb{Z}_{Y_\bullet}(r)).$$

Proof. According to the associated long exact sequence from the motivic fundamental triangle and 5-lemma, we only need to show the following two isomorphisms:

$$\bigoplus_{s=0}^{m} H^{r'-2s}(X_1, \mathbb{Z}_{X_1}(r-s)) \to H^{r'}(Y_1, \mathbb{Z}_{Y_1}(r))$$

and
$$\bigoplus_{s=0}^{m} H^{r'-2s}(X_1, p(r)\Omega^{<r}_{X_\bullet}) \to H^{r'}(Y_1, p(r)\Omega^{<r}_{Y_\bullet}).$$
The first one comes from [MVW] Corollary 15.5: $C_*\mathbb{Z}_{\mathrm{tr}}(\mathbb{P}^n) \cong \oplus_{i=0}^{n}\mathbb{Z}(i)[2i]$, and the second one comes from the fact that Y is a projective bundle of X. □

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