§8. Crystalline Hodge obstruction and motivic complex

① Introduction

Recall: k perfect field of char. $p>0$

$W = W(k)$

$X/_W$ smooth projective scheme $\quad X_n = X \otimes_W W/p^n. \quad (n \geq 1.)$

$K_0(X)$, $K_0(X_1)$ the K_0 of X & X_1 respectively

$\rightsquigarrow$ $\exists$ canonical map

$$\begin{aligned} K_0(X)_{\mathbb{Q}} &\longrightarrow K_0(X_1)_{\mathbb{Q}}. \\ \mathfrak{z} &\longmapsto \mathfrak{z}|_{X_1} \end{aligned}$$

Question: let $\mathfrak{z}_1 \in K_0(X_1)_{\mathbb{Q}}$, under what condition, one can lift $\mathfrak{z}_1$ to $\mathfrak{z} \in K_0(X)_{\mathbb{Q}}$? i.e. $\mathfrak{z}_1 \in \mathrm{Im}(K_0(X)_{\mathbb{Q}} \to K_0(X_1)_{\mathbb{Q}})$.

p-adic VHS:

$$\begin{tikzcd} K_0(X)_{\mathbb{Q}} \arrow[r, "\text{reduction}"] \arrow[d, "ch_{dR}"'] \arrow[dr, phantom, "\circlearrowright"] & K_0(X_1)_{\mathbb{Q}} \arrow[d, "ch_{cris}"] & \mathfrak{z}_1 \\ \bigoplus_r H^{2r}_{dR}(X/W)_{\mathbb{Q}} \arrow[r, "\Phi"] & \bigoplus_r H^{2r}_{cris}(X_1/W)_{\mathbb{Q}} & \end{tikzcd}$$

↑ de Rham-crystalline comparison.

Thm: the following two conditions are $\cong$

(a) $ch(\mathfrak{z}_1) \in \bigoplus_r H^{2r}_{cris}(X_1/W)_{\mathbb{Q}}$ can be lifted to $K_0(X)_{\mathbb{Q}}$

i.e. $\exists\, \mathfrak{z} \in K_0(X)_{\mathbb{Q}}$, s.t. $ch(\mathfrak{z}|_{X_1}) = ch(\mathfrak{z}_1)$

(b) $\mathfrak{z}_1$ is Hodge. i.e. $ch(\mathfrak{z}_1) \in \Phi\left(\bigoplus_r F^r H^{2r}_{dR}(X/W)_{\mathbb{Q}}\right)$

Main thm. of Bloch - Esnault - Kerz: under some technical condition (i.e. $p > \dim(X_1) + 6$)
i.e. for X of small dim.

the following are $\cong$

(i): z_1 can be lifted formally, i.e. $\exists\ \hat{z} \in \left(\varprojlim K_0(X_n)\right)_{\mathbb{Q}}$

s.t. $\hat{z}\big|_{X_1} = z_1$

(ii) z_1 is Hodge.

⚠ BEK literally lift the $K_0(X_1)_{\mathbb{Q}}$ class to an element in $\left(\varprojlim K_0(X_n)\right)_{\mathbb{Q}}$, not only the chern char. in crystalline cohomology (see their Remark iii in P677)

to this, the authors constructed the following diagram, with first line exact.

$$
\begin{tikzcd}
\bigoplus\limits_{0\le r\le d} CH^r_{\mathrm{cont}}(X_\bullet)_{\mathbb{Q}} \arrow[r] & \bigoplus\limits_{0\le r\le d} CH^r(X_1)_{\mathbb{Q}} \arrow[r, "ob."] & \bigoplus\limits_{0\le r\le d} H^{2r}_{\mathrm{cont}}(X_1, p(r)\Omega^{<r}_{X_\bullet}) \\
K_0^{\mathrm{cont}}(X_\bullet)_{\mathbb{Q}} \arrow[u, "\simeq", "ch_{\mathrm{cont}}"'] \arrow[d, "\text{surjective}"'] \arrow[r, phantom, "\circlearrowleft"] & & \\
\left(\varprojlim K_0(X_n)\right)_{\mathbb{Q}} \arrow[r, "\text{reduction}"'] & K_0(X_1)_{\mathbb{Q}} \arrow[uu, "\cong", "ch"'] &
\end{tikzcd}
\qquad (*)
$$

crystalline Hodge obstruction map

Where $X_\bullet = \hat{X}$ p-adic completion of X/W

$CH^r_{\mathrm{cont}}(X_\bullet)$ continuous Chow gp.

$K_0^{\mathrm{cont}}(X_\bullet)$ continuous K_0

then. z_1 Hodge $\overset{(1)}{\Longleftrightarrow}$ $ob(z_1) = 0$ $\overset{(2)}{\Longleftrightarrow}$ $ch(z_1)$ can be lifted to continuous Chow gp

$\overset{(3)}{\Longleftrightarrow}$ z_1 can be lifted formally to $\left(\varprojlim_n K_0(X_n)\right)_{\mathbb{Q}}$.

Aim of this talk: define the first line of the diagram and show (1) & (2).

next meeting: complete the construction of $(*)$ (the the equivalence (3) follows formally).

② continuous chow group and the obstruction sequence.

Recall: X_1/k proper smooth $\Rightarrow$ isomorphism

$$CH^r(X_1) \cong H^{2r}(X_1, \mathbb{Z}_{X_1}(r))$$

So continuous Chow group is defined analogously using the continuous version of $\mathbb{Z}_{X_1}(r)$ & the continuous cohomology

TO: · $X_\bullet/W$ p-adic smooth ~~proper~~ projective formal scheme over W

· $X_n = X_\bullet \hat{\otimes}_W W/p^n$, $n \in \mathbb{N}$

· $\mathbb{Z}_{X_1}(r) = \mathbb{Z}(r)\big|_{X_{1,Nis}} \in \widetilde{X_{1,Nis}}$, concentrated in degree $\leq r$

$$\log : \mathbb{Z}_{X_1}(r) \longrightarrow \mathcal{H}^r(\mathbb{Z}_{X_1}(r))[-r] = \mathcal{K}^M_{X_1,r}[-r] \xrightarrow{d\log} W_\bullet\Omega^r_{X_1,\log}[-r]$$

· choose an embedding such that

$$\begin{tikzcd} X_\bullet \arrow[r, hook] \arrow[d] & Y_\bullet \arrow[dl] \\ \mathrm{Spf}(W) & \end{tikzcd}$$

* $Y_\bullet/W$ smooth.
* $\exists$ lifting of Frobenius $F = F_{Y_\bullet} : Y_\bullet \to Y_\bullet$

(TO, one needs $X_\bullet/W$ projective instead of $X_\bullet/W$ proper !!)

let $D_\bullet$ = PD-envelope of $X_\bullet \hookrightarrow Y_\bullet$, compatible with the pd-structure on $(p) \subseteq W$ $\quad F \circlearrowright D_\bullet$

$J^{[i]} \subseteq \mathcal{O}_{D_\bullet}$ the corresponding PD-ideals $\quad i = 0, 1, \dots$

set (for $r < p$)

$$J(r)\Omega^\bullet_{D_\bullet} = \left(J^{[r]} \to J^{[r-1]} \otimes \Omega^1_{D_\bullet/W} \to \cdots \to J \otimes \Omega^{r-1}_{D_\bullet/W} \to \Omega^r_{D_\bullet/W} \to \cdots \right)$$

$$\mathfrak{S}_{X_\bullet}(r) = \tau_{\leq r} R\varepsilon_* \mathfrak{S}_{X_\bullet}(r)_{\acute{e}t}$$

$$= \tau_{\le r} R\varepsilon_* \left(J(r)\Omega^\bullet_{D.} \xrightarrow{1-f_r} \Omega^\bullet_{D.} \right)[-1]$$

where $f_r = \dfrac{\text{Frobenius}}{p^r}$ (so, need the lifting of Frobenius $F = F_{Y.}$)

<u>already seen:</u> fundamental triangle:

$$p(r)\Omega^{<r}_{X.}[-1] \longrightarrow \tilde{\mathfrak{S}}_{X.}(r) \xrightarrow{\Phi^J} W.\Omega^r_{X_1,\log}[-r] \xrightarrow{+1}$$

& the connecting homomorphism $W.\Omega^r_{X_1,\log}[-r] \longrightarrow p(r)\Omega^{<r}_{X.}[-1]$

is

$$\begin{tikzcd}
W.\Omega^r_{X_1,\log}[-r] \arrow[r] & W.\Omega^{\ge r}_{X_1} \arrow[r] & \mathfrak{f}(r)\, W.\Omega^\bullet_{X_1} \arrow[r, "\sim", "\Phi(F)"'] & p(r)\Omega^\bullet_{X_1} \arrow[d] \\
 & & & p(r)\Omega^{<r}_{X_1}
\end{tikzcd}$$

- define the <u>continuous motivic complex</u> $\mathbb{Z}_{X.}(r)$ by

$$\mathbb{Z}_{X.}(r) = \mathrm{Cone}\left(\mathbb{Z}_{X_1}(r) \oplus \tilde{\mathfrak{S}}_{X.}(r) \xrightarrow{-\log + \Phi^J} W\Omega^r_{X_1,\log}[-r] \right)[-1]$$

$\rightsquigarrow$ commutative diagram whose lines are exact triangles

$$\begin{tikzcd}
p(r)\Omega^{<r}_{X.} \arrow[r] \arrow[d, equal] & \mathbb{Z}_{X.}(r) \arrow[r] \arrow[d] & \mathbb{Z}_{X_1}(r) \arrow[r, "+1"] \arrow[d, "\log"] & {} \\
p(r)\Omega^{<r}_{X.} \arrow[r] & \tilde{\mathfrak{S}}_{X.}(r) \arrow[r, "\Phi^J"] & W.\Omega^r_{X_1,\log}[-r] \arrow[r, "+1"] & {}
\end{tikzcd}$$

motivic fundamental triangle.

<u>Def:</u> <u>continuous Chow group</u> of $X.$

$$CH^r_{\mathrm{cont}}(X.) := H^{2r}_{\mathrm{cont}}(X_1, \mathbb{Z}_{X.}(r))$$

properties of continuous cohomology $\Rightarrow$ short exact sequence

$$0 \to R^1\varprojlim H^{2r-1}(X_1, \mathbb{Z}_{X_n}(r)) \to CH^r_{cont}(X_\cdot) \to \varprojlim H^{2r}(X_1, \mathbb{Z}_{X_n}(r)) \to 0$$

<u>Example</u>: $r=1$

short exact sequence

$$0 \to R^1\varprojlim H^1(X_1, \mathbb{Z}_{X_n}(r)) \to CH^1_{cont}(X_\cdot) \to \varprojlim H^2(X_1, \mathbb{Z}_{X_n}(r)) \to 0$$

but we have seen in §7, proposition 7.2

$$\mathbb{Z}_{X_\cdot}(1) \cong \mathbb{G}_{m, X_\cdot}[-1]$$

so the sequence above becomes

$$0 \to R^1\varprojlim H^0(X_1, \mathbb{G}_{m,X_n}) \to CH^1_{cont}(X_\cdot) \to \varprojlim H^1(X_1, \mathbb{G}_{m,X_n}) \to 0$$

but $\begin{array}{c} X_n \\ \downarrow \text{smooth. proper} \\ \operatorname{Spec} W_n \end{array} \Rightarrow H^0(X_1, \mathcal{O}_{X_n}) \twoheadrightarrow H^0(X_1, \mathcal{O}_{X_{n-1}})$

with nilpotent kernel.

$$\Rightarrow H^0(X_1, \mathcal{O}_{X_n}^*) \twoheadrightarrow H^0(X, \mathcal{O}_{X_{n-1}}^*)$$

$$\Rightarrow R^1\varprojlim H^0(X_1, \mathbb{G}_{m,X_n}) = 0$$

$$\Rightarrow CH^1_{cont}(X_\cdot) \xrightarrow{\sim} \varprojlim H^1(X_1, \mathbb{G}_{m,X_n}) = \varprojlim \operatorname{Pic}(X_n)$$

that is $CH^1_{cont}(X_\cdot) \cong \operatorname{Pic}(X)$ if $X_\cdot = \hat{X}$ with X/W proper smooth.

Obstruction sequence:

motivic fundamental triangle

$$p(r)\Omega^{<r}_{X.}[-1] \to \mathbb{Z}_{X.}(r) \to \mathbb{Z}_{X_1}(r) \xrightarrow{+1}$$

$$\Rightarrow \quad CH^r_{cont}(X.) \to CH^r(X_1) \xrightarrow{ob} H^{2r-1}_{cont}(X_1, p(r)\Omega^{<r}_{X.})$$

exactness $\Rightarrow$ for $z_1 \in CH^r(X_1)$, z_1 can be lifted to $CH^r_{cont}(X.)$

$\Leftrightarrow ob(z_1) = 0.$

so one needs to compare the last condition (i.e. $ob(z_1)=0$)

with the one that z_1 is Hodge i.e.

$ch_{cris}(z_1) \in H^{2r}_{cris}(X_1/W)_{\mathbb{Q}}$ is contained in

crystalline de Rham comparison

$$\mathrm{Im}\left(F^r H^{2r}_{dR}(X./W)_{\mathbb{Q}} \to H^{2r}_{dR}(X./W)_{\mathbb{Q}} \xrightarrow{\sim} H^{2r}_{cris}(X_1/W)_{\mathbb{Q}} \right)$$

so, one needs to relate obstruction map with the crystalline cycle map

③ Crystalline cycle classes

$$d\log: \mathcal{K}^M_{X_1, r} \longrightarrow W.\Omega^r_{X_1, \log}$$

then the crystalline cycle map is given by

$$CH^r(X_1) \underset{\text{Bloch formula}}{\simeq} H^r(X_1, \mathcal{K}^M_{X_1, r}) \xrightarrow{d\log} H^r(X_1, W.\Omega^r_{X_1, \log})$$

Def. for $\mathfrak{z} \in CH^r(X_1)$, define $c(\mathfrak{z})$ to be the image of $\mathfrak{z}$ via the map

$$\begin{array}{c} CH^r(X_1) \longrightarrow H^r(X_1, W.\Omega^r_{X_1,\log}) = H^{2r}(X_1, W.\Omega^r_{X_1,\log}[-r]) \\ \downarrow \\ H^{2r}(X_1, W\Omega^{\geq r}_{X_1}) \\ \downarrow \\ H^{2r}(X_1, \mathfrak{f}(r) W\Omega_{X_1}) \end{array}$$

called the refine crystalline cycle class of $\mathfrak{z}_1$

the crystalline cycle class of $\mathfrak{z}_1$, denoted by $C_{cris}(\mathfrak{z})$ is the image of $c(\mathfrak{z})$ via

$$H^{2r}(X_1, \mathfrak{f}(r) W\Omega_{X_1}) \longrightarrow H^{2r}(X_1, W\Omega_{X_1})$$

Definition let $\mathfrak{z} \in CH^r(X_1)$

(1) one says that $C_{cris}(\mathfrak{z})$ (resp. $c(\mathfrak{z})$) is Hodge if

$$C_{cris}(\mathfrak{z}) \in \mathrm{Im}\left(H^{2r}_{cont}(X_1, \Omega^{\geq r}_{X_\bullet}) \to H^{2r}_{cont}(X_1, \Omega_{X_\bullet}) \right)$$

$$\left(\text{resp. } c(\mathfrak{z}) \in \mathrm{Im}\left(H^{2r}_{cont}(X_1, \Omega^{\geq r}_{X_\bullet}) \to H^{2r}_{cont}(X_\bullet, p(r)\Omega_{X_\bullet}) \simeq H^{2r}_{cont}(X_\bullet, \mathfrak{f}(r) W\Omega_{X_\bullet}) \right)\right)$$

(2) one say $C_{cris}(\mathfrak{z})$ is Hodge modulo torsion if

$$C_{cris}(\mathfrak{z})_{\mathbb{Q}} \in \mathrm{Im}\left(H^{2r}_{cont}(X_1, \Omega^{\geq r}_{X_\bullet})_{\mathbb{Q}} \to H^{2r}_{cont}(X_1, \Omega_{X_\bullet})_{\mathbb{Q}} \right)$$

Rem. in many cases, the maps in the definition above are injective

(i). degeneration of Hodge-to-de Rham spectral sequence modulo torsion

(ie. 特征0上. 对应谱序列退化)

$\Rightarrow$ the canonical map

$$H^{2r}_{cont}(X_1, \Omega^{\geq r}_{X_\bullet}) \otimes \mathbb{Q} \longrightarrow H^{2r}_{cont}(X_1, \Omega_{X_\bullet})$$

is injective

(ii) Assume $H^b_{cont}(X_1, \Omega^a_{X.})$ torsion free W-module for all a, b.

Spectral sequence

$$E_1^{a,b} = H^b_{cont}(X_1, \Omega^a_{X_1}) \Rightarrow H^{a+b}_{cont}(X_1, \Omega_{X.})$$

$(E_1^{a,b})_{\mathbb{Q}}$ degenerates at E_1

$\Rightarrow$ the differential $d: E_1^{a,b} \to E_1^{a+1,b}$

is zero after $-\otimes \mathbb{Q}$

but $E_1^{a,b}$ torsion free W-module

$\Rightarrow$ $d: E_1^{a,b} \to E_1^{a+1,b}$ is zero as well.

$$\Rightarrow \quad H^a(X_1, \Omega^{\geq r}_{X.}) \to H^a(X_1, \Omega_{X.}) \text{ injective !}$$

Rem: the map $H^{2r}_{cont}(X_1, p(r)\Omega^{\cdot}_{X.}) \otimes \mathbb{Q} \to H^{2r}_{cont}(X_1, \Omega^{\cdot}_{X.}) \otimes \mathbb{Q}$

is an isomorphism.

④ Main theorem

Thm: let $X./W.$ be a smooth projective p-adic formal scheme. let $z_1 \in CH^r(X_1)$

Then

(i) its refined crystalline cycle class $c(z_1) \in H^{2r}_{cont}(X_1, q(r) W.\Omega^{\cdot}_{X_1})$ is Hodge iff z_1 can be lifted to $CH^r_{cont}(X.)$ (via the surjective map

$$CH^r_{cont}(X.) \to CH^r(X_1)$$

(ii) its crystalline class $c_{crys}(\mathfrak{Z}_1) \in H^{2r}_{cont}(X_1, W.\Omega^{\cdot}_{X_1})$ is Hodge modulo torsion iff $\mathfrak{Z}_1 \otimes \mathbb{Q} \in \mathrm{Im}\left(CH^r_{cont}(X.)_{\mathbb{Q}} \to CH^r(X_1)_{\mathbb{Q}}\right)$ i.e. an integral multiple of $\mathfrak{Z}_1$ can be lifted to $CH^r_{cont}(X.)$

<u>Proof</u>: Motivic fundamental triangle

$$p(r)\Omega^{<r}_{X.} \to \mathbb{Z}_{X.}(r) \to \mathbb{Z}_{X_1}(r) \xrightarrow{+1}$$

or equivalently

$$\begin{tikzcd}
\mathbb{Z}_{X.}(r) \arrow[r] \arrow[ddd, dashed, "\substack{\exists \\ \text{by axioms} \\ \text{of triangulated} \\ \text{categories}}"'] & \mathbb{Z}_{X_1}(r) \arrow[r] \arrow[d] & p(r)\Omega^{<r}_{X.} \arrow[d, equal] \\
 & W.\Omega^r_{X_1,\log}[-r] \arrow[r] \arrow[d] & p(r)\Omega^{<r}_{X.} \arrow[dd, equal] \\
 & g(r)W.\Omega^{\cdot}_{X_1} \arrow[d] \arrow[r, phantom, "\circlearrowleft"] & \\
\Omega^{\geq r}_{X.} \arrow[r] & p(r)\Omega^{\cdot}_{X.} \arrow[r] & p(r)\Omega^{<r}_{X.}
\end{tikzcd}$$

commutative because of Thm. 6.1.

taking cohomology, find

$$\begin{tikzcd}
CH^r_{cont}(X.) \arrow[r] \arrow[d] & CH^r(X_1) \arrow[r, "ob"] \arrow[d, "\substack{\text{refined} \\ \text{crystalline cycle} \\ \text{class}}"] & H^{2r}_{cont}(X_1, p(r)\Omega^{<r}_{X.}) \arrow[d, equal] \\
H^{2r}_{cont}(X_1, \Omega^{\geq r}_{X.}) \arrow[r] & H^{2r}_{cont}(X_1, p(r)\Omega^{\cdot}_{X.}) \arrow[r] & H^{2r}_{cont}(X_1, p(r)\Omega^{<r}_{X.})
\end{tikzcd}$$

& its analogue up to torsion

$$
\begin{tikzcd}
CH^r_{cont}(X_\cdot)_{\mathbb{Q}} \arrow[r] \arrow[d] & CH^r(X_1)_{\mathbb{Q}} \arrow[r, "ob\otimes\mathbb{Q}"] \arrow[d] \arrow[dd, bend right, "c_{cris}\otimes\mathbb{Q}"'] & H^{2r}_{cont}(X_1, p(r)\Omega^{<r}_{X_\cdot})_{\mathbb{Q}} \arrow[d, equal] \\
H^{2r}_{cont}(X_1, \Omega^{\geq r}_{X_\cdot})_{\mathbb{Q}} \arrow[r] \arrow[d, equal] & H^{2r}_{cont}(X_1, p(r)\Omega^{\bullet}_{X_\cdot})_{\mathbb{Q}} \arrow[r] \arrow[d, "\cong"] & H^{2r}_{cont}(X_1, p(r)\Omega^{<r}_{X_\cdot})_{\mathbb{Q}} \arrow[d, "\cong"] \\
H^{2r}_{cont}(X_1, \Omega^{\geq r}_{X_\cdot})_{\mathbb{Q}} \arrow[r] & H^{2r}_{cont}(X_1, \Omega^{\bullet}_{X_\cdot})_{\mathbb{Q}} \arrow[r] & H^{2r}_{cont}(X_1, \Omega^{<r}_{X_\cdot})_{\mathbb{Q}}
\end{tikzcd}
$$

With these two diagrams, the main theorem follows by diagram chasing.